ON GLOBAL EXPONENTIAL STABILITY OF GENERALIZED COHEN-GROSSBERG NEURAL NETWORKS WITH TIME-VARYING DELAYS

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Abstract

In this paper, a generalized model of Cohen-Grossberg neural networks with time-varying delays is investigated. By employing the nonlinear measure approach, analytic methods, inequality technique and M-matrix theory, some sufficient conditions ensuring the existence, uniqueness and global exponential stability of equilibrium point for Cohen-Grossberg neural networks with time-varying delays are obtained. Two examples are given to show the effectiveness of the obtained results.

1. Introduction

Cohen-Grossberg neural network model was initially proposed by Cohen and Grossberg [12] in 1983 and soon has attracted considerable attention in theoretical research and engineering applications. In reality,
time delays inevitably exist in biological and artificial neural networks due to the finite switching speed of neurons and amplifiers. Thereby, it is very important to incorporate time delay in various neural networks. Usually, constant fixed time delays in models of delayed feedback systems serve as good approximation in simple circuits having a small number of cells. In most situations, delays are time-varying. Therefore, the studies of neural networks with time-varying delays are more important and actual than those with constant delays. In this paper, we consider a new CGNNs model with time-varying delays, which is described by the following functional differential equations:

\[
\frac{dx_i(t)}{dt} = -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(u_j x_j(t)) \right] - \sum_{j=1}^{n} d_{ij} g_j(v_j x_j(t - \tau_{ij}(t))) + I_i, \quad i = 1, 2, \ldots, n,
\]

(1)

where \( n \geq 2 \) is the number of neurons in the network, \( x_i(t) \) corresponds to the state of the \( i \)th neuron at time \( t \); \( a_i \) presents an amplification function; \( b_i \) is an appropriately behaved function; \( I_i \) denotes external input to the \( i \)th neuron; \( c_{ij}, d_{ij} \) denote the connection strengths of the \( j \)th neuron on the \( i \)th neuron, respectively; \( u_j > 0 \) and \( v_j > 0 \) are the neuronal gains associated with the neuronal activation; \( f_j \) and \( g_j \) denote the activation functions, respectively; \( \tau_{ij}(t) \) corresponds to the transmission delay and satisfies \( 0 \leq \tau_{ij}(t) \leq \tau \) (\( \tau \) is a constant). The initial conditions associated with (1) are of the following form:

\[
x_i(s) = \phi_i(s), \quad s \in [-\tau, 0], \quad i = 1, 2, \ldots, n.
\]

(2)

In particular, when \( u_j = v_j = 1 \) (\( j = 1, 2, \ldots, n \)), model (1) reduces to CGNNs with time-varying delays of the following form [1, 4, 17, 20, 21, 32, 33]

\[
\frac{dx_i(t)}{dt} = -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) \right] - \sum_{j=1}^{n} d_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i, \quad i = 1, 2, \ldots, n,
\]

(3)

when \( \tau_{ij}(t) = \tau_{ij} \), model (1) reduces the generalized CGNNs with discrete delays of the following form [28, 29]
\[ \frac{dx_i(t)}{dt} = -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}f_j(u_jx_j(t)) - \sum_{j=1}^{n} d_{ij}g_j(v_jx_j(t - \tau_{ij})) + I_i \right], \]

\[ i = 1, 2, \ldots, n. \] (4)

Moreover, the system (1) includes many famous models as special case, for example, Hopfield neural networks with discrete or time-varying delays (e.g., see [8, 9, 14, 31]), cellular neural networks with discrete or time-varying delays (e.g., see [10, 11], [2, 7, 23, 30, 34-36, 38, 39]) and bi-directional associate memory (BAM) neural network with discrete or time-varying delays (e.g., see [13, 18, 19, 22], [5, 6, 24, 25, 37]).

Motivated by the above discussions, the objective of this paper is to study the global exponential stability of a class of generalized Cohen-Grossberg neural networks with time-varying delays. Our methods, which does not make use of Lyapunov functional, is simple and valid for stability analysis of neural networks with variable delays, without assuming the boundedness of the activation functions and the differentiability of time-varying delays, as needed in most other papers.

The rest of this paper is organized as follows. Preliminaries are given in Section 2. In Section 3, we give main results and their proof. Examples are given to illustrate our theory in Section 4. Finally, in Section 5 we give the conclusion.

### 2. Preliminaries

Throughout this paper we assume that:

**(A1)** Each function \( a_i(x) \) is positive, continuous and there exist constants \( a_i \) such that

\[ 0 < a_i \leq a_i(x), \text{ for } u \in R, i = 1, 2, \ldots, n. \]

**(A2)** \( b_i(x) \) is monotone increasing, i.e., there exists a positive diagonal matrix \( B = \text{diag}(b_1, b_2, \ldots, b_n) \) such that

\[ \frac{b_i(x) - b_i(y)}{x - y} \geq b_i, \]

for all \( x, y \in R(x \neq y), i = 1, 2, \ldots, n. \)
For the activation functions \( f_i(x) \) and \( g_i(x) \), there exist positive diagonal matrices \( F = \text{diag}(F_1, F_2, \ldots, F_n) \) and \( G = \text{diag}(G_1, G_2, \ldots, G_n) \) such that
\[
F_i = \sup_{x \neq y} \frac{|f_i(x) - f_i(y)|}{x - y}, \quad G_i = \sup_{x \neq y} \frac{|g_i(x) - g_i(y)|}{x - y},
\]
for all \( x, y \in \mathbb{R}(x \neq y) \), \( i = 1, 2, \ldots, n \).

