ON OPERATORS WHOSE SQUARES ARE 2-NORMAL

ADNAN A. S. JIBRIL

King Faisal University
Saudi Arabia
e-mail: ajibril@kfup.edu.sa

Abstract

In this paper, we introduce the new class (SN) of operators acting on a Hilbert space $H$. In (SN), the square of each operator is 2-normal. We study some basic properties of operators in (SN). We study the relation between the class (SN) and some other classes of operators acting on $H$.

1. Introduction

Let $H$ be a complex Hilbert space and $L(H)$ be the algebra of all bounded linear operators acting on $H$. If $T \in L(H)$, then $T^*$ is its adjoint and $T = A + iB$ is its Cartesian decomposition. In [3], the author introduced the class of 2-normal operators: $T \in L(H)$ is called 2-normal, if $AB^2 = B^2A$ and $BA^2 = A^2B$. Several characterizations of 2-normal operators are given in [3] such as: $T \in L(H)$ is 2-normal if and only if $T^2T^* = T^*T^2$; if and only if $T^2$ is normal. The class of all 2-normal operators is denoted by (2N). In this paper, we enlarge the class (2N) by considering operators in $L(H)$, whose squares are 2-normal. We denote

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the set of all such operators by (SN). It is clear that (2N) is contained in (SN). Also, it is clear that an operator \( T \in L(H) \) is in (SN) if and only if \( T^4T^{s^2} = T^{s^2}T^4 \). In Section 2 of this paper, we investigate some basic properties of operators in (SN). In Section 3, we study the sum and the product of two operators in (SN). In Section 4, we study the relation between the class (SN) and some other classes of operators in \( L(H) \).

2. Some Basic Properties of Operators in (SN)

In this section of the paper, we investigate some basic properties of operators in (SN).

**Proposition 2.1.** If \( T \in (SN) \), then so are

(i) \( kT \) for any real number \( k \).

(ii) \( T^* \).

(iii) \( T^{-1} \) if it exists.

(iv) \( T/M \) (The restriction of \( T \) to any invariant subspace \( M \) of \( H \), which reduces \( T \)).

(v) Any operator \( S \in L(H) \) that is unitarily equivalent to \( T \).

**Proof.** The proofs of (i)-(ii) are straightforward.

(iii) \( (T^{-1})^4(T^{-1})^{s^2} = (T^4)^{-1}(T^{s^2})^{-1} = (T^{s^2}T^4)^{-1} = (T^4T^{s^2})^{-1} = (T^{s^2})^{-1}(T^4)^{-1} = (T^{-1})^{s^2}(T^{-1})^4. \)

Thus \( T^{-1} \in (SN) \).

(iv) Let \( M \) be a closed subspace of \( H \) that reduces \( T \). Then by ([1], Theorem 3, page 158), we have

\[
(T/M)^4(T/M)^{s^2} = (T^4|M)(T^{s^2}|M) = (T^{s^2}T^4|M) = (T^{s^2}T^{s^2}|M) = (T^{s^2}M)(T^{s^2}|M) = (T|M)^2(T|M)^4.
\]
Thus $T|M \in (SN)$.

(v) Let $S \in L(H)$, which is unitarily equivalent to $T$, then there is a unitary operator $U \in L(H)$ such that $S = U^*TU$, which implies that $S^* = U^*T^*U$. Thus

$$S^4S^*^2 = U^*TU^*TUU^*TUU^*T^*UU^*U^*T^*U$$

$$= U^*T^4T^*^2U \ldots \ldots; \quad (i)$$

$$S^*^2S^4 = U^*T^*UU^*T^*UU^*TUU^*TUU^*T^*U$$

$$= U^*T^*^2T^4U \ldots \ldots. \quad (ii)$$

Since the R.H.S's of (i) and (ii) are equal, then $S^4S^*^2 = S^*^2S^4$, which implies that $S \in (SN)$.

**Remark 2.1.** Unitarily equivalence in Proposition 2.1(v) cannot be replaced by similarity as shown by the following example:

**Example 2.1.** Consider the operators $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, and $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ acting on the two-dimensional Hilbert space $\mathbb{R}^2$.

Then direct calculations show that $T \in (SN)$, $S^4S^*^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = S^*^2S^4$, and $S = X^{-1}TX$, where $X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

**Proposition 2.2.** The class $(SN)$ is not convex.

**Proof.** We prove the proposition by an example.
Example 2.2. Consider the two operators $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acting on $\mathbb{R}^2$, then it is easy to show that both operators are 2-normal, thus in (SN). Now consider the operator $A = \frac{1}{2} T + \frac{1}{2} S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$, then by direct calculation, one can shows that $A^4A^{*2} = \begin{pmatrix} 69 & 7 \\ 64 & 32 \\ 1 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 17 \\ 64 & 32 \\ 1 & 21 \end{pmatrix}$. Thus $A \notin \text{SN}$, which means that (SN) is not convex.

