A SUPERLINEARLY CONVERGENT SSLE ALGORITHM FOR OPTIMIZATION PROBLEMS WITH LINEAR COMPLEMENTARITY CONSTRAINTS

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Abstract

In this paper, a sequential system of linear equations (SSLE) algorithm for solving mathematical problem with linear complementarity constraints is introduced, which uses Fischer-Burmeister (F-B) function and smoothing technique to rewrite

2010 Mathematics Subject Classification: 62Lxx.

Keywords and phrases: mathematics programs with equilibrium constraints, sequential system of linear equations, global convergence, superlinear convergence.

This work was supported in part by the NNSF (No. 10501009) of China and Guangxi Province Science Foundation (No. 0728206).

Received May 10, 2010

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under some suitable conditions without upper level complementarity, the proposed method is proved to possess global convergence and superlinear convergence.

1. Introduction

Mathematical programs with equilibrium constraints (MPEC) is an optimization problem, in which the constraints include variational inequalities or complementarity restrictions. This problem plays an important role in many fields such as engineering design, economic equilibrium, and multilevel game, see [6].

In this paper, we consider an important subclass of MPEC problem, which is called mathematical program with linear complementarity constraints (MPLCC):

\[
\begin{align*}
\min & \quad f(x, y), \\
\text{s.t.} & \quad Ax \preceq b, \quad w = Nx + My + q, \quad 0 \leq w^\top y \geq 0,
\end{align*}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a continuous differential function, \( A \in \mathbb{R}^{p \times n}, N \in \mathbb{R}^{m \times n}, M \in \mathbb{R}^{m \times m}, b \in \mathbb{R}^p, \) and \( q \in \mathbb{R}^m \). The development of various algorithms for mathematical programs with equilibrium constraints can be founded in [2, 4-8]. The major difficulty in solving (1.1) is that, its constraints fail to satisfy a standard constraint qualification at any feasible point [7], which is necessary for the regularity of a nonlinear program, so that standard methods are likely to fail for this problem. Therefore, designing algorithm for (1.1) is interesting. From the goal of reducing the computational amount of each iteration, Li and Jian [5] recently showed a superlinearly convergent SSLE type algorithm to (MPLCC) (1.1). In [5], by perturbed technique \( y_iw_i = \mu, i = 1 \sim m \) and generalized complementarity function \( \phi(a, b, \mu) = a + b - \sqrt{a^2 + b^2 + \lambda ab + (2 - \lambda)\mu}, \quad \lambda \in (-2, 2) \), the complementarity constraints are transformed into nonlinear equalities \( \phi(y_i, w_i, \mu) = 0, i = 1 \sim m \). However, due to the existence of the parameter \( \mu \), the SSLE algorithm in theoretically can not find an exact stationary point of (MPLCC) (1.1) in finite iteration.
In this paper, by means of Fischer-Burmeister function \( \phi(a, b) = a + b - \sqrt{a^2 + b^2}, (a, b) \in \mathbb{R}^2 \), we transform equivalently problem (1.1) into a nonsmooth optimization problem. Furthermore, introducing smoothing technique [8], we present a new SSLE method to solve the problem (1.1). Obviously, our approach does not adopt any artificial variable or parameter and only solve three systems of equations with same coefficients at each iteration. Under appreciate conditions, the proposed algorithm is proved to possess global convergence and superlinearly convergent rate. Moreover, we conclude that the current iterate point is an exact stationary point of (1.1), when the algorithm stops at current point \( z^k \).

2. Preliminaries and Algorithm

For the sake of convenience, we denote that

\[
X = \{ z | Ax \leq b, w = Nx + My + q, 0 \leq w \perp y \geq 0 \},
\]

\[
s = (x, y), t = (y, w),
\]

\[
X_0 = \{ z = (x, y, w) | w = Nx + My + q \}, A = (a_j, j = 1 \sim p),
\]

\[
b = (b_j, j = 1 \sim p),
\]

\[
dz = (dx, dy, dw), ds = (dx, dy), dt = (dy, dw),
\]

\[
dt_i = (dy_i, dw_i), t_i = (y_i, w_i),
\]

\[
L_1 = \{1, 2, \ldots, p\}, L_2 = \{1, 2, \ldots, m\},
\]

\[
\phi(x) = \max\{0; a_jx - b_j, j \in L_1\},
\]

\[
I(x) = \{ j \in L_1 | \phi(x) = a_jx - b_j \},
\]

\[
I_w(z) = \{ j \in L_2 | w_j = 0 < y_j \},
\]

\[
I_0(z) = \{ j \in L_2 | w_j = 0 = y_j \},
\]

\[
I_y(z) = \{ j \in L_2 | w_j > 0 = y_j \}.
\]
Throughout this paper, the following assumptions are assumed:

