ITERATIVE PROCESS FOR GENERALIZED $I$-ASYMPTOTICALLY QUASI-NONEXPANSIVE NONSELF MAPPINGS

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Abstract

Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset of $E$. Let $T : K \rightarrow E$ be a generalized $I$-asymptotically quasi-nonexpansive nonself mapping. In this paper, we establish iterative process for common fixed point of generalized $I$-asymptotically quasi-nonexpansive nonself mappings in Banach spaces. The results obtained in this paper improve and extend the corresponding results in the existing literatures.

1. Introduction

Let $E$ be a normed linear space, $K$ be a nonempty convex subset of $E$, and $T$ be a self map of $K$. Three most popular iteration procedures for obtaining fixed points of $T$, if they exist, are Mann iteration in [9], Ishikawa iteration in [4], and Noor iteration in [11].

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The multi-step iteration [14], arbitrary fixed order $p \geq 2$, defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n^1,$$

$$y_n^i = (1 - \beta_n^i)x_n + \beta_n^iT_y^{i+1}, i = 1, 2, \ldots, p - 2,$$

$$y_n^{p-1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1}Tx_n,$$  \hspace{1cm} (1.1)

where, for all $n \in N$,

$$\{\alpha_n\} \subset (0, 1), \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and for all $n \in N$,

$$\{\beta_n^i\} \subset [0, 1), 1 \leq i \leq p - 1, \lim_{n \to \infty} \beta_n^i = 0.$$

Taking $p = 3$ in (1.1), we obtain Noor iteration in [11]. Taking $p = 2$ in (1.1), we obtain Ishikawa iteration in [4]. Let $K$ be a subset of normed linear space $E$ and $T$ be a self-mapping of $K$. Then $T$ is called nonexpansive on $K$, if

$$\|Tx - Ty\| \leq \|x - y\|,$$  \hspace{1cm} (1.2)

for all $x, y \in K$. Let $F(T) := \{x \in K : Tx = x\}$ denotes the set of fixed points of a mapping $T$.

Let $K$ be a subset of normed linear space $E$, $T$ and $I$ be self-mappings of $K$. Then $T$ is called $I$-nonexpansive on $K$, if

$$\|Tx - Ty\| \leq \|Ix - Iy\|,$$  \hspace{1cm} (1.3)

for all $x, y \in K$ [16]. $T$ is called $I$-quasi-nonexpansive on $K$, if

$$\|Tx - p\| \leq \|Ix - p\|,$$  \hspace{1cm} (1.4)

for all $x \in K$ and $p \in F(T) \cap F(I)$. 
$T$ is called \textit{I-asymptotically quasi-nonexpansive} on $K$, if there exists a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n \to \infty} \lambda_n = 0$,
\begin{equation}
\| T^nx - p \| \leq (1 + \lambda_n^i) \| T^nx - p \|,
\end{equation}
for all $x \in K$ and $p \in F(T) \cap F(I)$ and $n = 1, 2, \ldots$.

Let $E$ be a real Banach space. A subset $K$ of $E$ is said to be a \textit{retract} of $E$, if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be \textit{retraction}, if $P^2 = P$. It follows that if a map $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$.

Recall that a Banach space $E$ is said to satisfy Opial’s condition [12], if for each sequence $\{x_n\}$ in $E$, the condition $x_n \rightarrow x$ implies that
\begin{equation}
\lim\inf_{n \rightarrow \infty} \| x_n - x \| < \lim\inf_{n \rightarrow \infty} \| x_n - y \|,
\end{equation}
for all $y \in E$ with $y \neq x$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [1] and Dotson [2] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces, which we adapt to a normed space as follows: $T$ is called a \textit{quasi-nonexpansive mapping} provided that
\begin{equation}
\| Tx - p \| \leq \| x - p \|,
\end{equation}
for all $x \in K$ and $p \in F(T)$.

