GENERALIZED VECTOR IMPLICIT VARIATIONAL INEQUALITY PROBLEM WITH APPLICATION TO BEST APPROXIMATION

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Abstract

We apply the KKM technique to study generalized vector implicit variational inequality problem. Our results generalize the works of Huang and Li [8], Li and Huang [11], and represent variant of result of Lee and Farajzadeh [10]. We also obtain a version of best approximations theorem in cone metric spaces.

1. Introduction and Preliminaries

The theory of variational inequalities is a very effective technique for studying a wide class of problems arising in different branches of pure and applied sciences (see Allen [2], Aubin and Ekeland [3], Gianessi [7], Baiocchi and Capelo [4], Blum and Oettli [5], Yuan [15], etc.).

2010 Mathematics Subject Classification: 90C33, 49J40.

Keywords and phrases: vector implicit variational inequality problem, KKM mapping, convex cone, best approximation.

Received September 14, 2010

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In recent years, variational inequalities have been generalized in various directions (see, for example, Agarwal and Verma [1], Noor [13], Wong et al. [14], Lin [12], etc.). In this paper, we introduced the following generalized vector equilibrium (GVEP) problem: Finding \( x_0 \in K \), such that

\[
G(x_0, s(y)) - G(x_0, g(x_0)) \in C(x_0), \text{ for all } y \in K, \quad (1.1)
\]

where \( G : K \times K \to Y \), \( g : K \to K \) are mappings, and \( C : K \to 2^Y \) is a multi-valued mapping with nonempty pointed convex cone values, \( X \) and \( Y \) are topological vector spaces, and \( K \) is a nonempty convex subset of \( X \).

(1) Let \( Y \) be an ordered Banach space induced by a pointed closed convex cone \( P \), \( L(X, Y) \) be the space of all continuous linear mappings from \( X \) into \( Y \), and \( \langle t, x \rangle \) be the value of the linear continuous mapping \( t \in L(X, Y) \) at \( x \), \( f : K \to L(X, Y) \), \( g : K \to K \), \( F : K \to Y \), \( s(x) = x \), \( C(x) = P \) for each \( x \in K \), and

\[
G(x, y) = \langle f(x), y \rangle + F(y), \text{ for all } x, y \in K,
\]

then problem (1.1) reduces to vector \( F \)-implicit variational inequality problem: Finding \( x_0 \in K \), such that

\[
\langle f(x_0), y - g(x_0) + F(y) - F(g(x_0)) \rangle \geq 0, \text{ for all } y \in K, \quad (1.2)
\]

which was introduced by Li and Huang [11].

(2) Let \( X \) be a real Banach space with dual \( X^\ast \) and \( K \) be a nonempty closed convex. Let \( f : K \to X^\ast \), \( g : K \to K \), and \( F : K \to \mathbb{R} \) be a function. If \( s(x) = x \), \( C(x) = [0, +\infty) \) for each \( x \in K \), and

\[
G(x, y) = \langle f(x), y \rangle + F(y), \text{ for all } x, y \in K,
\]

then (1.1) reduces to the \( F \)-implicit variational inequality problem: Finding \( x_0 \in K \), such that
\[ \langle f(x_0), y - g(x_0) \rangle \geq F(g(x_0)) - F(y), \text{ for all } y \in K, \quad (1.3) \]

which has been studied by Huang and Li [8].

The aim of this paper is to obtain the results of existence a solution of GVEP problem (1.1) by using the KKM technique.

**Theorem 1.1** [6]. Let \( Y \) be a convex set in a topological vector space \( X \) and \( K \) be a nonempty subset of \( Y \). Let \( G : K \to 2^Y \) be a KKM mapping with relatively closed values. If there is a nonempty subset \( K_0 \) of \( K \) such that the \( \bigcap_{x \in K_0} G(x) \) is compact and \( K_0 \) is contained in a compact convex subset of \( Y \), then \( \bigcap_{x \in K} G(x) \neq \emptyset \).

**Remark 1.1.** Let \( X \) be a topological vector space and \( K \) be a nonempty subset of \( X \). A mapping \( G : K \to 2^X \) is called a KKM mapping, if

\[
\text{co} \{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i),
\]

for each finite subset \( \{x_1, \ldots, x_n\} \) of \( K \).

2. Main Result

**Theorem 2.1.** Let \( X \) and \( Y \) be topological vector spaces and \( K \) be a nonempty convex subset of \( X \). Let \( F : K \times K \to Y \), \( g, s : K \to K \) be mappings, and \( C : K \to 2^Y \) be a multi-valued mapping with nonempty pointed convex cone values. Assume that:

1. the set \( \{x \in K : F(x, s(y)) - F(x, g(x)) \in C(x)\} \) is relatively closed in \( K \), for all \( y \in K \);

2. there exists a mapping \( H : K \times K \to Y \) such that
   
   (a) \( x \notin \text{co} \{y \in K : H(x, y) \notin C(x)\} \), for all \( x \in K \);
   
   (b) \( F(x, s(y)) - F(x, g(x)) - H(x, y) \in C(x) \), for all \( x, y \in K \);
(3) there exist a nonempty compact subset $B$ and a nonempty convex compact subset $D$ of $K$ such that, for each $x \in K \setminus B$, there exists $y \in D$ such that

$$F(x, s(y)) - F(x, g(x)) \in C(x),$$

then there exists $x_0 \in K$ such that

$$F(x_0, s(y)) - F(x_0, g(x_0)) \in C(x_0), \text{ for all } y \in K.$$

**Proof.** We define $G_1, G_2 : K \to 2^K$ by

$$G_1(y) = \{ x \in K : H(x, y) \in C(x) \},$$

$$G_2(y) = \{ x \in K : F(x, s(y)) - F(x, g(x)) \in C(x) \}.$$

From condition (a) of (2), we obtain $x \notin \{ y \in K : H(x, y) \notin C(x) \}$, so $H(x, x) \in C(x)$ and $G_1(y)$ is nonempty set for all $y \in K$.