**H 2.1.** The feasible set of (1.1) is nonempty, i.e., $X \neq \emptyset$.

**H 2.2.** $f$ is two-times continuously differentiable; $M$ is a $P$ matrix.

**H 2.3.** For any $z \in X_0$, the vector $\{a_j : j \in I(x)\}$ are linearly independent.

The following proposition can be refereed to in [2, 6].

**Proposition 2.1.** Suppose that $z^* \in X$ satisfies the so-called non-degeneracy condition:

$$
(y_i^*, w_i^*) \neq (0, 0), \quad i = 1, 2, \ldots, m. \tag{2.2}
$$

Then $z^*$ is a stationary point of problem (1.1), if and only if there exist multipliers $(\lambda^*, u^*, v^*) \in R^p \times R^m \times R^m$, such that

$$
\begin{bmatrix}
\nabla f(x^*, y^*) & 0 \\
0 & W^* \\
Y^*
\end{bmatrix}
\begin{bmatrix}
v^* \\
u^*
\end{bmatrix}
+
\begin{bmatrix}
A^T \\
0 \\
- I
\end{bmatrix}
\begin{bmatrix}
\lambda^* \\
- I
\end{bmatrix}
+
\begin{bmatrix}
N^T \\
M^T
\end{bmatrix}
\begin{bmatrix}
u^*
\end{bmatrix}
= 0,
$$

$$
\lambda^* \geq 0, \quad (Ax^* - b)^T \lambda^* = 0. \tag{2.3}
$$

Moreover, formula (2.3) is equivalent to the following conditions:

$$
\nabla f(x^*, y^*) + \begin{bmatrix}
N^T Y^* \\
W^* + M^T Y^*
\end{bmatrix}
\begin{bmatrix}
v^* \\
u^*
\end{bmatrix}
+ \begin{bmatrix}
A^T \\
0
\end{bmatrix}
\lambda^* = 0,
$$

$$
\lambda^* \geq 0, \quad (Ax^* - b)^T \lambda^* = 0, \tag{2.4}
$$

where diagonal matrices $W^* = \text{diag}(w_i^*, i = 1 \sim m)$, $Y^* = \text{diag}(y_i^*, i = 1 \sim m)$.

This algorithm proposed in this paper makes use of so-called Fischer-Burmeister function [1], $\phi : R^2 \to R$ defined by

$$
\phi(a, b) = a + b - \sqrt{a^2 + b^2}, \text{ for } (a, b) \in R^2. \tag{2.5}
$$
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By means of the above function \( \phi(a, b) \), like [8], we consider the following equivalent smooth optimization problem associated with problem (1.1):

\[
\begin{align*}
\min \quad & f(x, y), \\
\text{s.t.} \quad & Ax \geq b, w = Nx + My + q, \Phi_e(y, w) = 0, \quad (2.6)
\end{align*}
\]

where

\[
\Phi(y, w) = \Phi_e(y, w) + \Phi_e(y, w),
\]

\[
\Phi(y, w) = \begin{bmatrix} \phi(y_1, w_1) \\ \vdots \\ \phi(y_m, w_m) \end{bmatrix}, \Phi_e(y, w) = \begin{bmatrix} \phi_e(y_1, w_1) \\ \vdots \\ \phi_e(y_m, w_m) \end{bmatrix}, \Phi_e(y, w) = \begin{bmatrix} \phi_e(y_1, w_1) \\ \vdots \\ \phi_e(y_m, w_m) \end{bmatrix},
\]