\textbf{Remark 1.1.} There are many results of fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps were studied by Petryshin and Williamson [13]. Their analysis was related to the convergence of Mann iterates studied by Dotson [2]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in
Banach spaces were discussed by Ghosh and Debnath [3]. In [19], the weakly convergence theorem for \( I \)-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [20], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In [15], Rhoades and Temir considered \( T \) and \( I \) self-mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mapping and \( K \) is a nonempty closed convex subset of a uniformly convex Banach space. They established the weak convergence of the sequence of Mann iterates to a common fixed point of \( T \) and \( I \). However, if the domain \( K \) of \( T \) is a proper subset of \( E \) and \( T \) maps \( K \) into \( E \), then the Mann iteration in [9] may fail to be well defined. One method that has been used to overcome this in the case of single operator \( T \) is to introduce a retraction \( P : E \to K \) in the recursion formula as follows: \( u_1 \in K \),

\[
    u_{n+1} = (1 - \alpha_n)u_n + \alpha_n PTu_n, \quad n \geq 1.
\]

In [6], Kiziltunc and Ozdemir considered \( T \) and \( I \) nonself-mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mapping. They established the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of \( T \) and \( I \). In [7], Kiziltunc and Yildirim considered \( T \) and \( I \) nonself-mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mapping. They established the weak convergence of the sequence of modified multi-step iterative scheme \( \{x_n\} \) defined by, arbitrary fixed order \( p \geq 2 \),

\[
    x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n Ty_n^1),
\]

\[
    y_n^i = P((1 - \beta_n^i)x_n + \beta_n^i Ty_{n+1}^i), \quad i = 1, 2, \ldots, p - 2,
\]

\[
    y_n^{p-1} = P((1 - \beta_n^{p-1})x_n + \beta_n^{p-1} Tx_n), \quad (1.8)
\]

where, for all \( n \in N \),

\[
    \{\alpha_n\} \subset (0, 1), \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,
\]
and for all \( n \in N \),
\[
\{ \beta_n^i \} \subset [0, 1), \quad 1 \leq i \leq p - 1, \quad \lim_{n \to \infty} \beta^i_n = 0.
\]

Remark 1.2. Clearly, if \( T \) is a self-map, then (1.8) reduces to an iterative scheme (1.1).

In [10], Nantadilok established the weak convergence theorem of sequence of modified multi-step iterative scheme \( \{ x_n \} \) defined by (1.8) for \( I \)-quasi-nonexpansive nonself mapping \( T \), where \( I \) is a quasi-nonexpansive nonself mapping, \( T \) and \( I \) are nonself mappings of \( K \).

2. Preliminaries

Let \( E \) be a normed linear space, \( T \) be self-mapping on \( E \). Let \( \{ x_n \} \) be the sequence of the Ishikawa iterative scheme [4] associated with \( T \), \( x_0 \in E \),
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nTx_n.
\end{align*}
\]
for all \( n \in N \), where \( 0 \leq \{ \alpha_n \}, \{ \beta_n \} \leq 1 \).

Let \( S, T : K \to K \) be two mappings. In 2006, Lan [8] introduced the following iterative scheme with errors. The sequence in \( K \) defined by
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nS^n y_n + \psi_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nT^n x_n + \phi_n,
\end{align*}
\]
for all \( n \in N \), where \( 0 \leq \{ \alpha_n \}, \{ \beta_n \} \leq 1 \), and \( \{ \psi_n \}, \{ \phi_n \} \) are two sequences in \( K \).

Define the Ishikawa iterative process for generalized \( I \)-asymptotically quasi-nonexpansive nonself mapping in uniformly convex Banach space \( E \) as follows:
\[ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_nI^n'y_n), \]
\[ y_n = P((1 - \beta_n)x_n + \beta_nT^n'x_n), \quad (2.3) \]
for all \( n \in N \), where \( 0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1 \).

**Definition 2.1** [17]. Let \( E \) be a real normed linear space and \( K \) be a nonempty subset of \( E \). A mapping \( T: K \to K \) is called *generalized asymptotically quasi-noexpansive mapping*, if \( F(T) \neq \emptyset \) and there exist sequences of real numbers \( \{u_n\}, \{\varphi_n\} \) with \( \lim_{n \to \infty} u_n = 0 = \lim_{n \to \infty} \varphi_n \), such that
\[ \| T^n'x - p \| \leq \| x - p \| + u_n\| x - p \| + \varphi_n, \]
for all \( x \in K \), \( p \in F(T) \), and \( n \geq 1 \).

