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COMPUTATIONAL ALGEBRA TECHNIQUES IN ELECTROMAGNETISM

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Abstract

This paper shows how to apply some ideas of algebraic analysis to the study of a couple of systems of physical interest. In addition to giving an overview of what has already been done in [3] for the Maxwell's equations, in this paper we also consider the potentials. Finally, we treat the anti-self-dual Abelian Yang-Mills equations.

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1. Introduction

In recent years, techniques from computational algebra have become important to render effective some general results in the theory of partial differential equations.

In [2] the authors have shown how these tools can be used to discover and identify important properties of several systems of interest, such as the Cauchy-Fueter, the Moisil-Theodorescu, the Maxwell, and the Proca system.

In this paper we continue along these ideas. In particular, we revise the Maxwell's system, already treated in [3] and [2], in the Minkowski space $\mathbb{R}^{1,3}$. Our computations are intended to be consistent with a gauge theoretic approach of electromagnetism, and to bring us one step closer to an algebraic approach to the study of symmetries of field theory. We also further the same ideas to the Abelian instanton equations, both in the real Euclidean space $\mathbb{R}^4$ and in the complex Minkowski space $\mathbb{C}^{1,3}$.

While much more work will be needed to fully understand the implications of these results, our analysis of the syzygies of these systems should be considered a starting point for a new interest in algebraic methods in the theory of Yang Mills fields.

2. Algebraic Analysis of Maxwell's Equations in $\mathbb{R}^{1,3}$

We recall some results from [3] which explicitly highlight the connection between the study of syzygies and conservation laws in physics.

Consider the matrix of differential operators:

$$P(D) = \begin{bmatrix} P^e(D) \\ P^m(D) \end{bmatrix} = \begin{bmatrix} \partial_x & \partial_y & \partial_z & 0 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 & -\partial_z & \partial_y \\ 0 & -\partial_t & 0 & \partial_z & 0 & -\partial_x \\ 0 & 0 & -\partial_t & -\partial_y & \partial_x & 0 \\ 0 & 0 & 0 & \partial_x & \partial_y & \partial_z \\ 0 & -\partial_z & \partial_y & \partial_t & 0 & 0 \\ \partial_z & 0 & -\partial_x & 0 & \partial_t & 0 \\ -\partial y & \partial_x & 0 & 0 & 0 & \partial_t \end{bmatrix}, \quad (1)$$

where $P^e(D)$ and $P^m(D)$ are the first and the last four rows, respectively. Let us denote by F the vector

$$F = [E_1, E_2, E_3, B_1, B_2, B_3]^t$$

and let also

$$G = [G^e, G^m]^t = [\rho^e, j_1^e, j_2^e, j_3^e, \rho^m, -j_1^m, -j_2^m, -j_3^m]^t,$$

denote the vector corresponding to both electric and magnetic sources. Then for $R = \mathbb{C}[t, x, y, z]$, we can consider the polynomial map $P^t : R^8 \to R^6$ which one obtains by replacing, in $P(D)$, the derivatives $\partial_t, \partial_x, \partial_y, \partial_z$ by the variables t, x, y, z. The gist of algebraic analysis consists in studying the resolution of such polynomial map. In this case, as it was shown for example in [2], the resolution is given by

$$0 \to R^2 \xrightarrow{P_1^t} R^8 \xrightarrow{P^t} R^6 \to R^6 / P^t R^8 \to 0.$$

We will usually denote $M = R^6 / P^t R^8$, for simplicity of notation. It is important that we are in fact able to compute the map P_1, and it is

$$P_1(D) = \begin{bmatrix} P_1^e(D) \\ P_1^m(D) \end{bmatrix} = \begin{bmatrix} \partial_t & \partial_x & \partial_y & \partial_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_t & -\partial_x & -\partial_y & -\partial_z \end{bmatrix}.$$

In this setting, the first three authors proved the following results [2]