To begin with, we introduce some notation and recall some basic definitions.

For an \( n \times n \) matrix \( A \), \( |A| \) denotes the absolute value matrix given by \( |A| = (|a_{ij}|)_{n \times n} \). \( \lambda_{\max}(A^T A) \) and \( \lambda_{\min}(A^T A) \), respectively, denote the maximum and the minimum eigenvalue of matrix \( A^T A \). For any vector \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \), \( \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in C([-\tau, 0], \mathbb{R}^n) \), \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \), denote
\[
\|x\|_2 = \left( \sum_{i=1}^{m} x_i^2 \right)^{\frac{1}{2}}, \quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad \|\phi\|_\infty = \sup_{-\tau \leq s \leq 0} \{\|\phi\|_\infty\},
\]
\[
\|A\|_1 = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|, \quad \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)},
\]
\[
[x(t)]_\tau = ([x_1(t)]_\tau, [x_2(t)]_\tau, \ldots, [x_n(t)]_\tau)^T, \quad [x_1(t)]_\tau = \sup_{s \in [-\tau, 0]} \{x_1(t + s)\}, \quad |x_1(t)|_\tau = \|[x_1(t)]_\tau\|.
\]
Denote by \( \langle \cdot, \cdot \rangle \) the inner product of any two vectors in \( \mathbb{R}^n \), and for \( A, B \in \mathbb{R}^{n \times n} \) or \( A, B \in \mathbb{R}^n \), \( A \geq B(A > B) \) means that each pair of corresponding elements of \( A \) and \( B \) such that the inequality \( \geq (>) \).

**Definition 1.** An equilibrium point \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) of system (1) is said to be globally exponentially stable, if there exist positive constants \( \lambda > 0 \) and \( K > 0 \) such that
\[
|x(t) - x^*| \leq Ke^{-\lambda t}, \quad t \geq 0.
\]
Definition 2 [3]. A real matrix $D = (d_{ij})_{n \times n}$ is said to be a nonsingular $M$-matrix if $d_{ij} \leq 0$, $i, j = 1, 2, \ldots, n$, $i \neq j$, and all successive principal minors of $D$ are positive.

To the nonsingular $M$-matrix, we have

Lemma 1 [3]. Each of the following conditions is equivalent:

(i) $D$ is a nonsingular $M$-matrix.

(ii) $D^{-1}$ exists and $D^{-1}$ is a nonnegative matrix.

(iii) The diagonal elements of $D$ are all positive and there exists a positive vector $d$ such that $Dd > 0$ or $D^T d > 0$.

(iv) There exists a positive diagonal matrix $P = \text{diag}(p_1, p_2, \ldots, p_n)$ such that $PD + D^TP$ is a positive definite matrix.

Definition 3 [29]. Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$ and $H$ is an operator from $\Omega$ into $\mathbb{R}^n$. The constant

$$m_\Omega(H) = \sup_{x, y \in \Omega, x \neq y} \frac{\langle H(x) - H(y), x - y \rangle}{\|x - y\|_2^2}$$

is called the nonlinear measure of $H$ on $\Omega$.

Lemma 2 [29]. If $m_\Omega(H) < 0$, then $H : \Omega \to \mathbb{R}^n$ is a one-to one to mapping. In particular, if $\Omega = \mathbb{R}^n$, then $H$ is a homeomorphism of $\mathbb{R}^n$.

Lemma 3. Let $\gamma(t)$ be a continuous function in space $\mathbb{R}^n$, if the following conditions hold

(i) $D^+ \gamma(t) \leq a(y(t)) [P \gamma(t) + Q[\gamma(t)]_c]$, 

(ii) $-(P + Q)$ is a nonsingular $M$-matrix,

where $a(y(t)) = \text{diag}(a_1(y_1(t)), a_2(y_2(t)), \ldots, a_n(y_n(t)))$, $a_i(y_i(t)) \geq a_i > 0$, $\gamma(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$, $[\gamma(t)]_c = ([y_1(t)]_c, [y_2(t)]_c, \ldots, [y_n(t)]_c)^T$, $[y_i(t)]_c = \sup_{s \in [-\tau, 0]} \{y_i(t+s)\}$, $P = (p_{ij})_{n \times n}$, $p_{ij} \geq 0$ $(i \neq j)$, $Q = (q_{ij})_{n \times n}$, $q_{ij} \geq 0$. 
Then there exist a constant $\lambda$ and a constant vector $K \geq \lceil [y(0)]_c \rceil$ such that $y(t) \leq Ke^{-\lambda t}$.