If, in Definition 2.1, \( \varphi_n = 0 \) for all \( n \geq 1 \), then \( T \) becomes asymptotically quasi-nonexpansive mapping and hence the class of generalized asymptotically quasi-noexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings.

**Definition 2.2.** Let \( E \) be a real normed linear space and \( K \) be a nonempty subset of \( E \). A mapping \( T: K \to K \) is called *generalized I-asymptotically quasi-noexpansive mapping*, if \( F(T) \cap F(I) \neq \emptyset \) and there exist sequences of real numbers \( \{u_n\}, \{\varphi_n\} \) with \( \lim_{n \to \infty} u_n = 0 = \lim_{n \to \infty} \varphi_n \), such that
\[ \| T^n'x - p \| \leq \| I^n'x - p \| + u_n\| I^n'x - p \| + \varphi_n, \]
for all \( x \in K \), \( p \in F(T) \), and \( n \geq 1 \).

If, in Definition 2.2, \( \varphi_n = 0 \) for all \( n \geq 1 \), then \( T \) becomes I-asymptotically quasi-nonexpansive mapping and hence the class of generalized I-asymptotically quasi-noexpansive mappings includes the class of I-asymptotically quasi-nonexpansive mappings.
Lemma 2.1 [18]. Let \( \{a_n\}, \{b_n\}, \) and \( \{\sigma_n\} \) be sequences of nonnegative real numbers satisfying the following inequality:
\[
a_{n+1} \leq (1 + \sigma_n)a_n + b_n, \quad \forall n \geq 1.
\]
If \( \sum_{n=1}^{\infty} b_n < \infty \) and \( \sum_{n=1}^{\infty} \sigma_n < \infty \), then
\begin{enumerate}
\item \( \lim_{n \to \infty} a_n \) exists.
\item \( \lim_{n \to \infty} a_n = 0 \), if \( \{a_n\} \) has a subsequence converging to zero.
\end{enumerate}

Lemma 2.2. Let \( E \) be a uniformly convex Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T, I \) be nonself mappings of \( K \) with \( P \) a nonexpansive retraction, where \( T \) is a generalized I-asymptotically quasi-nonexpansive nonself mapping with \( \{u_n\}, \{\phi_n\} \subset [0, \infty] \) such that
\[
\sum_{n=1}^{\infty} u_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \phi_n < \infty \quad \text{and} \quad I \text{ is an asymptotically quasi-nonexpansive nonself mapping of } K \text{ with } \{v_n\} \subset [0, \infty] \text{ such that}
\]
\[
\sum_{n=1}^{\infty} v_n < \infty. \] Let \( \{x_n\} \) be the sequence defined by (2.3) with \( F = F(T) \cap F(I) \neq \emptyset \). Then \( \lim_{n \to \infty} \|x_n - p\| \) exists for any fixed point \( p \) of \( T \) and \( I \).

Proof. For any \( p \in F(T) \cap F(I) \).
\[
\|x_{n+1} - p\| = \|P((1 - \alpha_n)x_n + \alpha_n I^n y_n) - P(p)\|
\leq \|(1 - \alpha_n)x_n + \alpha_n I^n y_n - p\|
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|I^n y_n - p\|
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(1 + v_n)\|y_n - p\|,
\]
and
\[
\|y_n - p\| = \|P((1 - \beta_n)x_n + \beta_n T^n x_n) - P(p)\|
\leq \|(1 - \beta_n)x_n + \beta_n T^n x_n - p\|
\]
\[
\leq (1 - \beta_n) \| x_n - p \| + \beta_n \| T^n x_n - p \|
\]

\[
\leq (1 - \beta_n) \| x_n - p \| + \beta_n [(1 + u_n) \| I^n x_n - p \| + \varphi_n]
\]

\[
\leq (1 - \beta_n) \| x_n - p \| + \beta_n [(1 + u_n)(1 + v_n) \| x_n - p \| + \varphi_n]
\]

\[
= (1 + \beta_n u_n + \beta_n v_n + \beta_n u_n v_n) \| x_n - p \| + \beta_n \varphi_n. \quad (2.5)
\]