Remark 2.2. If $T \in \text{SN}$ such that $T^2 = 0$, then it is not necessarily that $T = 0$.

Example 2.3. Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $\mathbb{R}^2$, then $T^2 = 0$. Thus $T$ is in (SN). Clearly, $T \neq 0$.

If $T$ is in (SN), then it is not necessary that $T + \alpha I$ is in (SN).

Example 2.4. Consider the operator $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ acting on $\mathbb{R}^2$, then by direct calculations, $T$ is in (SN). Consider the operator $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, then direct calculations show that $A^4A^{*2} = \begin{pmatrix} 80 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 64 & 32 \\ 32 & 16 \end{pmatrix}$. Thus $A \notin \text{SN}$.

It was proved in ([6], Proposition 1.4, p.120) that, if $T \in (2N)$ such that $T^2$ is unitarily equivalent to $T^*$, then $T$ is normal. The result is still true for operators in (SN).
**Proposition 2.3.** If $T \in (SN)$ such that $T^2$ is unitarily equivalent to $T^*$, then $T$ is normal.

**Proof.** Since $T^2$ is unitarily equivalent to $T^*$, there is a unitary operator $U \in L(H)$ such that $T^2 = U^* T^* U$, which implies that $T^4 = U^* T^2 U$ and $T^{*2} = U^* T U$. Thus $T^4 T^{*2} = U^* T^2 T U$ and $T^{*2} T^4 = U^* T^{*2} U$. Since $T$ is in (SN), the last two equations implies that $T^2 T^* = T^* T^2$. Thus $T$ is 2-normal. Now, the result follows from ([6], Proposition 1.4, p. 120).

**Proposition 2.4.** The class $(SN)$ is closed in the strong operator topology.

**Proof.** Let $\{S_n\}$ be a sequence of operators in $(SN)$ such that $\{S_n\}$ converges strongly to $S \in (SN)$, i.e., $S_n \rightarrow^s S$. Since the product of operators in the strong operator topology is sequentially continuous ([2], p.62), $S_n^4 \rightarrow^s S^4$. Now since $\{S_n\}$ converges strongly to $S$, we have $\|S_n x - S x\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in H$. Thus,

$$\|S_n^* x - S^* x\| = \|(S_n^* - S^*) x\| = \|(S_n - S) x\|^2 \leq \|(S_n - S)^*\| \|x\|$$

$$= \|S_n^* x - S x\| = \|S_n x - S x\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$  

Thus $\{S_n^*\}$ converges strongly to $S^*$, which implies that $S_n^2 \rightarrow^s S^{*2}$. Thus $S_n^2 S_n^{*2} \rightarrow^s S^4 S^4$ and $S_n^{*2} S_n^2 \rightarrow^s S^{*2} S^4$. Since $\{S_n^2 S_n^{*2}\} = \{S_n^{*2} S_n^2\}$, we have $S^4 S^{*2} = S^{*2} S^4$, which means that $S \in (SN)$, thus (SN) is strongly closed.

### 3. Sum and Product of Two Operators in (SN)

In this section, we study the sum and the product of two operators in (SN).
Proposition 3.1. If \( S, T \in (SN) \) such that \( ST = TS \), then \( ST \in (SN) \).

Proof.

\[
(\text{ST})^4 (\text{ST})^{s_2} = S^4 T^4 T^{s_2} S^{s_2}
\]

\[
= S^4 T^4 S^{s_2} T^{s_2} \quad \cdots \cdots \cdots (i)
\]

Since \( T \in (SN) \), \( T^2 \) is 2-normal, which implies that \( T^4 \) is normal. Since
\( T^4 \) commute with \( S^2 \), then by Fuglede theorem, \( T^4 S^{s_2} = S^{s_2} T^4 \).
Similarly, we prove that \( S^4 T^{s_2} = T^{s_2} S^4 \). Now, the result follows from
the last two equations and (i) above.

Remark 3.1. If, in Proposition 3.1, \( ST \neq TS \), then the result is not
necessarily true. We show this by the following example:

Example 3.1. Consider the two operators \( S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) and
\( T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \) acting on \( R^2 \), then direct calculations show that both \( S \)
and \( T \) are 2-normal, thus \( S \) and \( T \) are in (SN). Now consider
\( ST = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} = K \) (say). Then by direct calculation, one can shows
that \( K^4 K^{s_2} = \begin{pmatrix} 6 & -20 \\ -16 & 64 \end{pmatrix} \), while \( K^{s_2} K^4 = \begin{pmatrix} 1 & -5 \\ -1 & 69 \end{pmatrix} \). Thus
\( K \notin (SN) \). It can be easily shown that \( ST = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \), while
\( TS = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \).

