NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF CERTAIN DELAY PARABOLIC EQUATIONS

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Abstract

In this paper, necessary and sufficient conditions for oscillation of Robin boundary problems of certain delay parabolic equations with variable coefficients are obtained.

1. Introduction

In the past few years, more and more scholars have become interested in studying the necessary and sufficient conditions for oscillation of...
partial functional differential equations. We refer the reader to Mishev and Bainov [6], Yu and Chen [9], Cui and Li [2, 3], Li and Cui [4, 5], and Cui and Liu [1].

Our aim in this paper is to obtain some necessary and sufficient conditions for oscillation of the following delay parabolic equations

\[
\frac{\partial \sigma}{\partial t} u(x, t) = a(t) \Delta u(x, t) + \sum_{k=1}^{s} a_k(t) \Delta u(x, t - \rho_k) - \sum_{j=1}^{m} q_j(t) u(x, t - \sigma_j),
\]

\[(x, t) \in \Omega \times [0, \infty) = G, \tag{1}\]

where \(\Omega\) is a bounded domain in \(R^n\) with a piecewise smooth boundary \(\partial \Omega\), and \(\Delta\) is the Euclidean \(n\)-space \(R^n\).

Throughout this paper, we always suppose that the following conditions hold:

(C1) \(a, a_k, q_j \in C([0, \infty); [0, \infty]), k \in I_s = \{1, 2, \ldots, s\}, j \in I_m = \{1, 2, \ldots, m\}\);

(C2) \(\rho_k\) and \(\sigma_j\) are nonnegative constants, \(k \in I_s, j \in I_m\).

We consider the Robin boundary condition

\[
\alpha(x) \frac{\partial u(x, t)}{\partial N} + \beta(x) u(x, t) = 0, (x, t) \in \partial \Omega \times [0, \infty), \tag{2}\]

where \(\alpha, \beta \in C(\partial \Omega, [0, \infty]), \alpha^2(x) + \beta^2(x) \neq 0, N\) is the unite exterior normal vector to \(\partial \Omega\).

As is customary, the solution \(u(x, t) \in C^2(G) \cap C^1(\overline{G})\) of the problem (1), (2) is said to be oscillatory in the domain \(G = \Omega \times [0, \infty)\) if for any positive number \(\mu\) there exists a point \((x_0, t_0) \in \Omega \times [\mu, \infty)\) such that the equality \(u(x_0, t_0) = 0\) holds.

In this paper, the following two lemmas are useful for our main results.

**Lemma 1** [8]. Suppose that \(\lambda_0\) is the smallest eigenvalue of the problem
\[
\begin{cases}
\Delta \phi(x) + \lambda \phi(x) = 0, \text{ in } \Omega, \\
\alpha(x) \frac{\partial \phi(x, t)}{\partial N} + \beta(x) \phi(x, t) = 0, \text{ on } \partial \Omega
\end{cases}
\]

and \( \phi(x) \) is the corresponding eigenfunction of \( \lambda_0 \). Then \( \lambda_0 = 0 \), \( \phi(x) = 1 \) as \( \beta(x) = 0(x \in \Omega) \) and \( \lambda_0 > 0 \), \( \phi(x) > 0(x \in \Omega) \) as \( \beta(x) \neq 0 \) \( (x \in \partial \Omega) \).

**Lemma 2** [7]. Every solution of the delay differential equation

\[
V'(t) + \sum_{j=1}^{m} q_j(t)V(t - \sigma_j) = 0, \ t \geq 0,
\]

is oscillatory if and only if the differential inequality

\[
V'(t) + \sum_{j=1}^{m} q_j(t)V(t - \sigma_j) \leq 0
\]

has no eventually positive solutions.

