

## ON THE LAGRANGIAN SUBSPACES OF COMPLEX MINKOWSKI SPACE

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### Abstract

In this article, we mainly investigate the spaces of  $\mathbb{C}^n$  with complex Finsler metrics and study their Lagrangian subspaces. For torus invariant complex Finsler metrics, we show that there is a topological torus of Lagrangians. For general complex Finsler metric, we show that the subspaces of  $\mathbb{C}^n$  with complex Finsler metric in the orbit of any Lagrangian acted by the isometry group of the metric are Lagrangians.

### 1. Introduction

As a generalization of Hermitian metric, the complex Finsler metric is defined in many literatures, for example, [1], [3] and [5], as follows.

**Definition 1.** A complex Finsler metric  $F$  is a function  $F : \mathbb{C}^n \rightarrow \mathbb{R}$  that satisfies the following conditions:

(1)  $F(v) \geq 0$  for any  $v \in \mathbb{C}^n$  and  $F(v) = 0$ , if and only if  $v = 0$ .

(2)  $F(\lambda v) = |\lambda| F(v)$  for any  $\lambda \in \mathbb{C}$  and  $v \in \mathbb{C}^n$ .

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(3)  $F$  is smooth on  $\mathbb{C}^n \setminus \{0\}$ .

Furthermore, a complex Finsler metric  $F$  is called to be *pseudoconvex*, if the (1.1)-real form

$$\frac{1}{2} \sqrt{-1} \partial \bar{\partial} (F^2) \quad (1)$$

is positive-definite, see [3], or equivalently, the Hermitian matrix

$$(g_{i\bar{j}}(\xi_1, \dots, \xi_n))_{n \times n}, \quad (2)$$

in which

$$g_{i\bar{j}} = \frac{1}{2} \frac{\partial^2 (F^2)}{\partial \xi_i \partial \bar{\xi}_j} \quad (3)$$

is positive-definite, see [6].

There is a class of complex Finsler metrics that attracts considerable interest, the torus invariant complex Finsler metric, which is defined as follows.

**Definition 2.** A complex Finsler metric  $F$  on  $\mathbb{C}^n$  is said to be *torus invariant*, if

$$F(e^{i\theta_1} \xi_1, \dots, e^{i\theta_n} \xi_n) = F(\xi_1, \dots, \xi_n), \quad (4)$$

for any  $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$  and any  $(e^{i\theta_1}, \dots, e^{i\theta_n}) \in \mathbb{T}^n := U(1)^n$ .

A complex space  $\mathbb{C}^n$  is called a *complex Minkowski space*, if it has a complex Finsler metric  $F$ , denoted as  $(\mathbb{C}^n, F)$ .

In a complex Minkowski space  $(\mathbb{C}^n, F)$ , the (1.1)-real form (1) gives a Kahler structure, and we would like to find the Lagrangian subspaces under the Kahler structure. For this purpose, we are able to obtain the following theorem:

**Theorem 3.** *The  $n$ -real planes in*

$$\mathbf{T}^n := \{ \text{span}_{\mathbb{R}}((e^{i\theta_1}, 0, \dots, 0), \dots, (0, \dots, 0, e^{i\theta_n})) : (e^{i\theta_1}, \dots, e^{i\theta_n}) \in U(1)^n \} \quad (5)$$

are Lagrangians for any complex Minkowski space  $(\mathbb{C}^n, F)$  with torus invariant complex Finsler metric  $F$ .

**Proof.** From (1), the Kahler form for the complex Minkowski space  $(\mathbb{C}^n, F)$  is

$$\kappa = \sqrt{-1} \sum_{i,j=1}^n \frac{1}{2} \frac{\partial^2(F^2)}{\partial \xi_i \partial \bar{\xi}_j} d\xi_i \wedge d\bar{\xi}_j. \quad (6)$$

Let  $F(r_1, \dots, r_n) := F(|\xi_1|, \dots, |\xi_n|)$ . The definition of torus invariant complex Finsler metric, (4), yields

$$F(\xi_1, \dots, \xi_n) = F(|\xi_1|, \dots, |\xi_n|). \quad (7)$$

Using the differential equalities

$$\frac{\partial}{\partial \xi_i} |\xi_i| = \frac{1}{2} \frac{|\xi_i|}{\xi_i}, \quad (8)$$

and

$$\frac{\partial}{\partial \bar{\xi}_i} |\xi_i| = \frac{1}{2} \frac{|\xi_i|}{\bar{\xi}_i}, \quad (9)$$

for  $i = 1, \dots, n$ , we have

$$\frac{\partial^2(F^2)}{\partial \xi_i \partial \bar{\xi}_j} = \frac{1}{4} \frac{\partial^2(F^2)}{\partial r_i \partial r_j} \frac{|\xi_i \xi_j|}{\xi_i \bar{\xi}_j}. \quad (10)$$