**Proof.** Since $-(P + Q)$ is an $M$-matrix, there exist a positive vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T > 0$ such that $(P + Q)\xi < 0$. Set $A = \text{diag}(a_1, a_2, \cdots, a_n)$, then $A > 0$, it follows that $A(P + Q)\xi < 0$ is an $M$-matrix, and there exists a $\lambda > 0$ such that

$$[\lambda E + A(P + Qe^{\lambda t})]\xi < 0. \quad (5)$$

Let $k = \frac{\|\phi\|}{\min_{1 \leq i \leq n}\{\xi_i\}}$, it is easily observed that

$$|y(t)| \leq k\xi e^{-\lambda t}, \quad -\tau \leq t \leq 0. \quad (6)$$

Now, we prove that $y(t) \leq Ke^{-\lambda t}$, where $K = k\xi \geq \|\phi\| = \lceil [y(0)]_c \rceil$. In order to do this, we shall first prove that for any positive constant $\varepsilon$,

$$y_i(t) \leq (k + \varepsilon)\xi_j e^{-\lambda t} \Delta z_i(t), \quad i = 1, 2, \cdots, n, t \geq 0. \quad (7)$$

If this is not true, then there exist a constant $t^* > 0$ and some integer $m$ such that

$$y_m(t^*) = z_m(t^*), \quad D^+y_m(t^*) \geq \hat{z}_m(t^*),$$

$$y_i(t) \leq z_i(t), \quad t \in [0, t^*], \quad i = 1, 2, \cdots, n. \quad (8)$$

From condition (i) of Lemma 3 and (7), we have

$$D^+y_m(t^*) \leq a_m(y_m(t^*))\sum_{j=1}^{n} \left[ p_{mj}y_j(t^*) + q_{mj}(y_j(t^*))_c \right]$$

$$\leq a_m(y_m(t^*))\sum_{j=1}^{n} \left[ p_{mj}(k + \varepsilon)\xi_j e^{-\lambda t^*} + q_{mj}(k + \varepsilon)\xi_j e^{-\lambda(t^* - t)} \right]$$

$$= a_m(y_m(t^*))\sum_{j=1}^{n} \left[ p_{mj} + q_{mj}e^{\lambda \tau} \right] (k + \varepsilon)\xi_j e^{-\lambda t^*}. \quad (9)$$
Since \( \lambda E + A(P + Qe^{\lambda t}) \leq 0 \), \( p_{ij} \geq 0 \) (\( i \neq j \)), and \( q_{ij} \geq 0 \), it follows that we have \( \sum_{j=1}^{n} \left[ p_{mj} + q_{mj} e^{\lambda t} \right] \xi_j < -\lambda \xi_m / a_m < 0 \).

Also, from \( a_m(y_m(t)) \geq a_m > 0 \), we get

\[
D^+ y_m(t^*) < -\lambda \xi_m (k + \varepsilon) e^{-\lambda t^*} = \dot{z}_m(t^*)
\]

which contradicts the inequality \( D^+ y_m(t^*) \geq \dot{z}_m(t^*) \) in (8). Thus (7) holds for all \( t \geq 0 \). Letting \( \varepsilon \to 0 \), we have

\[
y_i(t) \leq k \xi_i e^{-\lambda t}, \quad i = 1, 2, \ldots, n, t \geq 0.
\]

The proof is completed.

**Lemma 4 ([4]).** Let \( a \geq 0, b_k \geq 0 \) (\( k = 1, 2, \ldots, m \)), then

\[
a \prod_{k=1}^{m} b_k^q \leq \frac{1}{r} \left( a^r + \sum_{k=1}^{m} q_k b_k^r \right),
\]

where \( q_k \geq 0 \) (\( k = 1, 2, \ldots, m \)) are some constants, \( \sum_{k=1}^{m} q_k = r - 1 \), and \( r \geq 1 \).

### 3. Main Results

**Theorem 1.** Under assumptions (A1), (A2) and (A3), the system (1) has a unique equilibrium point, which is globally exponentially stable if \( W = B - (|C|U + |D|V) \) is a nonsingular M-matrix. Where

\[
B = \text{diag}(b_1, b_2, \ldots, b_n), \quad |C| = (|c_{ij}|)_{n \times n},
\]

\[
U = \text{diag}(u_1, u_2, \ldots, u_n), \quad F = \text{diag}(F_1, F_2, \ldots, F_n),
\]

\[
|D| = (|d_{ij}|)_{n \times n}, \quad V = \text{diag}(v_1, v_2, \ldots, v_n), \quad G = \text{diag}(G_1, G_2, \ldots, G_n).
\]