Theorem 1. *The system* $P(D)F = G$ *has a solution* $F \in \mathcal{S}'^{6}$ *if and only if* $G \in \mathcal{S}'^{8}$ *satisfies* $P_1(D)G = 0$, *i.e.*,

$$\partial_t \rho^e + \nabla \cdot \mathbf{j}^e = 0$$

$$\partial_t \rho^m + \nabla \cdot \mathbf{j}^m = 0. \tag{2}$$

Note that the electric compatibility equation is coming from $d(d * F) = d(J)$, where J is the charge-current vector (vanishing divergence equation), so that for any closed 3-surface Ω one has

$$\int_{\Omega} J = 0.$$

Theorem 2. *The characteristic variety of* $Ext^1(M, R)$, *as a set, is given by*

$$(t^2 - x^2 - y^2 - z^2)^2 = 0.$$

Therefore $Ext^1(M, R) \neq 0$ *and, moreover,* $Ext^2(M, R) = 0$.

This implies also that the compact singularities of its $\mathcal{C}^{\infty}$-solutions cannot be eliminated.

Theorem 3. *Let* $\Omega \in \mathbb{R}^4$ *be an open set and* $x \in \Omega$. *Then every solution of the Maxwell system in* $\Omega - \{x\}$, *whose components can be extended as distributions to all of* Ω, *is a distribution soloution to the Maxwell system to all of* Ω.

Finally, the particular structure of the Maxwell system (linearity and constant coefficients) allows us to write an integral representation for all its solutions, in accord with the Ehrenpreis-Palamodov Fundamental Principle.

Theorem 4. *Let V be the multiplicity variety of the system associated to* (1). *Let* $f : \mathbb{R}^4 \to R^6$ *be a solution of the Maxwell equations. Then there exist Noetherian operators* $N_j(x, z)$ *such that f can be represented as*

$$f(x) = \sum_j \int_V e^{i\langle x, z\rangle} N_j(x, z) dv_j(z),$$

where $dv_j(z)$ *are densities supported on V and satisfying, for every compact* $K \subset \Omega$

$$\int_V e^{\max_K(\langle x, z\rangle)} |dv| < +\infty.$$

We will now proceed to offer a similar algebraic analysis of Maxwell equations, but we will now consider them as equations in A^{μ}, rather than in F. Consider first the formulas (3), which are equivalent to $F = dA$, i.e., $F^{\mu\upsilon} = \partial^{\mu} A^{\upsilon} - \partial^{\upsilon} A^{\mu}$, and so equivalent to the homogenous Maxwell equations. We write them in terms of matrices as follows. Let

$$Q(D) = \begin{bmatrix} -\partial_x & -\partial_t & 0 & 0 \\ -\partial_y & 0 & -\partial_t & 0 \\ -\partial_z & 0 & 0 & -\partial_t \\ 0 & 0 & -\partial_z & \partial_y \\ 0 & \partial_z & 0 & -\partial_x \\ 0 & -\partial_y & \partial_x & 0 \end{bmatrix}$$

and consider the vectors $A = (A^0, A^1, A^2, A^3)^t$ and $F = (E^1, E^2, E^3, B^1, B^2, B^3)^t$. Then the formulas

$$E = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0$$

$$B = \nabla \times \mathbf{A}, \tag{3}$$

which are equivalent to the homogenous Maxwell equations are also equivalent to the system $Q(D)A = F$. Using CoCoA this gives the resolution:

$$0 \to R^1 \xrightarrow{Q_2^t} R^4 \xrightarrow{Q_1^t} R^6 \xrightarrow{Q^t} R^4 \to M \to 0 \tag{4}$$

and syzygies:

$$Q_1(D) = \begin{bmatrix} 0 & 0 & 0 & \partial_x & \partial_y & \partial_z \\ 0 & -\partial_z & \partial_y & \partial_t & 0 & 0 \\ \partial_z & 0 & -\partial_x & 0 & \partial_t & 0 \\ -\partial_y & \partial_x & 0 & 0 & 0 & \partial_t \end{bmatrix}$$

and

$$Q_2(D) = [\partial_t, -\partial_x, -\partial_y, -\partial_z].$$

We obtain a similar theorem as above.