and for \( e > 0 \), and the index set \( E(z, e) \equiv \left\{ j \in L_2 \mid \sqrt{y_j^2 + w_j^2} < e \right\} \),

\[
\phi_e(y_j, w_j) = \begin{cases} \phi(y_j, w_j), & j \in L_2 \setminus E(z, e), \\ \frac{2e - y_j}{2e} y_j + \frac{2e - w_j}{2e} w_j - \frac{e}{2}, & j \in E(z, e), \end{cases}
\]

\[
\phi_e(y_j, w_j) = \begin{cases} 0, & j \in L_2 \setminus E(z, e), \\ \frac{\left(\sqrt{y_j^2 + w_j^2} - e\right)^2}{2e}, & j \in E(z, e). \end{cases}
\]

It is clear that the function \( \Phi_e(y, w) \) is smooth, and

\[
\nabla_a \phi_e(y_j, w_j) = \begin{cases} \nabla_a \phi(y_j, w_j) = 1 - \frac{y_j}{\sqrt{y_j^2 + w_j^2}}, & j \in L_2 \setminus E(z, e), \\ 1 - \frac{y_j}{e}, & j \in E(z, e), \end{cases}
\]

\[
\nabla_b \phi_e(y_j, w_j) = \begin{cases} \nabla_b \phi(y_j, w_j) = 1 - \frac{w_j}{\sqrt{y_j^2 + w_j^2}}, & j \in L_2 \setminus E(z, e), \\ 1 - \frac{w_j}{e}, & j \in E(z, e). \end{cases}
\]
Like [5], we use the following penalty function as the merit function:

$$\theta(z, \alpha, e) = f(x, y) + \alpha \varphi(x) + \alpha \| \Phi_e(y, w) \|,$$

where $\alpha > 0$ is a penalty parameter.

Now, the algorithm for the solution of the problem (1.1) can be stated as follows.

**Algorithm A.**

**Step 0** (Initialization):

Given a starting point $z^0 = (x^0, y^0, w^0) \in X_0$, and an initial symmetric matrix $B_0 \in R^{(2m+n) \times (2m+n)}$ such that $H_0 \in R^{(n+m) \times (n+m)}$ with form of (2.24) is positive definite. Choose parameters $\alpha_0 > 0$, $\delta > 0$, $\varepsilon_0 > 0$, $0 < \beta < 1$, $0 < \sigma < 1$, $e \in (0, \infty)$. Set $k = 0$.

**Step 1** (Pivoting):

1.1 Let $i = 0$, $\varepsilon_{k,i} = \varepsilon_0$;

1.2 Compute $\bar{J}_{k,i} = \{ j \in L_1 : 0 \leq \varphi(x) - (a_j x - b_j) \leq \varepsilon_{k,i} \}$. If $\det (A_J A_J^T) \geq \varepsilon_{k,i}$, where $A_J = (a_j, j \in \bar{J}_{k,i})$, then let $J_k(x^k, \varepsilon_k) = J_{k,i}$, $\varepsilon_k = \varepsilon_{k,i}$, and go to Step 2. Otherwise, let $i := i + 1$, $\varepsilon_{k,i} = \frac{1}{2} \varepsilon_{k,i-1}$, and go back to 1.2.

**Step 2** (Direction generation):

2.1 Solve the following system of linear equations:

$$\begin{pmatrix} H_k & G_k^T \\ G_k & 0 \end{pmatrix} \begin{pmatrix} ds_k \\ \omega \end{pmatrix} = - \begin{pmatrix} \nabla f(s_k) \\ h(z^k, \varepsilon_k) \end{pmatrix}, \quad (2.12)$$

Let its solution be $ds^k_0$, $\omega^k_0 \equiv \begin{pmatrix} \lambda^k_0, J_k \\ v^k_0 \end{pmatrix}$, where $\lambda^k_{0,j} = (\lambda^k_{0,j}, j \in J_k)$, $v^k_0 = (v^k_{0,i}, i \in L_2)$. Set $dw^k_0 = N d\lambda^k_0 + M dy^k_0$, $dz^k_0 = (ds^k_0, dw^k_0)$. 
Case A. if $ds_0^k = 0, \lambda_{0,j}^k \geq 0, j \in J_k, E(z^k, e_k) = \emptyset$, stop;

Case B. if $ds_0^k = 0, \lambda_{0,j}^k \geq 0, j \in J_k, E(z^k, e_k) \neq \emptyset$, set $z^{k+1} = z^k, B_{k+1} = B_k, e_{k+1} = \frac{1}{2} e_k$, go to Step 6;

Case C. if $ds_0^k \neq 0$ or $\lambda_{0,j}^k < 0, j \in J_k$, then go to 2-2.