**Proof.** We shall prove this theorem in two steps.

**Step 1.** We first prove the existence and uniqueness of the equilibrium point.

Define an operator \( H : R^n \to R^n \) by
\[ H_i(x) = - \left[ b_i(x) - \sum_{j=1}^{n} c_{ij} f_j(u_j x_j) - \sum_{j=1}^{n} d_{ij} g_j(v_j x_j) + I_i \right], \]

where \( x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n, H(x) = (H_1(x), H_2(x), \cdots, H_n(x))^T. \)

we note that \( x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \) is an equilibrium point of (1) if and only if \( H(x^*) = 0. \) Since \( W = B - (|C|UF + |D|VG) \) is a nonsingular \( M \)-matrix, there exists a positive diagonal matrix \( P = \text{diag}(p_1, p_2, \cdots, p_n) \) such that \( PW + W^TP \) is a positive definite matrix. For all \( x, y \in \mathbb{R}^n, \) we have

\[
\langle PH(x) - PH(y), x - y \rangle = \sum_{i=1}^{n} \left\{ -p_i \left[ b_i(x_i) - b_i(y_i) - \sum_{j=1}^{n} c_{ij}(f_j(u_j x_j) - f_j(u_j y_j)) \right] \right. \\
\left. \sum_{j=1}^{n} d_{ij}(g_j(v_j x_j) - g_j(v_j y_j)) \right\} (x_i - y_i) \\
\leq \sum_{i=1}^{n} \left\{ -p_i \left[ b_i(x_i) - b_i(y_i) \right] (x_i - y_i) \\
+ p_i \sum_{j=1}^{n} \left[ |c_{ij}| \left| f_j(u_j x_j) - f_j(u_j y_j) \right| \right] (x_i - y_i) \\
+ p_i \sum_{j=1}^{n} \left[ |d_{ij}| \left| g_j(v_j x_j) - g_j(v_j y_j) \right| \right] (x_i - y_i) \right\} \\
\leq -\sum_{i=1}^{n} p_i b_i (x_i - y_i)^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i |c_{ij}| |F_{ij} u_j| |x_j - y_j| |x_i - y_i| \\
+ \sum_{i=1}^{n} \sum_{j=1}^{n} p_i |d_{ij}| |G_{ij} v_j| |x_j - y_j| |x_i - y_i| \\
= -\sum_{i=1}^{n} \sum_{j=1}^{n} \left( p_i b_i - p_i |c_{ij}| |F_{ij} u_j| - p_i |d_{ij}| |G_{ij} v_j| \right) \\
\times |x_i - y_i| |x_j - y_j|
\[ -(x - y)^T PW(x - y) \]
\[ = -\frac{1}{2} (x - y)^T (PW + W^T P)(x - y) \]
\[ \leq -\frac{1}{2} \lambda_{\min}(PW + W^T P)\|x - y\|^2. \]

It follows that \( m_R(PH) < 0 \), By Lemma 2, we know that \( PH \) is a homeomorphism of \( R^n \), this implies that there exists a unique \( x^* \in R^n \) such that \( PH(x^*) = 0 \). Also, \( P \) is a invertible, we derive that \( H(x) \) has a unique solution \( x^* \), hence model (1) has a unique equilibrium point \( x^* \).

**Step 2.** We prove that the unique equilibrium point \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) of model (1) is globally exponentially stable.

For \( i, j = 1, 2, \ldots, n \), set
\[ y_i(t) = x_i(t) - x_i^*, \quad \tilde{a}_i(y_i(t)) = a_i(y_i(t) + x_i^*), \quad \tilde{b}_i(y_i(t)) = b_i(y_i(t) + x_i^*) - b_i(x_i^*), \]
\[ \tilde{f}_j(u_jy_j(t)) = f_j(u_jy_j(t) + x_j^*) - f_j(u_jx_j^*), \quad \tilde{g}_j(v_jy_j(t)) \]
\[ = g_j(v_j(y_j(t) + x_j^*)) - g_j(v_jx_j^*), \]

then model (1) can be rewritten as
\[ \frac{dy_i(t)}{dt} = -\tilde{a}_i(y_i(t)) \left[ \tilde{b}_i(y_i(t)) - \sum_{j=1}^n c_{ij}\tilde{f}_j(u_jy_j(t)) - \sum_{j=1}^n d_{ij}\tilde{g}_j(v_jy_j(t) - \tau_{ij}(t)) \right]. \]

(10)

Define the initial condition of (10)
\[ \phi(s) = x(s) - x^*, \quad s \in [-\tau, 0]. \]

From (A1)-(A3), we calculate the upper right derivative along the solutions of (10)
\[ D^+|y_i(t)| = -\text{sgn}(y_i(t))\tilde{a}_i(y_i(t)) \left[ \tilde{b}_i(y_i(t)) - \sum_{j=1}^n c_{ij}\tilde{f}_j(u_jy_j(t)) - \sum_{j=1}^n d_{ij}\tilde{g}_j(v_jy_j(t) - \tau_{ij}(t)) \right] \]
\[ \leq \tilde{a}_i(y_i(t)) \left[ -b_i|y_i(t)| + \sum_{j=1}^n |c_{ij}|F_j|y_j(t)| + \sum_{j=1}^n |d_{ij}|G_j|y_j(t)|_\tau \right]. \]
or in matrix form,

\[ D^T|y(t)| \leq \tilde{a}(y(t))\{P|y(t)| + Q|y(t)|\}, \quad t \geq 0, \tag{11} \]

where \( P = -B + |C|UF, \) \( Q = |D|VG \) and \( \tilde{a}(y(t)) = \text{diag}(\tilde{a}_1(y_i(t)), \tilde{a}_2(y_2(t)), \ldots, \tilde{a}_n(y_n(t))) \geq A. \) Since \(-(P + Q)\) is an \( M\)-matrix, from Lemma 3, there exist a constant \( \lambda \) and a constant vector \( K \geq |y(0)| \), such that

\[ |y(t)| \leq Ke^{-\lambda t}, \quad t \geq 0, \]

i.e.,

\[ |x(t) - x^*| \leq Ke^{-\lambda t}, \quad t \geq 0. \]

The proof is completed.