From (2.4) and (2.5), we get

\[
\| x_{n+1} - p \| \leq \left[ 1 + \alpha_n \beta_n (u_n + v_n) + 2\alpha_n \beta_n u_n v_n + \alpha_n v_n \right.
\]

\[
+ \alpha_n \beta_n v_n^2 + \alpha_n \beta_n u_n^2 \| x_n - p \|
\]

\[
+ \beta_n \alpha_n (1 + v_n) \varphi_n. \quad (2.6)
\]

We can rewrite (2.6) as follows:

\[
\| x_{n+1} - p \| \leq (1 + \gamma_n) \| x_n - p \| + \psi_n,
\]

where

\[
\gamma_n = \alpha_n \beta_n (u_n + v_n) + 2\alpha_n \beta_n u_n v_n + \alpha_n v_n + \alpha_n \beta_n v_n^2 + \alpha_n \beta_n u_n^2
\]

with \( \sum_{n=1}^{\infty} \gamma_n < \infty \), and \( \psi_n = \beta_n \alpha_n (1 + v_n) \varphi_n \) with \( \sum_{n=1}^{\infty} \psi_n < \infty \).

By Lemma 2.1, \( \lim_{n \to \infty} \| x_n - p \| \) exists for each \( p \in F(T) \cap F(I) \).

\[ \square \]

**Lemma 2.3.** Let \( E \) be a uniformly convex Banach space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T, I \) be nonself mappings of \( K \), where \( T \) is a generalized \( I \)-asymptotically quasi-nonexpansive nonself mapping, \( I \) is an asymptotically quasi-nonexpansive nonself mapping of \( K \), with \( F = F(T) \cap F(I) \neq \emptyset \). Then \( F = F(T) \cap F(I) \) is closed.

**Proof.** Let \( \{ p_k \} \) be a sequence in \( F \) such that \( p_k \to p \) as \( k \to \infty \). Since \( K \) is closed and \( \{ p_k \} \) is a sequence in \( K \), we must have \( p \in K \). Since \( T : K \to E \) is a generalized \( I \)-asymptotically quasi-nonexpansive nonself mapping, we obtain
\[ \| T^n p_k - p \| \leq (1 + u_k) \| I^n p_k - p \| + \psi_k \]
\[ \leq (1 + u_k)(1 + v_k) \| p_k - p \| + \psi_k, \]
(2.7)
for all \( p_k \in K, p \in F = F(T) \cap F(I), \) and \( n \geq 1. \)

Taking limit both sides of (2.7), we get
\[ \lim_{k \to \infty} \| T^n p_k - p \| \leq 0, \quad n \geq 1, \]
which implies that
\[ \lim_{k \to \infty} \| T^n p_k - p \| = 0, \quad n \geq 1. \]

Therefore, we get \( \| T^n p - p \| = 0. \) In particular, we have \( \| Tp - p \| = 0. \)
Thus \( p \in F = F(T) \cap F(I). \) This completes our proof. \( \square \)

In this paper, we consider \( T \) and \( I \) nonself mappings of \( K, \) where \( T \) is a generalized \( I \)-asymptotically quasi-nonexpansive nonself mapping, and \( I \) is an asymptotically quasi-nonexpansive nonself mapping, a more general class of mappings than those mentioned in the existing literatures. We establish weak and strong convergence theorem of modified Ishikawa iterative scheme \( \{ x_n \} \) defined by (2.3) for generalized \( I \)-asymptotically quasi-nonexpansive nonself mapping.

3. Main Results

**Theorem 3.1.** Let \( K \) be a nonempty closed convex subset of uniformly convex Banach space \( E, \) which satisfies Opial’s condition, and let \( T, I \) be nonself mappings of \( K. \) Let \( T \) and \( I \) be the same as Lemma 2.2. If \( F = F(T) \cap F(I) \neq \emptyset, \) then the sequence \( \{ x_n \} \) defined by (2.3) converges weakly to a common fixed point \( p \) of \( F(T) \cap F(I). \)

**Proof.** Let \( p \in F(T) \cap F(I). \) Then, as in Lemma 2.2, it follows that \( \lim_{n \to \infty} \| x_n - p \| \) exists and so for \( n \geq 1, \) the sequence \( \{ x_n \} \) is bounded on \( K. \) Then by the reflexivity of \( E \) and the boundedness of \( \{ x_n \}, \) there
exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to p \) weakly. If \( p \in F(T) \cap F(I) \) is nonempty and a singleton, then the proof is complete.