2-2 Solve the following system of linear equations:

$$
\begin{pmatrix}
H_k & G_k^T \\
G_k & 0
\end{pmatrix}
\begin{pmatrix}
ds \\
\omega
\end{pmatrix}
= -\begin{pmatrix}
\nabla f(s^k) \\
h(z^k, e_k) - \eta^k
\end{pmatrix},
$$

(2.13)

where

$$
\eta_j^k = \begin{cases}
(\lambda_{0,j}^k)^3, & \text{if } \lambda_{0,j}^k \leq 0, j \in J_k, \\
0, & \text{otherwise.}
\end{cases}
$$

(2.14)

Let its solution be $ds^k, \omega^k = \begin{pmatrix} \lambda_{J_k}^k \\ v_k^k \end{pmatrix}$, set $dw^k = Ndx^k + Mdy^k, dz^k = (ds^k, dw^k)$.

2-3 Solve the following system of linear equations:

$$
\begin{pmatrix}
H_k & G_k^T \\
G_k & 0
\end{pmatrix}
\begin{pmatrix}
ds \\
\omega
\end{pmatrix}
= -\begin{pmatrix}
\nabla f(s^k) \\
h(z^k, e_k) + h(z^k + dz^k, e_k)
\end{pmatrix}.
$$

(2.15)

Let its solution be $\overline{ds}^k, \overline{\omega}^k = \begin{pmatrix} \overline{\lambda}_{J_k}^k \\ \overline{v}_k^k \end{pmatrix}$, set $\overline{dw}^k = N\overline{dx}^k + M\overline{dy}^k, \overline{dz}^k = (\overline{ds}^k, \overline{dw}^k)$. If $\|\overline{dz}^k - dz^k\| > \|dz^k\|$, then let $\overline{dz}^k = dz^k$.

**Step 3** (Penalty update):

Denote $\lambda_0^k = (\lambda_{0,j}^k, j \in L_1), \lambda^k = (\lambda_{j}^k, j \in L_1)$, where

$$
\lambda_{0,j}^k = \begin{cases}
\lambda_{0,j}^k, & j \in J_k, \\
0, & \text{otherwise,}
\end{cases}
\lambda_j^k = \begin{cases}
\lambda_{j}^k, & j \in J_k, \\
0, & \text{otherwise.}
\end{cases}
$$

(2.16)
Denote
\[ \zeta_k = \max\{\|\lambda_0^k\|_1, \|\lambda^k\|_1, \|\beta_0\|_1, \|v^k\|_1, \|2\lambda_0^k - \lambda^k\|_1, \|2v_0^k - v^k\|_1\}, \tag{2.17} \]
where \[ \|p\|_1 = \sum_{i=1}^{r}|p_i| \] for \( p = (p_1, p_2, \ldots, p_r) \in R^r \). The updating rule for \( \alpha \) is as follows:
\[ \alpha_k = \begin{cases} \alpha_{k-1}, & \text{if } \alpha_{k-1} \geq \zeta_k + \delta, \\ \max\{\alpha_{k-1} + 2\delta, \zeta_k + \delta\}, & \text{otherwise.} \end{cases} \tag{2.18} \]

**Step 4** (Step size determination):

Compute the step size \( \tau_k \), which is the first number \( \tau \) of the sequence \( \{1, \beta, \beta^2, \ldots\} \) satisfying
\[ \theta(z^k + \tau dz^k + \tau^2(\overline{d}z_k - dz^k), \alpha_k, e_k) \leq 0(z^k, \alpha_k, e_k) + \sigma \tau \psi(z^k, dz^k, \alpha_k, e_k), \tag{2.19} \]
where
\[ \psi(z^k, \alpha_k, e_k) = \nabla f(x^k)^T ds^k - \alpha_k \Phi_{e_k}(t^k)\|_{\|} + \frac{1}{2}(ds^k)^T H_k ds^k. \tag{2.20} \]