Remark 3.2. If, in Proposition 3.1, product is replaced by sum, then
the result is not, in general, true. We show this by the following example:

Example 3.2. Consider the two operators \( T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and
\( S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) acting on \( R^2 \), then direct calculations show that both \( S \) and
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T are 2-normal, thus both are in (SN). Now consider the operator

\[ S + T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = N \text{ (say)} \]

Then by direct calculation, one can shows that

\[ N^4N^{*^2} = \begin{pmatrix} 9 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \end{pmatrix} = N^{*^2}N^4. \text{ Thus } N \notin (SN). \]

It is clear that \( ST = TS \).

**Proposition 3.2.** The direct sum and the tensor product of two operators in (SN) are in (SN).

**Proof.** Let \( x = x_1 \oplus x_2 \in H \oplus H \) and let \( S, T \in (SN) \), then

\[
(T \oplus S)^4(T \oplus S)^{*^2} = (T \oplus S)^4(T \oplus S)^{*^2}(x_1 \oplus x_2)
\]

\[
= (T \oplus S)^4(T^{*^2} \oplus S^{*^2})(x_1 \oplus x_2)
\]

\[
= (T \oplus S)^4(T^{*^2}x_1 \oplus S^{*^2}x_2)
\]

\[
= T^{*^2}T^4x_1 \oplus S^{*^2}S^4x_2
\]

\[
= (T^{*^2} \oplus S^{*^2})(T^{*^2}x_1 \oplus S^{*^2}x_2)
\]

\[
= (T^{*^2} \oplus S^{*^2})(S^4x_2)
\]

\[
= (T^{*^2} \oplus S^{*^2})(T^{*^2} \oplus S^4)(x_1 \oplus x_2)
\]

\[
= (T^{*^2} \oplus S^{*^2})(T^{*^2} \oplus S^4)x
\]

\[
= (T + S)^{*^2}(T \oplus S)^4x.
\]

Thus \( (T \oplus S)^4(T \oplus S)^{*^2} = (T \oplus S)^{*^2}(T \oplus S)^4 \), which means that \( T \oplus S \in (SN) \). Also

\[
(T \oplus S)^4(T \oplus S)^{*^2}x = (T \oplus S)^4(T \oplus S)^{*^2}(x_1 \oplus x_2)
\]

\[
= (T^4 \oplus S^4)(T^{*^2}x_1 \oplus S^{*^2}x_2)
\]
Thus $(T \otimes S)^4(T \otimes S)^x = (T \otimes S)^x(T \otimes S)^4$, which means that $T \otimes S \in (SN)$.

4. Relation Between the Class $(SN)$ and Some Other Classes of Operators in $L(H)$

In this section, we study the relation between the class $(SN)$ and some other classes of operators in $L(H)$.

**Proposition 4.1.** If $T \in (SN)$ is isometry, then $T$ is unitary.

**Proof.** Since $T \in (SN)$, $T^4T^{x^2} = T^{x^2}T^4$. Multiplying this equation on the left by $T^{x^2}$, we get $T^{x^2}T^4T^{x^2} = T^{x^4}T^4$. Since $T$ is isometry, the last equation becomes $T^{2T^{x^2}} = T^{x^2}T^2$. Thus $T^2$ is normal, which implies that $T \in (2N)$. Now, the result follows from ([3], Proposition 3.1, p. 193).

**Proposition 4.2.** If $T \in (SN)$ is similar to an idempotent $S \in L(H)$, then $T$ is a projection.

**Proof.** Since $T$ is similar to $S$, there is an invertible operator $N \in L(H)$ such that $T = N^{-1}SN$, which implies that $T^2 = N^{-1}S^2N = N^{-1}SN$. Thus $T$ is an idempotent, which implies that $T^4 = T$ and $T^{x^2} = T^x$. Thus, the relation $T^4T^{x^2} = T^{x^2}T^4$ implies that $TT^* = T^*T$, which implies that $T$ is normal. The result now follows from ([2], p. 111).
In [10], Kutkut introduced a new class of operators, which he called the class of parahyponormal operators: \( T \in L(H) \) is called parahyponormal, if \( \|Tx\|^2 \leq \|TT^*x\| \), for all \( x \) in \( H \) with \( \|x\| = 1 \), or equivalently ([10], Theorem 1.1, p. 74), if and only if \( (TT^*)^2 - 2\lambda T^*T + \lambda^2 I \geq 0 \), for every \( \lambda > 0 \).

