Journal of Mathematical Sciences: Advances and Applications
Volume 4, Number 2, 2010, Pages 395-411

OPTIMAL CONTROL OF A PRODUCTION INVENTORY SYSTEM WITH GENERALIZED EXPONENTIAL DISTRIBUTED DETERIORATION

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Abstract

This paper is concerned with the optimal control of a production inventory system with time-varying deteriorating items. It is assumed that the deterioration rate follows the two parameters generalized exponential distribution. The continuous review policy is investigated. The optimality conditions are derived using Pontryagin maximum principle. The optimal total cost, the optimal inventory level, and the optimal production are derived. The sensitivity analysis of the system will be analyzed against the monetary and non-monetary parameters. Extensive numerical simulation study will be presented.

1. Introduction

In the recent years, the problem of inventory models with deteriorating items has received a great attention. The deterioration process can be defined as change, damage, decay, evaporation, pilferage,

2010 Mathematics Subject Classification: 49J15.

Keywords and phrases: Optimal control, inventory system, optimal production, momentary and non-momentary parameters.

This research was supported by the College of Science Research Center at King Saud University under project No. Stat/2009/xx.

Received December 7, 2009

spoilage, obsolescence, and loss of utility or loss of marginal value of a commodity that results in decreasing usefulness from the original one. Products such as vegetables, fish, medicine, blood, alcohol, gasoline, and radioactive chemicals have finite shelf life, and start to deteriorate once they are produced. Most researches in deteriorating inventory assumed constant rate of deterioration. In fact, the time-varying deteriorating item is more realistic than constant deteriorating item systems. However, the Weibull distribution is used to represent the product in stock deteriorates with time. The deterioration rate increases with age.

Maintenance of inventory systems with deteriorating items is a problem of major concern in the supply chain of almost any business organizations. Many of the physical goods undergo decay or deterioration over time. Commodities such as fruits, vegetables, foodstuffs are subject to direct deterioration while kept in store.

Also, highly volatile liquids such as gasoline, alcohol, turpentine, undergo physical depletion over time through the process of evaporation. Electronic goods, radioactive substances, photographic film, grain, deteriorate through a gradual loss of potential or utility with the passage of time. A model with exponentially decaying inventory was initially proposed by Ghare and Schrader [9]. Covert and Philip [3] have developed an economic order quantity model with Weibull and Gamma distributed deterioration rates, respectively. El-Gohary and El-Syed [7] have discussed the problem of optimal control of multi-items inventory system with different type of deteriorations. Tadj et al. [19] have discussed the problem of optimal control of an inventory system with ameliorating and deteriorating items. Most of the previous works on inventory models did not consider simultaneously the above-mentioned factors. This is not true in real life since the above factors are significant. The main contribution of our study is to consider the factors such as generalized exponential distribution, multiple delivery, and limited resources simultaneously in a model. To the best of our knowledge, this type of model has not been developed before. Our objective is to minimize the total cost per unit time.

Since then, several researchers have studied deteriorating inventory models with time varying demand under a variety of modelling assumptions see, for example, the references [4, 5, 12, 21]. Two excellent surveys on recent trends in modelling of continuously deteriorating inventory are those by Raafat [15] and Goyal and Giri [12].

We consider in this paper, a production inventory system that produces a single product at a certain production rate and seeks an alternative production rate. The system is dynamic behavior by nature and an optimal control approach seems particularly well suited to achieve its optimization.

One fascinating aspect of optimal control theory is its wide range of applications. It has found successful applications in various areas of management science and operationals research, see Sethi and Thompson [16] and Sulem [18]. We are especially interested in applications of optimal control theory to inventory theory. Among the most recent literature on the subject, one might cite Tadj et al. [19] and El-Gohary and El-Syed [7].

In the inventory system, we are studying here, the product is supposed to deteriorate while in stock. Items deterioration is a matter of prime importance in inventory theory, and this feature has received a great deal of attention from researchers, as shown by the surveys of Raafat [15], and Goyal and Giri [12].

In the recent years, interest in optimal control systems has increased rapidly along with interest and progress in control topics. The optimal control has a variety of specific meanings, but it often implies that the system is capable of accommodating unpredictable environmental changes, whether these changes arise within the system or external to it [6, 13].

Besides adjusting its production rate, we also assume that the firm has set an inventory goal level and penalties are incurred for the inventory level to deviate from its goal, and for the desired production rate to deviate from the actual production rate.

The problem of this paper can be considered as an optimal control problem with one control variable and one state variable, and a closed form solution is obtained by using the maximum principle.

The rest of the paper is organized as follows. Following this introduction, the model development is presented in Section 2. In Section 3, we present the mathematical model of the inventory system with generalized exponential deteriorating items. The control problem of this system is presented in Section 4. In Section 5, we present some illustrative examples and discuss the sensitivity of our model against the system parameters.