**Step 5** (Reupdating): Update \( B_k \) by some given method to yield a new matrix \( B_{k+1} \).

Set
\[ z^{k+1} = z^k + \tau dz^k + \tau^2(\overline{d}z_k - dz^k), \]
\[ r_{k+1} = \min\left\{ \sqrt{(y_{j-1}^{k+1})^2 + (w_{j-1}^{k+1})^2} \mid j = 1, 2, \ldots, m \right\}, \tag{2.21} \]
\[ e_{k+1} = \begin{cases} \frac{1}{2}e_k, & \text{if } r_{k+1} \leq e_k, \\ e_k, & \text{otherwise.} \end{cases} \tag{2.22} \]

**Step 6.** Set \( k = k + 1 \), return to Step 1.
Remark. (i) In Algorithm A,

\[ B_k = \begin{bmatrix} C_k & 0 \\ 0 & D_k \end{bmatrix}, \quad C_k \in \mathbb{R}^{(n+m) \times (n+m)}, \quad D_k \in \mathbb{R}^{m \times m}, \quad k = 0, 1, \ldots \]

is an approximation of the Hesse matrix \( H(z, v, e) \) of the Lagrange function of problem (2.6), where

\[
H(z, v, e) = \begin{pmatrix}
\nabla^2_{xx} f(s) & \nabla^2_{xy} f(s) & 0_{n \times m} \\
\nabla^2_{yx} f(s) & \nabla^2_{yy} f(s) + \text{diag}(v_i \frac{\partial^2 \phi_a(t)}{\partial a^2}) & \text{diag}(v_i \frac{\partial^2 \phi_a(t)}{\partial a \partial b}) \\
0_{m \times n} & \text{diag}(v_i \frac{\partial^2 \phi_b(t)}{\partial b \partial a}) & \text{diag}(v_i \frac{\partial^2 \phi_b(t)}{\partial b^2}) \\
\end{pmatrix}.
\]

(ii) \( H_k = C_k + (N M)^T D_k (N M) \),

\[ G_k = \begin{pmatrix} A_{J_k} & 0 \\ D_a^k N & D_a^k + D_b^k M \end{pmatrix}, \]

\[ h(z^k, e_k) = \begin{pmatrix} A_{J_k} x^k - b_{J_k} \\ \Phi_{e_k}(t^k) \end{pmatrix}. \]

(iii) \( D_a^k = \text{diag}(\nabla a \phi_e(y_j^k, w_j^k), j = 1, 2, \ldots, m) \),

\[ D_b^k = \text{diag}(\nabla b \phi_e(y_j^k, w_j^k), j = 1, 2, \ldots, m). \]

In the remainder of this section, similar to the analysis in [3, 5], we give some results to show that Algorithm A is correctly stated.

Lemma 2.1. Let assumption H 2.3 hold, then pivoting in the Step 1 is terminated after a finite number of computations.

Lemma 2.2. Let assumptions H 2.2-H 2.3 hold, and \( H_k \) is positive definite, then for any \( z^k \in X_0 \), the coefficient matrix of (2.12) is nonsingular. Furthermore, systems (2.12), (2.13), and (2.15) have a unique solution, respectively. In addition to, it holds that

\[ ds_0^k = -P_k \nabla f(s^k) - F_k h(z^k, e_k), \]
\[ \omega_0^k = -F_k^T \nabla f(s^k) + (G_k H_k^{-1} G_k^T)^{-1} h(z^k, e_k), \]
\[ ds^k = ds_0^k + F_k \eta^k, \quad \omega^k = \omega_0^k - (G_k H_k^{-1} G_k^T)^{-1} \eta^k, \] \hspace{1cm} (2.25)

where
\[ P_k = H_k^{-1} - H_k^{-1} G_k^T (G_k H_k^{-1} G_k^T)^{-1} G_k H_k^{-1}, \]
\[ F_k = H_k^{-1} G_k^T (G_k H_k^{-1} G_k^T)^{-1}. \]

**Lemma 2.3.** Suppose that assumptions H 2.2-H 2.3 hold and \( H_k \) is symmetric positive definite, then the following statements are true:

(i) For any \( k \), \( \varphi(z^k, dz^k, \alpha_k, e_k) < 0 \),

(ii) For any \( k \), there exists a constant \( \tau > 0 \) satisfying (2.19), that is, Step 4 in Algorithm A is well defined.