**Example 4.1.** Consider the operator \( T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) acting on \( R^2 \). It is shown in ([10], Example 2.1, p. 81) that \( T \) is not parahyponormal and direct calculation shows that \( T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), which means that \( T \in (SN) \).

Next, we give an example of a parahyponormal operator, which is not in (SN).

**Example 4.2.** Consider the operator \( T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) acting on \( R^2 \), then it is shown in ([10], Example 2.1, p. 80) that \( T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) is parahyponormal and by direct calculation, one can shows that \( T^4T^* = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} \) is not \( T^2T^* = \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix} \), which means that \( T \not\in (SN) \). Thus, the two classes (SN) and parahyponormal operators are independent. In [8], the author introduced the class of \( n \)-power normal operators: \( T \in L(H) \) is \( n \)-power normal, if \( T^nT^* = T^*T^n \) for some positive integer \( n \). The class of all \( n \)-power normal operators is denoted by \((nN)\). For \( n = 2 \), \((2N) \subset (SN)\). For \( n = 3 \), we show by examples that the classes \((3N)\) and \((SN)\) are independent.

**Example 4.3.** Consider the operator \( T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \) acting on \( R^2 \), then by direct calculation, one can shows that \( T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Thus
$T \in (SN)$. However, and also by direct calculation, one can shows that

$$T^3 T^* = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = T^* T^3.$$ Thus $T \notin (3N)$.

In the following, we give an example of an operator in $(3N)$, which is not in $(SN)$:

**Example 4.4.** Consider the operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ acting on $R^2$, then direct calculations show that $T^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which implies that $T^3 T^* = T^* T^3$. Thus $T \in (3N)$. However, direct calculations show that $T^4 T^* = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \neq \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} = T^* T^4$. Thus $T \notin (SN)$. From the last two examples, we conclude that the classes $(3N)$ and $(SN)$ are independent.

Now, we discuss the relation between the class $(SN)$ and the class of almost subprojection, which was introduced by the author in [5]: $T \in L(H)$ is called almost subprojection, if $T^4 = T^{*2}$.

**Proposition 4.3.** If $T \in L(H)$ is almost subprojection, then $T \in (SN)$.

**Proof.** Since $T$ is almost subprojection, $T^4 = T^{*2}$. Multiplying the last equation on the left and then on the right by $T^{*2}$, we get $T^4 T^{*2} = T^{*4} = T^{*2} T^4$. Thus $T \in (SN)$.

The following is an example of an operator in $(SN)$, which is not almost subprojection:

**Example 4.5.** Consider the operator $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ acting on $R^2$, then it is clear that $T \in (SN)$. Now, by direct calculation,
$T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} \ast \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = T^{*2}$.

**Definition 4.1.** $T \in L(H)$ is called *partial isometry,* if

$$T^*T = (T^*T)^2.$$  

**Example 4.6.** Consider the operator $T = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{2}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ acting on $\mathbb{R}^2$, then direct calculations show that $T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (T^*T)^2$. Thus $T$ is a partial isometry. Now and again, by direct calculation, one can shows that $T^4T^{*2} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = T^{*2}T^4$, which implies that $T \notin (SN)$.

**Example 4.7.** Consider the operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ acting on $\mathbb{R}^2$, then by direct calculation, one can shows that $T^4T^{*2} = \begin{pmatrix} 0 & 8 \\ -8 & 0 \end{pmatrix} = T^{*2}T^4$, which means that $T \in (SN)$. Also and again, by direct calculation, one can shows that $T^*T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = (T^*T)^2$. Thus $T$ is not a partial isometry. From the last two examples, we conclude that the class (SN) and the class of all partial isometric operators are independent.

In [9], Kamei introduced the class of skew normal operators acting on $H$. If $T = A + iB \in L(H)$, then $T$ is called *skew-normal,* if $AB = -BA$. It follows immediately from the definition that $T$ is skew normal, if and only if $T^2$ is Hermitian.

**Proposition 4.4.** If $T \in L(H)$ is skew normal, then $T \in (SN)$.  

$$T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} \ast \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = T^{*2}.$$
**Proof.** The proof follows immediately from ([3], Proposition 2.1, p. 192). The converse of the last proposition is not, in general, true as the following example shows:

**Example 4.8.** Consider the operator \( T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) acting on \( \mathbb{R}^2 \), then by direct calculation, one can shows that \( T^4T^{*2} = \begin{pmatrix} 0 & 8 \\ -8 & 0 \end{pmatrix} = T^{*2}T^4 \). Thus \( T \in \text{(SN)} \). However, \( T^2 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \), which means that \( T^2 \) is not Hermitian. Thus \( T \) is not skew normal.

**References**