We will assume that \( F(T) \cap F(I) \) is not a singleton. We show that \( \{x_n\} \) converges weakly to a common fixed point of \( T \) and \( I \). Let \( \{x_{n_k}\} \) and \( \{x_{n_j}\} \) be two subsequences of \( \{x_n\} \), which converge weakly to \( p \) and \( q \), respectively. We show that \( p = q \). Suppose that \( E \) satisfies Opial’s condition and that \( p \neq q \) belong to weak limit set of the sequence \( \{x_n\} \).

Then \( x_{n_k} \to p \) and \( x_{n_j} \to q \), respectively. Since \( \lim_{n \to \infty} \|x_n - p\| \) exists for any \( p \in F(T) \cap F(I) \), by Opial’s condition, we conclude that

\[
\lim_{n \to \infty} \|x_n - p\| = \lim_{k \to \infty} \|x_{n_k} - p\| < \lim_{k \to \infty} \|x_{n_k} - q\| = \lim_{j \to \infty} \|x_{n_j} - q\| < \lim_{j \to \infty} \|x_{n_j} - p\| = \lim_{n \to \infty} \|x_n - p\|.
\]

This is a contradiction. Thus \( \{x_n\} \) converges weakly to a common fixed point of \( F(T) \cap F(I) \).

\( \square \)

**Theorem 3.2.** Let \( E \) be a real Banach space and \( K \) be a nonempty closed convex nonexpansive retract of \( E \) with \( P \) a nonexpansive retraction. Let \( T : K \to E \) be a generalized \( I \)-asymptotically quasi-nonexpansive nonself mapping with \( \{u_n\}, \{\varphi_n\} \subset [0, \infty) \) such that \( \sum_{n=1}^{\infty} u_n < \infty \) and \( \sum_{n=1}^{\infty} \varphi_n < \infty \), and \( I \) be an asymptotically quasi-nonexpansive nonself mapping of \( K \) with \( \{v_n\} \subset [0, \infty) \) such that \( \sum_{n=1}^{\infty} v_n < \infty \). Let \( \{x_n\} \) be the sequence defined by (2.3) with \( F = F(T) \cap F(I) \neq \emptyset \). Then \( \{x_n\} \) converges strongly to a common fixed point of \( T \) and \( I \), if and only if
where \(d(x, F) = \inf \{\|x - p\| : p \in F = F(T) \cap F(I)\}\).

**Proof.** The necessity is obvious, so it is omitted. We now prove the sufficiency. Let \(p \in F = F(T) \cap F(I)\). Furthermore, from Lemma 2.2, we obtain

\[
\|x_{n+1} - p\| \leq (1 + \gamma_n)\|x_n - p\| + \psi_n,
\]

where

\[
\gamma_n = \alpha_n \beta_n (u_n + v_n) + 2 \alpha_n \beta_n u_n v_n + \alpha_n v_n + \alpha_n \beta_n v_n^2 + \alpha_n \beta_n u_n v_n^2
\]

with \(\sum_{n=1}^{\infty} \gamma_n < \infty\), and \(\psi_n = \beta_n \alpha_n (1 + v_n) \phi_n\) with \(\sum_{n=1}^{\infty} \psi_n < \infty\). By Lemma 2.1, \(\lim_{n \to \infty} \|x_n - p\|\) exists for each \(p \in F(T) \cap F(I)\).

By (3.1), we get

\[
|d(x_{n+1}, F)| \leq (1 + \gamma_n) d(x_n, F) + \psi_n.
\]

Then, by Lemma 2.1, \(\lim_{n \to \infty} d(x_n, F)\) exists.