(iii) The sequence \( \{z^k\} \) generated by Algorithm A satisfies \( w^k = N z^k + M y^k + q \) for all \( k \).

3. Global Convergence of Algorithm

In this section, we analyze the global convergence of the algorithm. Firstly, we show that \( z^k \) is an exact stationary point of (1.1), if the algorithm stops at \( z^k \).

**Lemma 3.1.** For system (2.12), if \( ds_0^k = 0, z_{0, j}^k \geq 0, \) and \( E(z^k, e_k) = \emptyset, \) then \( z^k \) is an exact stationary point of problem (1.1).

According to Step 2 (2-1) of Algorithm A, if Case B appears, that is, updating is used as follows: \( z^{k+1} = z^k, e_{k+1} = \frac{1}{2} e_k, B_{k+1} = B_k \). At next iteration, two case will occur: (i) one has \( dz^{k+1} = 0, \lambda^{k+1}_{0, j} \geq 0, j \in J_k, \) \( E(z^{k+1}) = \emptyset, \) then \( z^{k+1} = z^k \) is a stationary point of (1.1); (ii) another case is similar to Case C, under which situation, algorithm generates an
infinite sequence \(\{z^k\}\). For the sake of convenience, we only consider the Case C in the remainder of this section. Moreover, we make some assumptions as follows:

**H 3.1.** There exist constants \(c_2 \geq c_1 > 0\), such that
\[
c_1\|s\|^2 \leq s^TH_k s \leq c_2\|s\|^2, \forall s \in \mathbb{R}^{n+m}, k = 0, 1,\ldots
\]

**H 3.2.** The point sequence \(\{z^k\}\) produced by Algorithm A is bounded, and every accumulation point \(z^* = (x^*, y^*, w^*)\) of \(\{z^k\}\) satisfies the following conditions:

(i) The (lower level) nondegeneracy condition (2.2) holds;

(ii) The submatrix \(M_{J^*J^*}\) is nondegenerate, i.e., all of its principal minors are nonnegative, where the index set \(J^* = \{i : w_i^* = 0\}\).

**Lemma 3.2.** Suppose that assumptions H 2.2-H 3.2 hold, then

(1) There exists a constant \(c > 0\), such that \(\|(D^h_a + D^h_b M)^{-1}\| < c\),

\[
\left| \begin{pmatrix} H_k & G_k^T \\ G_k & 0 \end{pmatrix} \right|^{-1} < c, \text{ for all } k \in \kappa.
\]

(2) The sequences \(\{dz^k, k \in \kappa\}, \{d^a z^k, k \in \kappa\}, \{\lambda^k_0, k \in \kappa\}, \{\lambda^k, k \in \kappa\}, \{u^k_0, k \in \kappa\}, \{v^k, k \in \kappa\}\) are all bounded.

(3) There exists a positive integer \(k_0\) such that \(\alpha_k = \alpha_{k_0} = \alpha, \forall k \geq k_0\).

(4) There exists a constant \(e > 0\) such that \(e_k \geq e\) for all \(k\).

(5) \(\lim_{k \to \infty} \psi(z^k, dz^k, \alpha, e) = 0\).

According to assumptions H 3.1-H 3.2 and Lemma 3.2, we might as well assume that there exists a subsequence \(\kappa\), such that
Theorem 3.1. Suppose that assumptions H 2.1-H 3.2 hold, and \( z^* \) is an accumulation point of sequence \( \{z^k\} \) generated by proposed algorithm, then \( z^* \) is a stationary point of (1.1).

Proof. Let \( z^* \) be a given accumulation point of \( \{z^k\} \) and \( z^k \rightarrow z^* \).