Now, let us take any  $e_{i_0} := (0, \dots, e^{i\theta_{i_0}}, \dots, 0)$  and  $e_{j_0} := (0, \dots, e^{i\theta_{j_0}}, \dots, 0)$ ,  $1 \leq i_0, j_0 \leq n$ , and evaluate the Kahler form  $\kappa$  on these two vectors at any point  $v = (\lambda_1 e^{i\theta_1}, \dots, \lambda_n e^{i\theta_n}) \in \text{span}_{\mathbb{R}}((e^{i\theta_1}, 0, \dots, 0), \dots, (0, \dots, 0, e^{i\theta_n}))$ . It follows from (6) that

$$\kappa_v(e_{i_0}, e_{j_0}) = \frac{\sqrt{-1}}{2} \left( \frac{\partial^2(F^2)}{\partial \xi_{i_0} \partial \bar{\xi}_{j_0}} e^{i(\theta_{i_0} - \theta_{j_0})} - \frac{\partial^2(F^2)}{\partial \xi_{j_0} \partial \bar{\xi}_{i_0}} e^{i(\theta_{j_0} - \theta_{i_0})} \right). \quad (11)$$

Applying (10), we obtain that

$$\begin{aligned} \kappa_v(e_{i_0}, e_{j_0}) &= \frac{\sqrt{-1}}{8} \frac{\partial^2(F^2)}{\partial r_i \partial r_j} \left( \frac{\lambda_{i_0} \lambda_{j_0}}{\lambda_{i_0} e^{i\theta_{i_0}} \lambda_{j_0} e^{-i\theta_{j_0}}} e^{i(\theta_{i_0} - \theta_{j_0})} \right. \\ &\quad \left. - \frac{\lambda_{j_0} \lambda_{i_0}}{\lambda_{j_0} e^{i\theta_{j_0}} \lambda_{i_0} e^{-i\theta_{i_0}}} e^{i(\theta_{j_0} - \theta_{i_0})} \right) \\ &= 0. \end{aligned} \quad (12)$$

Therefore,  $\kappa$  vanishes at every point on the  $n$ -plane

$$\text{span}_{\mathbb{R}}((e^{i\theta_1}, 0, \dots, 0), \dots, (0, \dots, 0, e^{i\theta_n})) \quad (13)$$

in  $\mathbf{T}^n$ , that finishes the proof.  $\square$

To generalize the results from complex  $L^p$  spaces, we would like to ask whether the  $n$ -real planes in

$$\mathbf{T}^1 := \{\text{span}_{\mathbb{R}}((e^{i\theta}, 1, \dots, 1), \dots, (1, \dots, 1, e^{i\theta})) : e^{i\theta} \in U(1)\} \quad (14)$$

are Lagrangians for any complex Minkowski space  $(\mathbb{C}^n, F)$  with torus invariant complex Finsler metric  $F$  or not. However, it turns out that the answer is no in general. For this, one can take the example of  $\mathbb{C}^2$  with  $F(\xi_1, \xi_2) = \sqrt[4]{|\xi_1|^4 + 2|\xi_2|^4}$ . In this case, one can see that  $\text{span}(e^{i\theta}, 1)$ ,  $(e^{i\theta}, 1)$  is not a Lagrangian subspace under the Kahler form (6).

For non-torus invariant complex Finsler metrics, we have the following remark to Theorem 3:

**Remark 4.** In general, the  $n$ -real planes in

$$\mathbf{T}^n := \{\text{span}_{\mathbb{R}}((e^{i\theta_1}, 0, \dots, 0), \dots, (0, \dots, 0, e^{i\theta_n})) : (e^{i\theta_1}, \dots, e^{i\theta_n}) \in U(1)^n\} \quad (15)$$

are not necessarily Lagrangian subspaces in complex Minkowski space  $(\mathbb{C}^n, F)$ . One example is the following space,  $\mathbb{C}^2$  with non-torus invariant complex Finsler metric:

$$F(\xi_1, \xi_2) = \sqrt{|\xi_1 \bar{\xi}_2 + 2\bar{\xi}_1 \xi_2| + 2|\xi_1|^2 + 2|\xi_2|^2}. \quad (16)$$

In this example, a direct computation gives

$$\begin{aligned} \frac{\partial^2(F^2)}{\partial \xi_1 \partial \bar{\xi}_2} &= \frac{1}{4|\xi_1 \bar{\xi}_2 + 2\bar{\xi}_1 \xi_2|^3} (2|\xi_1 \bar{\xi}_2 + 2\bar{\xi}_1 \xi_2|^2 (5\xi_2 \bar{\xi}_1 + 8\xi_1 \bar{\xi}_2) \\ &\quad - (5\bar{\xi}_1 |\xi_2|^2 + 4\xi_1 \bar{\xi}_2^2) (5|\xi_1|^2 \xi_2 + 4\xi_1^2 \bar{\xi}_2)), \end{aligned} \quad (17)$$

and

$$\frac{\partial^2(F^2)}{\partial \xi_2 \partial \bar{\xi}_1} = \overline{\frac{\partial^2(F^2)}{\partial \xi_1 \partial \bar{\xi}_2}}. \quad (18)$$