**Remark 1.** In Theorem 1, the conditions that ensure existence, uniqueness and global exponential stability of equilibrium point of model (1) are independent of amplification function and delays, this implies that the strong self-regulation is dominant in the networks. However, those criteria about exponential stability in [1, 15, 16, 20, 21] rely on the upper bound or the below bound of amplification function, hence our results are much less conservative and restrictive.

**Corollary 1.** Under assumptions (A1), (A2) and (A3), the system (1) has a unique equilibrium point, which is globally exponentially stable if any one of the following conditions is true:

(i) \( b_i > u_i F_i \sum_{j=1}^{n} |c_{ij}| + v_i G_i \sum_{j=1}^{n} |d_{ji}|, \quad i = 1, 2, \ldots, n. \)

(ii) \( b_i > \sum_{j=1}^{n} (|c_{ij}| u_j F_j + |d_{ij}| v_j G_j), \quad i = 1, 2, \ldots, n. \)

(iii) There exists a positive vector \( l = (l_1, l_2, \ldots, l_n)^T > 0 \) such that

\[ l_i b_i > \sum_{j=1}^{n} l_j (|c_{ij}| u_j F_j + |d_{ij}| v_j G_j), \quad i = 1, 2, \ldots, n. \]
(iv) There exists a positive vector \( l = (l_1, l_2, \cdots, l_n)^T > 0 \) such that

\[
l_i b_i > u_i F_i \sum_{j=1}^{n} l_j |c_{ji}| + v_j G_i \sum_{j=1}^{n} l_j |d_{ji}|, \quad i = 1, 2, \cdots, n.
\]

**Proof.** In fact, any one of the conditions (i)-(iv) in Corollary 1 can assure \( B - (|C| F + |D| G) \) is a nonsingular \( M \)-matrix. The proof is completed.

**Corollary 2.** Under assumptions (A1), (A2) and (A3), the system (3) has a unique equilibrium point, which is globally exponentially stable if \( W = B - (|C| F + |D| G) \) is a nonsingular \( M \)-matrix.

**Remark 2.** Corollary 2 is the same as Corollary 2 in [27], which implies Theorem 1 and Theorem 3 in [1] as well as the main results in [15, 33].

**Corollary 3.** Suppose that (A1)- (A3) hold and there exist twelve sets of real numbers \( e_i > 0, h_i > 0, k_i > 0, l_i > 0, m_i > 0, r_i > 0, \alpha_{ij} > 0, \beta_{ij} > 0, \gamma_{ij} > 0, \delta_{ij} > 0, \eta_{ij} > 0, \theta_{ij} > 0 \) such that

\[
\sum_{j=1}^{n} \left[ \frac{h_j k_j}{h_i l_i} F_i^{\alpha_{ji}} u_i^{\theta_{ji}} |c_{ji}|^{\gamma_{ji}} + F_j^{2-\alpha_{ij}} u_j^{2-\beta_{ij}} \frac{h_i e_i}{h_j e_j} |c_{ij}|^{2-\gamma_{ij}} \\
+ \frac{h_j m_j}{h_i r_i} G_i^{\delta_{ji}} u_i^{\eta_{ji}} |d_{ji}|^{\theta_{ji}} + G_j^{2-\delta_{ij}} v_j^{2-\eta_{ij}} \frac{h_i e_i}{h_j e_j} m_i |d_{ij}|^{2-\theta_{ij}} \right] < 2b_i, \quad (12)
\]

for \( i = 1, 2, \cdots, n \). Then model (1) has a unique equilibrium point \( x^* \), which is globally exponentially stable.

**Proof.** Set \( P = \text{diag}(h_1 / \sqrt{e_1}, h_2 / \sqrt{e_2}, \cdots, h_n / \sqrt{e_n}) \), then \( P^{-1} = \text{diag}(\sqrt{e_1} / h_1, \sqrt{e_2} / h_2, \cdots, \sqrt{e_n} / h_n) \). We consider the following linear system

\[
\frac{dz(t)}{dt} = \left[ P( - B + |C| F + |D| G) P^{-1} \right] z(t). \quad (13)
\]