Theorem 5. *The system* $Q(D)A = F$ *has a solution* $A \in \mathcal{S}'^{r_0}$ *if and only if* $F \in \mathcal{S}'^{r_1}$ *satisfies* $Q_1(D)F = 0$, *i.e.,*

$$\nabla \cdot B = 0$$

$$\nabla \times E + \partial_t B = \mathbf{0}.$$

Therefore the potential A *satisfies the formulas* (3) *if and only if the magnetic source-free Maxwell equations* $dF = 0$ *are satisfied. Moreover, if we consider magnetic sources in the form* $G^m = (\rho^m, \mathbf{j}^m)$, *then the system* $Q_1(D)F = G^m$ *has a solution* $F \in \mathcal{S}'^{r_0}$ *if and only if* $G \in \mathcal{S}'^{r_1}$ *satisfies* $Q_2(D)G^m = 0$, *i.e.,*

$$\partial_t \rho^m + \nabla \cdot \mathbf{j}^m = 0.$$

In order to obtain a similar interpretation of the electric source-free part of Maxwell's equations $(d * F = 0)$, one could start with the following matrix $Q^*(D)$:

$$Q^*(D) = \begin{bmatrix} 0 & 0 & \partial_z & -\partial_y \\ 0 & -\partial_z & 0 & \partial_x \\ 0 & \partial_y & -\partial_x & 0 \\ -\partial_x & -\partial_t & 0 & 0 \\ -\partial_y & 0 & -\partial_t & 0 \\ -\partial_z & 0 & 0 & -\partial_t \end{bmatrix}$$

and then consider the equation $Q^*(D)A = *F = (-B^1, -B^2, -B^3, E^1, E^2, E^3)^t$. The module associated to this equation has the same resolution (4), as $Q^*(D)$ is obtained from $Q(D)$ by interchanging the first three rows with the last three rows and multiplying the new first three rows by -1. Moreover, the syzygy $Q_1^*(D)$ is basically the same as $Q_1(D)$ modulo a minus sign and switching the first two with the last two rows. The second syzygy is $Q_2^*(D) = -Q_2(D)$, and we obtain a similar theorem:

Theorem 6. *The system* $Q^*(D)A = *F$ *has a solution* $A \in \mathcal{S}^{r_0}$ *if and only if* $*F \in \mathcal{S}^{r_1}$ *satisfies* $Q_1^*(D) * F = 0$, *i.e.,*

$$\nabla \cdot E = 0$$

$$\nabla \times B - \partial_t E = \mathbf{0}.$$

Therefore the potential A *satisfies the formulas* (3) *if and only if the electric source-free Maxwell equations* $d * F = 0$ *are satisfied. Moreover, if we consider electric sources in the form* $G^e = (\rho^e, \mathbf{j}^e)$, *then the system* $Q_1^*(D) * F = G^e$ *has a solution* $*F \in \mathcal{S}^{r_0}$ *if and only if* $G^e \in \mathcal{S}^{r_1}$ *satisfies* $Q_2^*(D)G^e = 0$, *i.e.,*

$$\partial_t \rho^e + \nabla \cdot \mathbf{j}^e = 0.$$

Our results above can be understood in terms of products of matrices of differential operators as follows. Note that considering the system $P(D)F = G$, where $F = Q(D)A$, we notice that $P^m(D)Q(D) \equiv 0$, so the system becomes only $P^e(D)Q(D) = G^e$. The physical explanation is that in the presence of the gauge potential A and electric sources G^e, magnetic monopoles cannot exist. This can be seen directly, as $F = dA$ leads to $dF = 0$.

Similarly, the dual system $P^*(D) * F = G$, where $*F = Q^*(D)A$ will simplify to $P^m(D)Q^*(D)A = G^m$, as $P^e(D)Q^*(D) \equiv 0$. The physical explanation is that in the presence of the gauge potential A and magnetic sources G^m, electric sources cannot exist [2].