**2. Main Results**

**Theorem 1.** The necessary and sufficient condition for all solutions of problem (1), (2) to oscillate is that all solutions of the delay differential equation

\[
y'(t) + \lambda_0 a(t)y(t) + \lambda_0 \sum_{k=1}^{s} a_k(t)y(t - \rho_k) + \sum_{j=1}^{m} q_j(t)y(t - \sigma_j) = 0, \ t \geq 0,
\]

to oscillate, where \( \lambda_0 \) is the smallest eigenvalue of (3).

**Proof.** (i) **Sufficiency.** Suppose to the contrary that there is a non-oscillatory solution \( u(x, t) \) of the problem (1), (2) which has no zero in \( \Omega \times [t_0, \infty) \) for some \( t_0 \geq 0 \). Without loss of generality we assume that \( u(x, t) > 0, u(x, t - \rho_k) > 0, u(x, t - \sigma_j) > 0, (x, t) \in \Omega \times [t_1, \infty) \), \( k \in I_s, j \in I_m \).

Multiplying both sides of (1) by \( \phi(x) > 0 \) and integrating with respect to \( x \) over the domain \( \Omega \), we have
\[
\frac{d}{dt} \left( \int_{\Omega} u(x, t) \phi(x) dx \right) = a(t) \int_{\Omega} \Delta u(x, t) \phi(x) dx + \sum_{k=1}^{s} \alpha_k(t) \int_{\Omega} \Delta u(x, t - \rho_k) \phi(x) dx \\
- \sum_{j=1}^{m} q_j(t) \int_{\Omega} u(x, t - \sigma_j) \phi(x) dx, \quad t \geq t_1. \quad (7)
\]

From Green's formula and boundary condition (2), it follows that
\[
\int_{\Omega} \Delta u(x, t) \phi(x) dx = \int_{\partial \Omega} \left( \phi(x) \frac{\partial u(x, t)}{\partial N} - u(x, t) \frac{\partial \phi(x)}{\partial N} \right) dS + \int_{\Omega} u(x, t) \Delta \phi(x) dx \\
= \int_{\partial \Omega} \left( \phi(x) \frac{\partial u(x, t)}{\partial N} - u(x, t) \frac{\partial \phi(x)}{\partial N} \right) dS - \lambda_0 \int_{\Omega} u(x, t) \phi(x) dx \quad (8)
\]
and
\[
\int_{\Omega} \Delta u(x, t - \rho_k) dx = -\lambda_0 \int_{\Omega} u(x, t - \rho_k) \phi(x) dx, \quad t \geq t_1, \quad k \in I_s, \quad (9)
\]
where \(dS\) is the surface element on \(\partial \Omega\).

Set
\[
V(t) = \int_{\Omega} u(x, t) \phi(x) dx, \quad t \geq t_1.
\]

Combining (7)–(9) we have
\[
V'(t) + \lambda_0 a(t)V(t) + \lambda_0 \sum_{k=1}^{s} \alpha_k(t)V(t - \rho_k) + \sum_{j=1}^{m} q_j(t)V(t - \sigma_j) = 0, \quad t \geq t_1. \quad (10)
\]

It follows from (10) that \(V(t)\) is a positive solution of Equation (6), which contradicts the fact that all solutions of Equation (6) is oscillatory.

(ii) **Necessity.** Suppose that Equation (6) has a non-oscillatory solution \(\tilde{V}(t) > 0\). Without loss of generality we assume \(\tilde{V}(t) > 0\) for \(t \geq t_* \geq 0\), where \(t_*\) is some large number. From (6), we have
\( \ddot{V}(t) + \lambda_0 a(t) \dot{V}(t) + \lambda_0 \sum_{k=1}^{g} a_k(t) V(t - \rho_k) + \sum_{j=1}^{m} q_j(t) V(t - \sigma_j) = 0, \ t \geq t_* . \) \tag{11}