Constructing a Lyapunov function

\[
L(z(t)) = \frac{1}{2} \sum_{j=1}^{n} |z_j(t)|^2.
\]
Calculating the right derivative $D^+L(z(t))$ of $L(z(t))$ along the solution of (13), we have

$$D^+L(z(t)) = \sum_{i=1}^{n} |z_i(t)| \left[ \operatorname{sgn}(z_i(t)) \left\{ -b_i z_i(t) + \frac{h_i}{\sqrt{e_i}} \sum_{j=1}^{n} \left( |c_{ij}| F_j u_j + |d_{ij}| G_j v_j \right) \sqrt{\frac{e_j}{h_j}} z_j(t) \right\} \right]$$

$$\leq \sum_{i=1}^{n} -b_i z_i^2(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( |c_{ij}| F_j u_j + \frac{h_i}{\sqrt{e_i}} \sqrt{\frac{e_j}{h_j}} |z_i(t)||z_j(t)| \right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}| \sqrt{\frac{e_j}{h_j}} |z_i(t)||z_j(t)|$$

$$\leq \sum_{i=1}^{n} -b_i z_i^2(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{e_i}{e_j} r_i G_j^{2-\delta_{ij}} v_j^{2-\eta_{ij}} d_{ij} |z_i^2(t)| + \frac{m_i}{r_j} G_j^{\gamma_{ij}} v_j^{\eta_{ij}} d_{ij} |z_j^2(t)| \right]$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \left\{ 2h_i - \frac{h_i h_j}{h_i h_j} F_i^{a_{ji}} u_i^{b_{ji}} |c_{ji}|^{\gamma_{ji}} + F_j^{2-\alpha_{ij}} u_j^{2-\beta_{ij}} \right.$$  

$$\left. + \frac{h_i m_j}{h_i m_j} G_i^{\gamma_{ji}} v_i^{\eta_{ji}} d_{ji}^{b_{ji}} + G_j^{2-\delta_{ij}} v_j^{2-\eta_{ij}} \frac{h_i e_j}{h_j e_i m_i} |d_{ij}|^{2-\theta_{ij}} \right\} z_i^2(t) < 0, \quad t > 0.$$

From (13) and Lyapunov stability theorem [26], we know that the zero solution of (13) is globally asymptotically stable, furthermore, the real parts of all eigenvalues of matrix $P(B - |C|UF - |D|VG)P^{-1}$ is positive, so is $B - |C|UF - |D|VG$. Therefore, $B - |C|UF - |D|VG$ is an $M$-matrix.

From Theorem 1, we know model (1) has a unique equilibrium point, which is globally exponentially stable. The proof is completed.
Remark 3. It is difficult to check the condition of Corollary 3 in this paper, at the same time, it has also extensive flexibility and applicability. Through specific choice of the parameters, a series of new criteria can be obtained for the global asymptotic stability and global exponential stability of those models included by (1).

Corollary 4. Suppose that (A1)- (A3) hold and there exist ten sets of real numbers $e_i > 0, h_i > 0, k_i > 0, l_i > 0, m_i > 0, r_i > 0, s_i > 0, t_i > 0, p_i > 0, q_i > 0$ such that

$$F_i u_i \sum_{j=1}^{n} \frac{h_j k_j}{h_i l_i} |c_{ji}|^{s_j / t_i} + \sum_{j=1}^{n} F_j u_j \frac{h_i e_j l_j}{h_j e_i k_i} |c_{ij}|^{2-s_i / t_j} + G_i v_i \sum_{j=1}^{n} \frac{h_j m_j}{h_i n_i} |d_{ji}|^{p_j / q_i}$$

$$+ \sum_{j=1}^{n} G_j v_j \frac{h_i e_j r_j}{h_j e_i m_i} |d_{ij}|^{2-p_i / q_j} < 2b_i,$$

for $i = 1, 2, \cdots, n$. Then model (1) has a unique equilibrium point $x^*$, which is globally exponentially stable.

Proof. Let $\alpha_{ij} = \beta_{ij} = \delta_{ij} = \eta_{ij} = 1, \gamma_{ij} = s_i / t_j, \theta_{ij} = p_i / q_j, i, j = 1, 2, \cdots, n$ in (12), we obtain (14). The proof is completed.

Remark 4. The condition of Theorems 1 and 2 in [28] is the same as of Corollary 4 except from the difference of symbols, but the result in [28] is the special cases of Corollary 3 when $\tau_{ij}(t) = \tau_{ij}$. According to Remark 3 in [28], we generalize and improve many existing results such as those in [2], [5]-[8], [9, 13, 20, 24, 25], [28]-[31], [34]-[37].

Corollary 5. Suppose that (A1)- (A3) hold and there exist four sets of real numbers $\alpha_{ij} > 0, \gamma_{ij} > 0, \delta_{ij} > 0, \theta_{ij} > 0$ such that

$$\sum_{j=1}^{n} \left[ F_j^{2-\alpha_{ij}} |c_{ij}|^{2-\gamma_{ij}} + F_i^{\alpha_{ij}} |c_{ji}|^{2-\gamma_{ij}} + G_j^{2-\delta_{ij}} |d_{ij}|^{2-\theta_{ij}} + G_i^{\delta_{ij}} |d_{ji}|^{2-\theta_{ij}} \right] < 2b_i,$$

for $i = 1, 2, \cdots, n$. Then model (3) has a unique equilibrium point $x^*$, which is globally exponentially stable.
Proof. Set \( e_i = h_i = k_i = l_i = m_i = r_i = u_i = v_i = 1, i = 1, 2, \ldots, n \) in (12), we obtain (15). The proof is completed.