From condition (1), we have that $G_2(y)$ is relatively closed for all $y \in K$.

We prove that $G_1$ is KKM mapping. Suppose that there exists a finite subset $\{y_1, \ldots, y_n\}$ of $K$ and $\lambda_i \geq 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$y_0 = \sum_{i=1}^{n} \lambda_i y_i \notin \bigcup_{i=1}^{n} G_1(y_i).$$

Then

$$H(y_0, y_i) \notin C(y_0), i = 1, \ldots, n,$$

and

$$y_i \in \{ y \in K : H(y_0, y) \notin C(y_0), i = 1, \ldots, n.\}$$

So,

$$y_0 \in \text{co} \{ y \in K : H(y_0, y) \notin C(y_0) \},$$
which is a contradiction to assumption (a) of (2). Hence $G_1$ is a KKM mapping. From condition (b) of (2), we have $G_1(y) \subseteq G_2(y)$ for all $y \in K$. This implies that $G_2$ is a KKM mapping.

From condition (3), we obtain $\bigcap_{y \in D} G_2(y) \subseteq B$. Therefore, $G_2$ satisfy the conditions of Theorem 1.1, so,

$$\bigcap_{y \in K} G_2(y) \neq \emptyset.$$

Let $x_0 \in \bigcap_{y \in K} G_2(y)$, then we have

$$F(x_0, s(y)) - F(x_0, g(x_0)) \in C(x_0), \text{ for all } y \in K.$$

\[\square\]

**Remark 2.1.** (1) Observe that condition (a) of (2) is automatically fulfilled, if $H(x, x) \in C(x)$ and the set $\{y \in K : H(x, y) \notin C(x)\}$ is convex for all $x \in K$.

(2) Let $X$ be a real Banach space with dual space $X^*$, $K$ be a nonempty closed convex cone of $X$. Let $f : K \to X^*$, $g : K \to K$, $h : K \times K \to \mathbb{R}$, and $F : K \to \mathbb{R}$ be a function. If we put

$$F(x, y) = \langle f(x), y - g(x) + F(g(x)) - F(y) \rangle,$$

$s(x) = x$ and $C(x) = [0, +\infty)$ for each $x \in K$, Theorem 2.1 reduces to result of Huang and Li [8] (Theorem 3.2).

(3) Note that, if in Theorem 2.1, $Y$ is an ordered Banach space induced by a pointed closed convex cone,

$$F(x, y) = \langle f(x), y - g(x) + F(g(x)) - F(y) \rangle,$$

$s(x) = x$ and $C(x) = P$ for each $x \in K$, we obtain the result of Li and Huang [11] (Theorem 3.2).
(4) Observe that, by Theorem 2.1, we can obtain the result of Lee and Farajzadeh [10] (Theorem 2.2) and Fan [6] (Theorem 6). Also, from Theorem 2.1, we obtain the result of Allen [2].

3. Applications to Best Approximations in Cone Metric Space

In this section, from Theorem 2.1, we obtain the best approximation theorem in cone metric spaces.

Let $E$ be a vector space and $P$ is pointed closed convex cone in $E$. We define partial ordering $\leq$ with respect to $P$ by $x \leq y$, if and only if $y - x \notin P$, write $x \not\leq y$.

**Definition 3.1** [9]. Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(d$_1$) $0 < d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;

(d$_2$) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d$_3$) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

**Example 3.1** [9]. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Theorem 3.1.** Let $(Y, d)$ be cone metric space and $K$ be a nonempty convex subset of $Y$. Let $f : K \to Y$, $g, s : K \to K$ be mappings. Assume that:

1. the set $\{x \in K : d(f(x), g(x)) \leq d(f(x), s(y))\}$ is relatively closed in $K$, for all $y \in K$;

2. $x \notin \text{co}\{y \in K : d(f(x), g(x)) \leq d(f(x), s(y))\}$, for all $x \in K$.
(3) there exist a nonempty compact subset $B$ and a nonempty convex compact subset $D$ of $K$ such that, for each $x \in K \setminus B$, there exists $y \in D$ such that

$$d(f(x), g(x)) \preceq d(f(x), s(y)),$$

then there exists $x_0 \in K$ such that

$$d(f(x_0), g(x_0)) \preceq d(f(x_0), s(y)), \text{ for all } y \in K.$$

In particular, if $f(K) \subseteq s(K)$, then $f(x_0) = g(x_0)$.

**Proof.** Put

$$F(x, y) = d(f(x), y), \text{ for } x, y \in K,$$

$$H(x, y) = d(f(x), s(y)) - d(f(x), g(x)), \text{ for all } x, y \in K,$$

and $C(x) = P$ for each $x \in K$. Then $F$ and $H$ satisfy all of the requirements of Theorem 2.1. Therefore, there exists $x_0 \in K$ such that

$$F(x_0, g(x_0)) \preceq F(x_0, s(y)), \text{ for all } y \in K,$$

i.e.,

$$d(f(x_0), g(x_0)) \preceq d(f(x_0), s(y)), \text{ for all } y \in K.$$

$\square$

**References**


