MATRICES WITH CONVOLUTIONS OF BINOMIAL FUNCTIONS, THEIR DETERMINANTS, AND SOME EXAMPLES

NORMAN C. SEVERO and PAUL J. SCHILLO

150 Meadowview Lane
Williamsville, NY 14221
U.S.A.
e-mail: severo@buffalo.edu

288 May Street
Buffalo, NY 14211
U.S.A.

Abstract

The purpose of this article is to obtain the determinant of any member of a class of matrices, whose entries are convolutions of binomial functions. We also give here some examples in the form of problems and solutions, which use some of the properties of these matrices.

1. Introduction

We discuss matrices defined in [3], which have elements that are convolutions of binomial terms. The basis of the paper is a well-known technique from invariant theory, the symmetric representation of matrices, i.e., the representation of the multiplicative semigroup tensors. The general case for matrices of any size is well-known, (see [1], [4]).
For complex numbers $\alpha, \beta, \gamma, \text{ and } \delta$, we denote by

$$M_N = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_N,$$

the $N + 1$ by $N + 1$ matrix, whose $h$-th row and $k$-th column entry, $h, k = 0, \ldots, N$, is

$$\lambda_{hk}(\alpha, \beta, \gamma, \delta; N) = \sum_{t=0}^{N} \binom{N - k}{h - t} \binom{k}{t} \alpha^{N - h - k + t} \beta^{h - t} \gamma^{h} \delta^{k - t}. \tag{1}$$

The matrix $M_N$ equals 1, for $N = 0$, the ordinary 2 by 2 matrix $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$.

For $N = 1$, and for $N = 2$,

$$\begin{bmatrix} \alpha^2 \\ 2\alpha \beta \\ \beta^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma^2 \\ 2\gamma \delta \\ \delta^2 \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha \gamma & \gamma^2 \\ 2\alpha \beta & \beta \gamma + \alpha \delta & 2\gamma \delta \\ \beta^2 & \beta \delta & \delta^2 \end{bmatrix},$$

which illustrates another method for generating the 3 by 3 matrix (see [3]). (In the Appendix, we display the matrices for $N = 3, 4, \text{ and } 5$.)

In [3], the following multiplication theorem is proved for $N = 0, 1, \ldots$, and for any complex numbers $\alpha, \beta, \gamma, \delta, a, b, c, \text{ and } d$:

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}_N \begin{bmatrix} a & c \\ b & d \end{bmatrix}_N = \begin{bmatrix} \alpha a + \gamma b & \alpha c + \gamma d \\ \beta a + \delta b & \beta c + \delta d \end{bmatrix}_N. \tag{2}$$

To fully appreciate the power of this homomorphism, we encourage the reader to verify (2) for at least $N = 2$ and 3, and for some entries for $N = 4$ and 5.

Although, the determinant calculation of any matrix can be done by writing the matrix in upper-triangular form (see, e.g., [2]), we derive the result directly by using generating functions of the matrices.

### 2. The Determinant of $M_N$

In the following theorem, we use an alternative definition of $M_N$, which is also given in [3]. The $k$-th column, $k = 0, \ldots, N$, of $M_N$ has $h$-th
element $\lambda_{hk}(\alpha, \beta, \gamma, \delta; N)$, which is the coefficient of $u^k$, for $h = 0, \ldots, N$, in the generating function

$$G_k(N; u) = (\alpha + \beta u)^{N-k}(\gamma + \delta u)^k.$$  

**Theorem.** The determinant of $M_N$, for $N = 0, 1, 2, \ldots$, is

$$D_N = (\alpha\delta - \beta\gamma)^{N(N+1)/2}. \quad (3)$$

**Proof.** We define $E_N$ as the $N+1$ by $N+1$ determinant with column 0 equal to that of $D_N$, and $k$-th column generating function, $k = 1, \ldots, N$, equal to

$$G_k(N; u) - G_{k-1}(N; u) (\gamma / \alpha)$$

$$= (\alpha + \beta u)^{N-k}(\gamma + \delta u)^k - (\alpha + \beta u)^{N-(k-1)}(\gamma + \delta u)^{k-1}(\gamma / \alpha)$$

$$= (\alpha + \beta u)^{N-k}(\gamma + \delta u)^{k-1}[(\gamma + \delta u) - (\alpha + \beta u)\gamma / \alpha]$$

$$= G_{k-1}(N - 1; u) cu,$$

where

$$c = \frac{\alpha\delta - \beta\gamma}{\alpha}.$$  

Thus, $D_N$ is equal to the determinant $E_N$ with the column $k = 0$, the same as that of $D_N$, and with $k$-th column, $k = 1, \ldots, N$, having entries 0, for $h = 0$, and $c\lambda_{h-1,k}(\alpha, \beta, \gamma, \delta; N-1)$, for $h = 1, \ldots, N$. Now, using Laplace's formula to expand $E_N$ along the row $h = 0$, we get

$$D_N = \alpha^N |C(N)| D_{N-1},$$

where $|C(N)|$ is the determinant of the $N$ by $N$ diagonal matrix, each of whose entries is $c$.

Finally, using the induction hypothesis,

$$D_{N-1} = (\alpha\delta - \beta\gamma)^{(N-1)N/2},$$

which is trivially true for $N = 1$, we get
This completes the proof.

Note that, for example, the determinant of $M_5$, given in the Appendix, is simply $(\alpha \delta - \beta \gamma)^{15}$.