From (2.20) and H 3.1, we obtain

\[
\psi(z^k, dz^k, \alpha, e) \leq -\frac{1}{2} c_1 \|ds_0^k\|^2 - \sum_{j \in J, \lambda_{0,j}^k < 0} (\lambda_{0,j}^k)^4 \leq 0, \quad \forall k \in \kappa.
\]

From Lemma 3.2, we have \( \lim_{k \rightarrow \infty} \psi(z^k, dz^k, \alpha, e) = 0 \), so

\[
\|ds_0^k\| \rightarrow 0, \quad \sum_{j \in J, \lambda_{0,j}^k \leq 0} (\lambda_{0,j}^k)^4 \rightarrow 0 (k \rightarrow \infty).
\]

Notice that \( \{\lambda_{0}^k, k \in \kappa\} \) and \( \{v_0^k, k \in \kappa\} \) are bounded from Lemma 3.2 (2), without loss of generality, we suppose that \( \lambda_{0,j}^k \rightarrow \lambda_j^*, \quad v_0^k \rightarrow \tilde{v}_j^* \). Hence, we can verify by (3.2) that \( \tilde{\lambda}_j^* \geq 0, \forall j \in J \). If not, then there exists a \( \tilde{\lambda}_t^* < 0, \quad t \in J \), so \( \lambda_{0,t}^k \rightarrow \tilde{\lambda}_t^* > 0 \), furthermore,

\[
\sum_{j \in J, \lambda_{0,j}^k \leq 0} (\lambda_{0,j}^k)^4 \geq (\lambda_{0,t}^k)^4 \rightarrow (\tilde{\lambda}_t^*)^4 > 0,
\]

which is contradictory with (3.2). So passing to \( k \rightarrow \infty \) in (2.12), we obtain

\[
\nabla f(s^*) + \left[ \begin{array}{c} N^T D_b^* \\ D_a^* + M^T D_b^* \end{array} \right] \tilde{v}^* + \left[ \begin{array}{c} A_J^T \\ 0 \end{array} \right] \tilde{\lambda}^* = 0,
\]

\[
(\tilde{\lambda}^*)_J \geq 0, \quad A_J x^* - b_J = 0, \quad \Phi_e(y^*, w^*) = 0,
\]

(3.3)
where
\[ D_a^* = \text{diag}(\nabla a \Phi_e(y_j^*, w_j^*)), \quad \text{if } j = 1 \sim m, \]
\[ D_b^* = \text{diag}(\nabla b \Phi_e(y_j^*, w_j^*)), \quad \text{if } j = 1 \sim m. \]

In addition, from (2.9) and (2.8), it holds that
\[
\phi(y_j^*, w_j^*) = \begin{cases} 
\frac{2e - y_j^*}{2e} y_j^* + \frac{2e - w_j^*}{2e} w_j^* - \frac{e}{2}, & j \in E(z^*, e), \\
\phi(y_j^*, w_j^*), & j \in L_2 \setminus E(z^*, e), 
\end{cases}
\]
\[
\overline{\phi}(y_j^*, w_j^*) = \begin{cases} 
0, & j \in L_2 \setminus E(z^*, e), \\
\frac{(y_j^* - w_j^*)^2 - e^2}{2e}, & j \in E(z^*, e).
\end{cases}
\]

While, in view of Lemma 3.2 (4), one has that
\[
\sqrt{\left(y_j^*\right)^2 + \left(w_j^*\right)^2} \geq e, \quad \forall j = 1 \sim m,
\]
so, it holds that
\[
\overline{\Phi}(y^*, w^*) = 0, \quad \Phi(y^*, w^*) = \Phi_e(y^*, w^*) + \overline{\Phi}(y^*, w^*) = \Phi_e(y^*, w^*) = 0,
\]
thereby, we have that
\[
0 \leq y^* \perp w^* \geq 0, \quad I_0(z^*) = \emptyset.
\]