Consider now Maxwell's equation $d * F = J$ in the variables A, which reads:

$$\Box A^v - \partial_v(\partial_\mu A^\mu) = j^v.$$

In order to write its matrix, note that we can also multiply the matrix obtained from the first four rows of the matrix $P(D)$ above (1), say

$$\widetilde{P}(D) = \begin{bmatrix} \partial_x & \partial_y & \partial_z & 0 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 & -\partial_z & \partial_y \\ 0 & -\partial_t & 0 & \partial_z & 0 & -\partial_x \\ 0 & 0 & -\partial_t & -\partial_y & \partial_x & 0 \end{bmatrix}$$

and the matrix $Q(D)$. We obtain the matrix:

$$S(D) = \begin{bmatrix} -\partial_x^2 - \partial_y^2 - \partial_z^2 & -\partial_x\partial_t & -\partial_y\partial_t & -\partial_z\partial_t \\ \partial_x\partial_t & -\partial_y^2 - \partial_z^2 + \partial_t^2 & \partial_x\partial_y & \partial_x\partial_z \\ \partial_y\partial_t & \partial_x\partial_y & -\partial_x^2 - \partial_z^2 + \partial_t^2 & \partial_y\partial_z \\ \partial_z\partial_t & \partial_x\partial_z & \partial_y\partial_z & -\partial_x^2 - \partial_y^2 + \partial_t^2 \end{bmatrix}$$

Consider now the new system

$$S(D)A = G^e = [\rho, j^1, j^2, j^3]^t.$$

By running CoCoA on S and we get the resolution:

$$0 \to R^1 \xrightarrow{S_1^t} R^4 \xrightarrow{S^t} R^4 \to M \to 0$$

with one syzygy

$$S_1(D) = [\partial_t, \partial_x, \partial_y, \partial_z],$$

which gives again the conservation law for the electric current, as expected.

Theorem 7. *The system* $S(D)A = G^e$ *has a solution* $F \in \mathcal{S}^{r_0}$ *if and only if* $G^e \in \mathcal{S}^{r_1}$ *satisfies* $S_1(D)G^e = 0$. *i.e.,*

$$\partial_t \rho^e + \nabla \cdot \mathbf{j}^e = 0.$$

Finally, we note that in the case of Maxwell's equations it is not useful to combine both Q and S together in a single matrix because in this case the potentials are a tool to determine the fields. Note here that the resulting characteristic variety is zero, which leads to non-trivial implications for this system.

3. Algebraic Analysis of the Abelian ASD Equations

In this section, we apply our computational algebra techniques for the Abelian anti-self- dual (ASD) equations written in local coordinates using as variables *E*, the electric field, and then *A*, the gauge potential. In the real Minkowski space $\mathbb{R}^{1,3}$ both the ASD and the self-dual (SD) equations are satisfied only for trivial *F*. Therefore, one must either complexify *F* and consider this setup embedded in the complex Minkowski space $\mathbb{C}^{1,3}$, or work with real forms on (locally) either the Euclidean space $\mathbb{R}^4$ or the ultrahyperbolic space $\mathbb{R}^{2,2}$. For simplicity of presentation, we study only the cases of Euclidean and the complex Minkowski spaces separately below. Also, for consistency of the presentation, we will focus only on the ASD equations.

3.1. The Euclidean case.

In the standard Euclidean metric on $\mathbb{R}^4$, the ASD equations are:

$$F_{01} = -F_{23}, \quad F_{02} = -F_{31}, \quad F_{03} = -F_{31},$$

which are equivalent to $E = -B$. Because $dF = 0$, it follows that $d * F = 0$, thus *F* is also a solution to the source-free Maxwell's system. In local coordinates in variable *E*, the ASD equations in $\mathbb{R}^4$ are:

$$-\nabla \cdot E = 0, \quad \nabla \times E + \partial_t E = \mathbf{0}, \tag{5}$$

and they have real solutions. As in the previous section, we consider the matrix of differential operators:

$$V(D) = \begin{bmatrix} -\partial_x & -\partial_y & -\partial_z \\ \partial_t & \partial_z & -\partial_y \\ -\partial_z & \partial_t & \partial_x \\ \partial_y & -\partial_x & \partial_t \end{bmatrix}$$

and consider the electric vector $E = (E^1, E^2, E^3)^t$. Then the equations (5) are equivalent to the system $V(D)E = 0$. For an electric source $J = (\rho, j_1, j_2, j_3)$, we consider the non-homogenous system $V(D)E = J$.