Remark 5. The main results (Theorems 1, 2 and 3) of [21] are involved in our Corollaries 2 and 5. In [21], the author required that the time-varying delay \( \tau_{ij}(t) \) is bounded and differentiable. Also, the author demanded that \( a_i(x) \) has the upper bound in Theorem 2. However, in this paper, we do not assume that \( \tau_{ij}(t) \) is non-negative and differentiable, for the amplification function \( a_i(x) \), we only require that \( a_i(x) \) is positive and below-bounded, so results in [21] are improved and generalized.

At last, let us consider the following linear system

\[
\frac{dz(t)}{dt} = \left(-B + [C][U]F + [D][V]G\right)z(t).
\]

(16)

Constructing a Lyapunov function

\[
L(z(t)) = \frac{1}{r} \sum_{j=1}^{n} |z_j(t)|^r.
\]

Calculating the right derivative \( D^+L(z(t)) \) of \( L(z(t)) \) along the solution of (16), applying Lemma 4, similar to the proof of Corollary 3, we can obtain.

Corollary 6. Under assumptions (A1), (A2) and (A3), model (1) has a unique equilibrium point, which is globally exponentially stable if there exist constants \( t_k > 0 (k = 1, 2, \ldots, K_1) \), \( s_k > 0 (k = 1, 2, \ldots, K_2) \), \( r_i > 0 \) \((i = 1, 2, \ldots, n)\), \( \alpha_{ij}, \alpha_{ij}^*, \beta_{ij}, \beta_{ij}^*, \gamma_{ij}, \gamma_{ij}^*, \lambda_{ij}, \lambda_{ij}^*, \mu_{ij}, \mu_{ij}^*, v_{ij}, \nu_{ij}^* \in R \) \((i, j = 1, 2, \ldots, n)\) such that

\[
\sum_{j=1}^{n} \left| \sum_{k=1}^{K_1} t_k |c_{ij}| \sigma_{ij} \sigma_{ij}^* \sigma_{ij} \sigma_{ij} = \frac{r_j}{r_i} |c_{ji}| \sigma_{ji} \sigma_{ji}^* \sigma_{ji} \sigma_{ji} \right. \\
+ \sum_{k=1}^{K_2} s_k |d_{ij}| \sigma_{ij} \sigma_{ij} \sigma_{ij} \sigma_{ij} = \frac{r_j}{r_i} |d_{ji}| \sigma_{ji} \sigma_{ji} \sigma_{ji} \sigma_{ji} \right) < \sigma b_i,
\]

(17)
where $\sigma = \sum_{k=1}^{K_1} t_k + 1 = \sum_{k=1}^{K_2} s_k + 1; K_1 a_{ij} + a_{ij}^* = 1, K_1 b_{ij} + b_{ij}^* = 1, K_1 \gamma_{ij} + \gamma_{ij}^* = 1, K_2 \lambda_{ij} + \lambda_{ij}^* = 1, K_2 \mu_{ij} + \mu_{ij}^* = 1, K_2 v_{ij} + v_{ij}^* = 1$.

Remark 6. Those results in [4, 37, 38, 39] are some special cases of Corollary 6.

4. Examples

In order to illustrate the feasibility of our above-established criteria in the preceding sections, we provide concrete examples. Although the selection of the coefficients and functions in the examples is somewhat artificial, the possible application of our theoretical theory is clearly expressed.

Example 1. Consider the following model

$$\frac{dx_i(t)}{dt} = -a_i(x_i(t)) b_i(x_i(t)) - \sum_{j=1}^{2} c_{ij} f_j(x_j(t))$$

$$- \sum_{j=1}^{2} d_{ij} f_j(x_j(t - \tau_{ij}))) + I_i, \quad i = 1, 2, \quad (18)$$

where the coefficients and functions are taken as

$a_1(x) = 2 + \sin x, a_2(x) = 2 + \cos x, a_1 = a_2 = 1, b_1(x) = b_2(x) = 5x, b_1 = b_2 = 5,$

$f_1(x) = f_2(x) = \frac{1}{2} (|x + 1| - |x - 1|), F_1 = F_2 = 1,$

$C = (c_{ij})_{2 \times 2} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, D = (d_{ij})_{2 \times 2} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, (\tau_{ij}) = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