Also from [3], the Krawtchouk matrix is

$$B_N = \begin{pmatrix} 1 & 1 \\ 1 & 1 - p^{-1} \end{pmatrix}_N,$$

so that its determinant is

$$|B_N| = \left(-\frac{1}{p}\right)^{N(N+1)/2}.$$  

This simple result appears to be new.

3. Some Identities

The following identities are derived using either Equations (1), (2), or (3).

**Problem 1.** For non-negative integers $N$, $h$, and $k$, and complex numbers $w$ and $z$, show that

$$\sum_{t=0}^{N} {t \choose h} {k \choose t} w^{t-h} z^{k-t} = {k \choose h} (w + z)^{k-h}.$$

**Note.** Use the equation

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}_N \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}_N = \begin{pmatrix} 1 & w + z \\ 0 & 1 \end{pmatrix}_N,$$

and the fact that the entry in the $h$-th row, and $k$-th column of the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}_N$$
is $${k \choose h} x^{k-h}.$$
Problem 2. From the relation
\[
\begin{bmatrix}
i & 1 \\0 & 1
\end{bmatrix}_N \begin{bmatrix}
i & 1 \\1 & 0
\end{bmatrix}_N \begin{bmatrix}
i & 0 \\1 & 1
\end{bmatrix}_N = \begin{bmatrix}
i & 0
\end{bmatrix}_N,
\]
in which \(i\) is the imaginary unit, and \(N\) is a non-negative integer, show that for \(h, k, 0, \ldots, N,\)
\[
\sum_{s=0}^{N} \sum_{t=0}^{N} \binom{N-k}{s-k} \binom{N-s}{t} \binom{t}{h} (-1)^{s+t} = \binom{N-k}{h} (-1)^N.
\]

Problem 3. For non-negative integer \(N,\) and any real numbers \(x\) and \(y \neq 1,\) let \(A_N\) and \(B_N\) be the \(N+1\) by \(N+1\) matrices, whose \(h\)-th rows and \(k\)-th columns entries are, respectively,
\[
a_{hk} = (\sqrt{y-1})^{h+k} (\sqrt{x+y} - \sqrt{x})^{N-h-k} \sum_{t=0}^{N} \binom{N-k}{h-t} \binom{t}{y/y-1}^t,
\]
and
\[
b_{hk} = (\sqrt{y-1})^{h+k} (\sqrt{x+y} + \sqrt{x})^{N-h-k} \sum_{t=0}^{N} \binom{N-k}{h-t} \binom{t}{y/y-1}^t.
\]
Show that the determinants of \(A_N\) and \(B_N\) are each equal to 1.

Note.
\[
A_N = \begin{bmatrix}
\sqrt{x+y} - \sqrt{x} \\
\sqrt{y-1}
\end{bmatrix}_N
\begin{bmatrix}
\sqrt{y-1} \\
\sqrt{x+y} + \sqrt{x}
\end{bmatrix}_N.
\]
Furthermore, if
\[
H_N = \begin{bmatrix}
0 & 1 \\1 & 0
\end{bmatrix}_N,
\]
then \(B_N = H_N A_N H_N.\)

Problem 4. Let \(D_N\) be the determinant, whose \(h\)-th row, \(k\)-th column element is
\[ \left( \frac{\sqrt{2}}{2} \right)^{N-h} \sum_{k=0}^{N} \binom{N-k}{h-k} i^{h-t}, \]

where \( i \) is the imaginary unit. Show that the value of \( D_N \) is

\[ \cos \left( \frac{7\pi N(N+1)}{8} \right) + i \sin \left( \frac{7\pi N(N+1)}{8} \right). \]

**Note.** \( D_N \) is the determinant of the matrix

\[ \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ i & 1 \end{pmatrix} \]

### 4. Appendix

For \( N = 3, 4, \) and \( 5, \) the matrices \( M_N \) are, respectively,

\[ \begin{pmatrix} \alpha^3 & \alpha^2 \gamma & \alpha \gamma^2 & \gamma^3 \\ \beta^3 + \gamma^2 & 3\alpha^2 \beta + 2\alpha \gamma \delta & 2\beta \gamma + \alpha \delta^2 & 3\gamma \delta^2 \\ \beta \gamma & \beta^2 \gamma + 2\alpha \delta^2 & 2\beta \gamma \delta + \alpha \delta^2 & 3\gamma \delta^2 \\ \beta^3 & \beta^2 \delta & \beta \delta^2 & \delta^3 \end{pmatrix} \]

and

\[ \begin{pmatrix} \alpha^4 & \alpha^3 \gamma & \alpha^2 \gamma^2 & \alpha \gamma^3 & \gamma^4 \\ 4\alpha^3 \beta + \gamma^2 & 3\alpha^2 \beta \gamma + \alpha \gamma \delta & 2\alpha \beta \gamma^2 + 2\alpha \gamma \delta & 3\beta \gamma^2 \delta + 3\alpha \gamma \delta^2 & 4\gamma \delta^2 \\ 6\alpha \beta^2 & 3\alpha \beta \gamma + 3\alpha \gamma \delta & \beta^2 \gamma^2 + 4\alpha \beta \gamma \delta + \alpha \gamma \delta^2 & 3\beta \gamma^2 \delta + 3\alpha \gamma \delta^2 & 6\gamma \delta^2 \\ 4\alpha \beta^3 & \beta^3 \gamma + 3\alpha \beta \delta & 2\beta^2 \gamma \delta + 2\alpha \beta \delta^2 & 3\beta \gamma \delta^2 + \alpha \delta^3 & 4\gamma \delta^3 \\ \beta^4 & \beta^3 \delta & \beta^2 \delta^2 & \beta \delta^3 & \delta^4 \end{pmatrix} \]
References