In view of \( I(x^+) \subseteq J \), it follows from (3.3) that \( \varphi(x^*) = 0 \), furthermore, \( Ax^* - b \leq 0 \). Passing to the limit \( k \to \infty \) in \( w^k = Nx^k + My^k + q \), we get \( w^* = Nx^* + My^* + q \), so \( z^* = (x^*, y^*, w^*) \in X \). From (2.10) and (2.11), it is easy to see that
\[
(D_a^*)_{jj} = \begin{cases} 
1, & j \in I_x(z^*), \\
0, & j \in I_y(z^*) \cup I_w(z^*),
\end{cases}
\]
\[
(D_b^*)_{jj} = \begin{cases} 
1, & j \in I_y(z^*), \\
0, & j \in I_w(z^*) \cup I_x(z^*).
\end{cases}
\]

Now, we define
\[ v^*_j = \begin{cases} 
\tilde{v}^*_j, & j \in I_y(z^*), \\
w_j^*, & j \in I_w(z^*), \\
|y_j^*, & j \in I_w(z^*), 
\end{cases} \]

\[ \lambda^*_j = \tilde{\lambda}^*, \quad j \in J, \quad \lambda^*_j = 0, \quad j \notin J. \]

According to above-mentioned analysis, we have

\[
\nabla f(s^*) + \begin{pmatrix} N^T y^* \\
W^* + M^T y^* \end{pmatrix} v^* + \begin{pmatrix} A^T \\
0 \end{pmatrix} \lambda^* = 0,
\]

\[ 0 \leq \lambda^* \perp (b - Ax^*) \geq 0, \quad z^* \in X, \]

\[ (y_j^*, w_j^*) \neq (0, 0), \quad j = 1 \sim m, \]

which shows that \( z^* \) is an exact stationary point of problem (1.1).

### 4. Superlinear Convergence

Denote

\[ i^*_y = i_y(z^*) \overset{\text{def}}{=} \{ i \in L_2 : y_i^* = 0 \}, \quad i^*_w = i_w(z^*) \overset{\text{def}}{=} \{ i \in L_2 : w_i^* = 0 \}, \]

\[ L^*_I = \{ j \in L_1 : \lambda_j^* > 0 \}, \quad I^* = I(x^*), \quad A_{L^*_I} = (a_j, j \in L^*_I). \]

In order to prove the strong and superlinear convergence, the following assumptions are necessary:

**H 4.1.** There exists an accumulation point \( z^* \) of \( \{ z^k \} \), so from Lemma 3.1, there exist multipliers \((\lambda^*, v^*)\) such that the stationary point pair \((z^*, \lambda^*, v^*)\) satisfying (2.3), suppose that second order sufficient conditions as follows hold true: \( (ds)^T \nabla^2 f(x^*, y^*) ds > 0, \quad \forall ds \in \Omega, \)

where

\[ \Omega \overset{\text{def}}{=} \{ 0 \neq ds \in \mathbb{R}^{n+m} : N_{i^*_w} dx + (M_{i^*_w})_w dy^* = 0, A_{L^*_I} dx = 0, \quad dy^*_y = 0 \}. \]

(4.1)
The following lemma can be refereed to in [5].

**Lemma 4.1.** Suppose that assumptions H 2.1-H 4.1 hold, then

(a) \( J_k = I(x^*) = I^* \) for \( k \) large enough.

(b) \( \lim_{k \to \infty} z^k = z^* \), \( \lim_{k \to \infty} d_0^k = \lim_{k \to \infty} d^k = 0 \), \( \lim_{k \to \infty} \lambda_0^k = \lambda^* \), \( \lim_{k \to \infty} v_0^k = v^* \).

(c) \( \eta^k = 0 \) for all sufficiently large \( k \), moreover, \( d_0^k = d^k \).

(d) \( \|dz^k - dz^k\| = O(\|d_0^k\|^2) \).

**Theorem 4.1.** Suppose that assumptions H 2.1-H 4.2 hold, then \( \tau_k = 1 \) for all sufficiently large \( k \).

**Theorem 4.2.** Suppose the assumptions H 2.1-H 4.2 hold, if \( e_k - e = o(\|dz^k\|) \), then the sequence \( \{z^k\} \) produced by Algorithm A superlinearly converges to a stationary point \( z^* \) of (MPLCC) (1.1), i.e., \( \|z^{k+1} - z^*\| = o(\|z^k - z^*\|) \).

**References**