It is not hard to verify the validity of (A1)-(A3), and it is easy to calculate that

$$B - (|C| + |D|)F = \begin{pmatrix} 3 & \ast & \ast \\ -4 & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix}.$$
Therefore $B - (|C| + |D|)F$ is a nonsingular $M$-matrix, from Theorem 1, we know that system (15) has one unique equilibrium point, which is globally exponentially stable. But we note that there do not exist positive numbers $d_1$ and $d_2$ such that

$$
2 > \frac{1}{b_1d_1} \left[ \frac{2\sigma_2d_1}{a_1} + d_1 \sum_{j=1}^{2} \left( F_j^{2\sigma_1}|c_{1j}| + F_j^{2\sigma_2}|d_{1j}|e^{2\sigma_2t} \right) \right] \\
+ \sum_{j=1}^{2} d_j \left( F_j^{2(1-\eta_1)}|c_{j1}| + F_j^{2(1-\eta_2)}|d_{j1}| \right) \\
\geq \frac{1}{b_1d_1} \left[ \sum_{j=1}^{2} d_{1j} \sum_{j=1}^{2} |c_{1j}| + |d_{1j}| \right] + \sum_{j=1}^{2} d_j \sum_{j=1}^{2} |c_{j1}| + |d_{j1}| \right] = \frac{1}{5d_1} \left( 4d_1 + 2d_1 + 4d_2 \right) = \frac{6}{5} + \frac{4}{5} \frac{d_2}{d_1},
$$

and

$$
2 > \frac{1}{b_2d_2} \left[ \frac{2\sigma_2d_2}{a_2} + d_2 \sum_{j=1}^{2} \left( F_j^{2\sigma_1}|c_{2j}| + F_j^{2\sigma_2}|d_{2j}|e^{2\sigma_2t} \right) \right] \\
+ \sum_{j=1}^{2} d_j \left( F_j^{2(1-\eta_1)}|c_{j2}| + F_j^{2(1-\eta_2)}|d_{j2}| \right) \\
\geq \frac{1}{b_2d_2} \left[ \sum_{j=1}^{2} d_{2j} \sum_{j=1}^{2} |c_{2j}| + |d_{2j}| \right] + \sum_{j=1}^{2} d_j \sum_{j=1}^{2} |c_{j2}| + |d_{j2}| \right] \\
= \frac{1}{5d_2} \left( 6d_2 + 2d_1 + 2d_2 \right) = \frac{8}{5} + \frac{2}{5} \frac{d_1}{d_2}.
$$

Hence, the conditions of Theorem 2 and Corollary 2 in [20] cannot hold, this implies that Theorem 2 and Corollary 2 in [20] are not applicable to ascertain the stability of model (18).

In addition, from model (18), we have $a_{ii} = 1, \overline{a}_i = 3, i = 1, 2$. According to the condition of Theorem 3 in [16], we note that there do not exist constants $h_{ij} > 0, l_{ij} > 0, h_{ij}^* > 0, l_{ij}^* > 0, r > 1, \omega_i > 0, (i, j = 1, 2)$ such that
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\[ r_{\omega_1} b_1 a_1 - \sum_{j=1}^{2} \omega_j \left( (r - 1) |c_{1j}| \frac{r^{-h_j}}{r^{r-1}} F_j^{r} + |c_{1j}| \frac{b_{h_j}}{r} F_j^{h_j} \right) + (r - 1) |d_{1j}| \frac{r^{-\tau_{ij}}}{r^{r-1}} F_j^{\tau_{ij}} \]

\[ = 5r_{\omega_1} - 3 \sum_{j=1}^{2} \omega_j \left( (r - 1) + 1 + (r - 1) + 1 \right) = 5r_{\omega_1} - 6\omega_1 > 0. \]

Hence, the condition of Theorem 3 in [16] is not applicable to ascertain the stability of model (18).

**Remark 7.** Theorem 1 and Theorem 3 in [1], Theorem 1 in [21] and Theorem 2 in [15] are also not applicable to ascertain the existence of the equilibrium point or the stability of model (18).

**Example 2.** Consider the following model:

\[ \frac{dx_i(t)}{dt} = -a_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{2} c_{ij} f_j(u_j x_j(t)) \right. \]

\[ \left. - \sum_{j=1}^{2} d_{ij} g_j(v_j x_j(t - \tau_{ij}(t))) + I_i \right], \quad i = 1, 2, \quad (19) \]

where the coefficients and functions are taken as

\( b_1(x) = b_2(x) = 5x, \quad b_1 = b_2 = 5, \quad a_1(x) = 2 + \cos x, \quad a_2(x) = 2 + \sin x, \quad a_1 = a_2 = 1, \)

\( f_j(x) = g_j(x) = |x + 1| + |x - 1|, \quad F_j = G_j = 2, \quad u_j = v_j = \frac{1}{2}, \quad j = 1, 2, \)

\( C = (c_{ij})_{2 \times 2} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \quad D = (d_{ij})_{2 \times 2} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \)

\( (\tau_{ij}(t)) = \begin{pmatrix} \cos^2 t & 2 \cos^2 t \\ 3 \sin^2 t & 4 \sin^2 t \end{pmatrix}, \quad (I_1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \)

It is easy to check that assumptions (A1)-(A3) hold, and it is easy to calculate that
Therefore, \( B - (|C|U F + |D|V G) \) is a nonsingular \( M \)-matrix, from Theorem 1, we know that system (19) has one unique equilibrium point, which is globally exponentially stable.

5. Conclusions

In this paper, a class of generalized Cohen-Grossberg neural networks with both variable delays have been studied. Some sufficient conditions for assuring the existence, uniqueness and exponential stability of the equilibrium point have been established. These obtained results are new and they improve previously known results. Moreover, two examples are given to illustrate the effectiveness of the new results.

References

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