Multiplying both sides of (11) by \( \varphi(x) \), we obtain

\[
\frac{\partial}{\partial t} (\ddot{V}(t) \varphi(x)) + \lambda_0 a(t) \dot{V}(t) \varphi(x) + \lambda_0 \sum_{k=1}^{g} a_k(t) V(t - \rho_k) \varphi(x) + \sum_{j=1}^{m} q_j(t) V(t - \sigma_j) \varphi(x) = 0, \ t \geq t_*, \ x \in \Omega . \tag{12}
\]

Let \( \bar{u}(x, t) = \ddot{V}(t) \varphi(x), (x, t) \in \Omega \times [0, \infty) \). By Lemma 1, we have \( \Delta \varphi(x) = -\lambda_0 \varphi(x), x \in \Omega \). Then (12) implies

\[
\frac{\partial}{\partial t} \bar{u}(x, t) = a(t) \Delta \bar{u}(x, t) + \sum_{k=1}^{g} a_k(t) \Delta \bar{u}(x, t - \rho_k) - \sum_{j=1}^{m} q_j(t) \bar{u}(x, t - \sigma_j), \ t \geq t_*, x \in \Omega , \tag{13}
\]

which shows that \( \bar{u}(x, t) = \ddot{V}(t) \varphi(x), (x, t) \in \Omega \times [t_*, \infty) \), satisfies (1).

From Lemma 1, we get

\[
\alpha(x) \frac{\partial \varphi(x)}{\partial N} + \beta(x) \varphi(x) = 0, \ x \in \partial \Omega ,
\]

which implies

\[
\alpha(x) \frac{\partial \bar{u}(x, t)}{\partial N} + \beta(x) \bar{u}(x, t) = 0, \ (x, t) \in \partial \Omega \times [0, \infty) . \tag{14}
\]

Hence \( \bar{u}(x, t) = \ddot{V}(t) \varphi(x) > 0 \) is a non-oscillatory solution of the problem (1), (2), which is a contradiction. The proof is complete.

**Remark 1.** Using Theorem 1, we can obtain many necessary and sufficient conditions for oscillation of problem (1), (2). For example, by Lemma 2 and Theorem 1, we easily obtain the following result.
Theorem 2. The necessary and sufficient condition for all solutions of problem (1), (2) to oscillate is that the delay differential inequality

\[ y'(t) + \lambda_0 a(t)y(t) + \lambda_0 \sum_{k=1}^{s} \alpha_k(t)y(t - \rho_k) + \sum_{j=1}^{m} q_j(t)y(t - \sigma_j) \leq 0, \quad t \geq 0, \quad (15) \]

has no eventually positive solutions, where \( \lambda_0 \) is the smallest eigenvalue of (3).

Noting the Corollary 3 in [8], from Theorem 1, we can obtain the following conclusion.

Theorem 3. Let \( P_i(t) \geq q_i \geq 0, \quad i \in I_{1+s+m} \). Assume that one of the following conditions

\[ \sum_{i=1}^{1+s+m} q_i \tau_i > \frac{1}{e}, \quad (16) \]

\[ \sum_{i=1}^{1+s+m} \tau_i \left( \prod_{i=1}^{1+s+m} q_i \right)^{\frac{1}{1+s+m}} > \frac{1}{e}, \quad (17) \]

is satisfied, where

\[
P_i(t) = \begin{cases} 
\lambda_0 a(t), & i = 1, \\
\lambda_0 a_{i-1}(t), & 2 \leq i \leq s + 1, \\
q_{i-1-s}(t), & s + 2 \leq i \leq 1 + s + m, \\
0, & i = 1, \\
\rho_{i-1}, & 2 \leq i \leq s + 1, \\
\sigma_{i-1-s}, & s + 2 \leq i \leq 1 + s + m, 
\end{cases}
\]

and \( \lambda_0 \) is the smallest eigenvalue of (3). Then every solution of the problem (1), (2) is oscillatory in \( G \).

Remark 2. By taking \( \beta(x) = 0, \quad x \in \Omega, \) in Theorems 1-3, we immediately obtain the all results in [2]. Therefore, our work extend the results in [2].
References