Using CoCoA we compute the resolution corresponding to the associated module:

$$0 \to R^1 \xrightarrow{V_1^t} R^4 \xrightarrow{V^t} R^3 \to M \to 0 \tag{6}$$

and the syzygy:

$$V_1(D) = [\partial_t, \partial_x, \partial_y, \partial_z].$$

We obtain a similar theorem as above.

Theorem 8. *The system* $V(D)E = J$ *corresponding to the ASD equations in* $\mathbb{R}^4$, *has a solution* $E \in \mathcal{S}'^0$ *if and only if* $E \in \mathcal{S}'^1$ *satisfies* $V_1(D)J = 0$, *i.e., the conservation law*

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0.$$

If we consider the ASD equations in variable A, as we did in the case of Maxwell's system, we obtain the system $W(D)A = J$, where:

$$W(D) = \begin{bmatrix} -\partial_x^2 - \partial_y^2 - \partial_z^2 & \partial_x\partial_t & \partial_y\partial_t & \partial_z\partial_t \\ \partial_x\partial_t & -\partial_t^2 & -\partial_z\partial_t & \partial_y\partial_t \\ \partial_y\partial_t & \partial_z\partial_t & -\partial_t^2 & -\partial_x\partial_t \\ \partial_z\partial_t & -\partial_y\partial_t & \partial_x\partial_t & -\partial_t^2 \end{bmatrix}$$

with the corresponding resolution:

$$0 \to R^1 \xrightarrow{W_1^t} R^4 \xrightarrow{W^t} R^4 \to M \to 0 \tag{7}$$

and W_1 is basically the same syzygy as V_1.

3.2. The complex Minkowski case.

In $\mathbb{C}^{1,3}$ the ASD equations are $B = -iE$, which, in local (complex) coordinates are equivalent to

$$\nabla \cdot E = 0, \quad \nabla \times E - i\partial_t E = \mathbf{0}. \tag{8}$$

The solutions in real space-time are automatically complex valued. If we denote the complex electric 3-vector by $E = E_1 + iE_2$, where both E_1 and E_2 are real 3-vectors, then the system obtained from (8) is

$$\nabla \cdot E_1 = 0, \quad \nabla \times E_1 + \partial_t E_2 = \mathbf{0}$$

$$\nabla \cdot E_2 = 0, \quad \nabla \times E_2 - \partial_t E_1 = \mathbf{0}, \tag{9}$$

system which is analog to the real Maxwell system. We keep the same notation for the matrices involved, but we consider variables (E_1, E_2) instead of (E, B), and the vector $G = [G_1^e, G_2^e]^t = [\rho_1^e, j_{11}^e, j_{12}^e, j_{13}^e, \rho_2^e, j_{21}^e, j_{22}^e, j_{23}^e]^t$, formed by the real components of the complex vector $(\rho, \mathbf{j})$. Our computation algebra techniques will yield then, in a similar fashion, the following theorem:

Theorem 9. *The system* $P(D)E = G$ *has a solution* $E = E_1 + iE_2 \in \mathcal{S}'^{r_0}$ *if and only if* $G \in \mathcal{S}'^{r_1}$ *satisfies* $P_1(D)G = 0$. *i.e., the compatibility law for the complex electric source* $(\rho, \mathbf{j})$:

$$\partial_t \rho^e + \nabla \cdot \mathbf{j}^e = \mathbf{0}. \tag{10}$$

A similar analysis for (complex) variables A will yield a similar result as Theorem 5.

It is important to notice that in our algebraic approach the real-Minkowski Maxwell's system of equations yields the same matrices as the complex-Minkowski Abelian instanton equations. In both cases, the characteristic variety of $Ext^1(M, R)$ is a cone, therefore, in both cases, the compact singularities cannot be eliminated, [2].

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