If we evaluate the Kahler form (6) on  $e_1 := (0, e^{i\theta_1})$  and  $e_2 := (0, e^{i\theta_2})$  at any point  $v = (\lambda_1 e^{i\theta_1}, \lambda_2 e^{i\theta_2})$  in the plane  $\text{span}(e_1, e_2)$ , we then have

$$\begin{aligned} \kappa_v(e_1, e_2) &= -\frac{2}{|e^{i \cdot 2(\theta_1 - \theta_2)} + 2|^3} \text{Im}((2|e^{i \cdot 2(\theta_1 - \theta_2)} + 2|^2 + 5)e^{i(\theta_1 - \theta_2)} \\ &\quad + 2e^{i \cdot 3(\theta_1 - \theta_2)})e^{i(\theta_1 - \theta_2)}. \end{aligned} \quad (19)$$

Let  $\theta := \theta_1 - \theta_2$ , then

$$\kappa_v(e_1, e_2) = -\frac{2}{|e^{i \cdot 2\theta} + 2|^3} \text{Im}(2|e^{i \cdot 2\theta} + 2|^2 + 5)e^{i \cdot 2\theta} + 2e^{i \cdot 4\theta}). \quad (20)$$

One can see from (20) that, the plane  $\text{span}(e_1, e_2)$  must satisfy that  $\theta_1 - \theta_2 = \frac{k\pi}{2}$ ,  $k = 0, \pm 1, \pm 2, \pm 3$  in order to have  $\kappa_v(e_1, e_2) = 0$ . So, in the torus  $\mathbf{T}^2 := \{\text{span}_{\mathbb{R}}((e^{i\theta_1}, 0), (0, e^{i\theta_2})) : (e^{i\theta_1}, e^{i\theta_2}) \in U(1)^2\}$ , we only have 4 circles of planes, which are Lagrangian subspaces of the complex Minkowski space  $\mathbb{C}^2$  with non-torus invariant complex Finsler metric (16).

From the above discussions, we know that the Lagrangian subspaces depend greatly on Finsler metrics. Complex Minkowski spaces with Hermitian metrics as special Finsler metrics can have much more Lagrangian subspaces than complex Minkowski spaces with non-Hermitian Finsler metrics do, for instance, complex  $L^2$  space and  $L^p$  space,  $1 \leq p < \infty$ ,  $p \neq 2$ . Moreover, by perturbing some coefficients in Finsler metric, one will have different Lagrangian subspaces for the space, for example,  $L_{(\alpha, \beta)}^p(\xi_1, \xi_2) := (\alpha|\xi_1|^p + \beta|\xi_2|^p)^{1/p}$ ,  $\alpha, \beta \in \mathbb{R}^+$ . Of course, for  $L_{(\alpha, \beta)}^p$  metric, one can express the Lagrangians in terms of the parameters  $\alpha$  and  $\beta$ , and the space of Lagrangians for  $L_{(\alpha, \beta)}^p$  will have the same topology as the one for  $L^p$ , but in the class of torus invariant complex Finsler metrics, one can have spaces of Lagrangians with very different topologies, though all the spaces of Lagrangians contain the torus (5).

However, we can generalize Theorem 3 to general complex Finsler metrics in another direction. First, we know that the space of Lagrangian subspaces in any complex Minkowski space is closed, and then:

**Lemma 5.** *The space of Lagrangian subspaces in a complex Minkowski space  $(\mathbb{C}^n, F)$  is compact in  $Gr(n, \mathbb{C}^n)$ .*

One can define a group

$$G := \{g \in GL(n, \mathbb{C}^n) : F(gv) = F(v) \text{ for any } v \in \mathbb{C}^n\} \quad (21)$$

is called the *isometry group* of a Finsler metric  $F$  on  $\mathbb{C}^n$ , see [2]. Aikou in [2] points out that, one can use the method in [7] that treats real Finsler manifolds to show the following:

**Lemma 6.** *The isometry group  $G$  of a complex Finsler metric is a compact Lie group.*

Based on the above lemma, we can establish the following theorem on the Lagrangians of a space with complex Finsler metric. Thanks to J. Fu for pointing out a functorial proof as well.

**Theorem 7.** *Suppose an  $n$ -real plane  $P$  is a Lagrangian subspace of a complex Minkowski space  $(\mathbb{C}^n, F)$ . Then for any  $g$  in the isometry group  $G$ , the  $n$ -real plane  $gP$ , which is  $P$  acted by  $g$  is also a Lagrangian subspace.*

**Proof.** One can prove it by evaluating the Kahler form (6) on transformed vectors and carrying out a direct computation.

The other proof is functorial. Using the commutativity of the holomorphic map  $g$  with the complex differential operators  $\partial$  and  $\bar{\partial}$ , in other words,  $g^*\partial = \partial g^*$  and  $g^*\bar{\partial} = \bar{\partial} g^*$ , see [4], we have by (21) that

$$g^*\kappa = g^*\left(\frac{1}{2}\sqrt{-1}\partial\bar{\partial}(F^2)\right) = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}(g^*F^2) = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}F^2 = \kappa. \quad (22)$$

So if  $\kappa$  vanishes at any pair of vectors in  $P$ , it also does in  $gP$ .  $\square$

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