Next, we show that \(\{x_n\}\) is a Cauchy sequence in \(E\). In fact, \(\sum_{n=1}^{\infty} \gamma_n < \infty\), and \(1 + x \leq e^x\), for all \(x > 0\). From (3.1), for \(m, n \geq 1\) and \(p \in F(T) \cap F(I)\), we have

\[
\|x_{n+m} - p\| \leq (1 + \gamma_{n+m-1}) \|x_{n+m-1} - p\| + \psi_{n+m-1}
\]

\[
\leq (1 + \gamma_{n+m-1}) (1 + \gamma_{n+m-2}) \|x_{n+m-2} - p\|
\]

\[
+ (1 + \gamma_{n+m-1}) \psi_{n+m-1}
\]

\[
\leq (1 + \gamma_{n+m-1}) (1 + \gamma_{n+m-2}) \|x_{n+m-2} - p\|
\]

\[
+ (1 + \gamma_{n+m-1}) [\psi_{n+m-1} + \psi_{n+m-2}]
\]

\[
\leq \exp(\gamma_{n+m-1} + \gamma_{n+m-2}) \|x_{n+m-2} - p\|
\]

\[
+ \exp(\gamma_{n+m-1}) [\psi_{n+m-1} + \psi_{n+m-2}]
\]

\[
\vdots
\]
\[
\leq \exp \left( \sum_{i=n}^{n+m-1} \gamma_i \right) \| x_n - p \| + \exp \left( \sum_{i=n}^{n+m-1} \sum_{i=n}^{n+m-1} \gamma_i \right) \sum_{i=n}^{n+m-1} \psi_i
\]

\[
\leq M \| x_n - p \| + M \sum_{i=n}^{n+m-1} \psi_i, \quad (3.2)
\]

where \( M = \exp \left( \sum_{i=n}^{\infty} \gamma_i \right) < \infty. \)

The assumption \( \lim_{n \to \infty} d(x_n, F) = 0 \) implies that there exists a sequence \( \{d(x_n, F)\} \) converging to zero. Therefore, by Lemma 2.1, we get

\[
\lim_{n \to \infty} d(x_n, F) = 0.
\]

Let \( \epsilon > 0 \), there exists a positive number \( n_0 \) such that for all \( n \geq n_0 \),

\[
d(x_n, F) < \frac{\epsilon}{6M} \quad \text{and} \quad \sum_{i=n}^{\infty} \psi_i < \frac{\epsilon}{3M}. \]

From the first inequality, there exists \( p_0 \in F = F(T) \cap F(I) \) such that \( \| x_{n_0} - p_0 \| < \frac{\epsilon}{6M} \). Hence, for all \( n \geq n_0 \) and \( m \geq 1 \), and from (3.2), we have

\[
\| x_{n+m} - x_n \| \leq \| x_{n+m} - p_0 \| + \| x_n - p_0 \|
\]

\[
\leq M \| x_{n_0} - p_0 \| + M \left( \sum_{i=n}^{n+m-1} \psi_i \right)
\]

\[
+ M \| x_{n_0} - p_0 \| + M \left( \sum_{i=n_0}^{n-1} \psi_i \right)
\]

\[
\leq 2M \| x_{n_0} - p_0 \| + M \left( \sum_{i=n}^{\infty} \psi_i \right) + M \left( \sum_{i=n_0}^{\infty} \psi_i \right)
\]

\[
\leq 2M \frac{\epsilon}{6M} + M \frac{\epsilon}{3M} + M \frac{\epsilon}{3M} = \epsilon.
\]
This means that \( \{x_n\} \) is a Cauchy sequence in \( E \). Since \( E \) is a Banach space, which is complete, implies that \( \{x_n\} \) is convergent. Assume that \( x_n \to p \in E \). Since \( K \) is closed and \( \{x_n\} \) is a sequence in \( K \) converging to \( p \), we have that \( p \in K \). Therefore, by Lemma 2.3, \( F = F(T) \cap F(I) \) is closed. Now, \( \lim_{n \to \infty} d(x_n, F) = 0 \) and \( x_n \to p \) as \( n \to \infty \), the continuity of \( d(x, F) \) implies that \( d(p, F) = 0 \). Then \( p \in F \). This completes our proof.

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References