2. Model Development

The problem of inventory system with time-varying deteriorating item is very important, since the time varying deteriorating item is more realistic than constant deteriorating items. In this section, we discuss the development of an inventory system, which consists of one item that deteriorates while in the store. The novelty we will take into account in this paper is that, the time to deterioration is a random variable following two-parameters generalized exponential distribution. The probability density function for two-parameters generalized exponential distribution is given by

$$f(t) = \alpha\beta(1 - e^{-\alpha t})^{\beta-1} e^{-\alpha t}, \ t > 0, \tag{2.1}$$

where the parameters α and β are positive real constants. The probability distribution function is

$$F(t) = (1 - e^{-\alpha t})^{\beta}, \ t > 0. \tag{2.2}$$

The instantaneous deterioration rate $\theta(t)$ as a function of time is given by

$$\theta(t) = \frac{f(t)}{1 - F(t)} = \frac{\alpha\beta(1 - e^{-\alpha t})^{\beta-1} e^{-\alpha t}}{1 - (1 - e^{-\alpha t})^{\beta}}, \ t > 0. \tag{2.3}$$

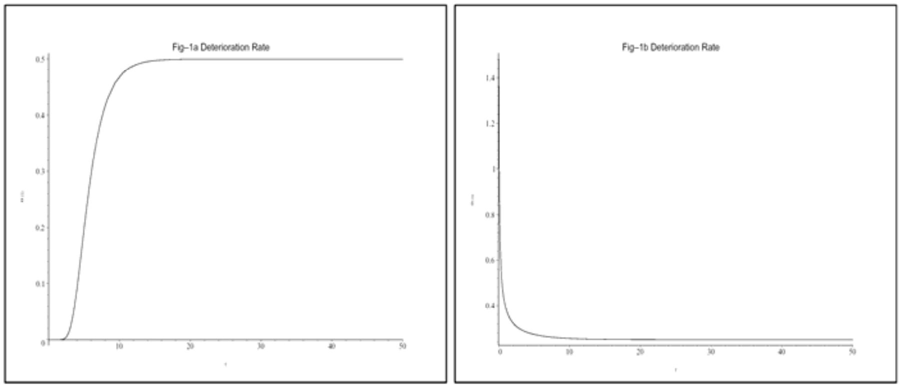

Figure 1. (a) The deterioration rate for $\alpha = 0.25$ and $\beta = 2$, (b) The deterioration rate for $\alpha = 0.25$ and $\beta = 0.5$. From graphical representtation of the deterioration rate, we observe that, the deterioration rate increased for the set of values $\beta > 1$ and decreased for the set of values $\beta < 1$.

To build the mathematical model for the production inventory system with generalized exponential distributed deterioration, we will assume that the demand rate is a general function of time. We will also assume that, the firm has set an inventory goal level and a production goal rate. The main objective of this study is to derive the optimal production rate that will keep the inventory level and production rate as close as possible to inventory goal level and production goal rate, respectively. In this research, we will study the continuous review policy. In this case, the necessary and sufficient conditions will be derived by using the Pontryagin maximum principle.

3. Mathematical Model and Notation

Assume that $T > 0$ represents the length of the planing horizon, and consider a firm manufactures a certain product, selling some and stocking the rest in warehouse. The firm has set an inventory goal level $\bar{I}$ and a penalty $h \geq o$ is incurred for inventory level to deviate from its goal. At any instant of time $t \in [0, T]$, the firm manufactures units of the product are a rate $P(t)$ at instant t. It has set a production goal rate $\overline{P}$ and a penalty $c > 0$ is incurred for the production rate to deviate from its goal. The production of new units at rate $P(t)$ increases the inventory level, while the demand from the product at rate $S(t)$ and deterioration at rate $\theta(t)$ decreases the inventory level.

The change in the inventory level in stock is therefore given by the state equation

$$\dot{I}(t) = P(t) - S(t) - \frac{f(t)}{1 - F(t)} I(t), \ t \in [0, T]. \tag{3.1}$$

We assume that the initial stock $I(0) = I_0$ is known and the production goal rate $\hat{P}$ can be computed by using the state equation (3.1) as

$$\hat{P}(t) = S(t) + \frac{f(t)}{1 - F(t)} I(t), \ t \in [0, T]. \tag{3.2}$$

To study the optimal control problem of the inventory model with generalized exponential deterioration, we assume that $I(t)$ represents the state variable, while $P(t)$ represents the control variable which needs to be nonnegative:

$$P(t) \geq 0, \ t \in [0, T]. \tag{3.3}$$

Next, we look for the optimal production rate, that is, the rate that minimizes the performance measure

$$2J = \int_0^T \left\{ h[I(t) - \hat{I}]^2 + c[P(t) - \hat{P}(t)]^2 \right\} dt. \tag{3.4}$$

Now to solve this problem, it is well-known that, historically, there have been two basic inventory systems: the first is the continuous-review system and the second is the periodic system. In this paper, we will concerned with the continuous-review system, which needs the optimal inventory level of a company's inventory is monitored at all times and the inventory position is constantly adjusted.

4. Continuous Review System

We assume that the firm adopts a continuous-review system. The necessary optimality conditions are derived by using Pontryagin maximum principle, for more details about this principle see, for example, Sethi and Thompson [1, 14, 16].

We now go through the procedure of applying Pontryagin minimum principle. First, we replace the cost integral (3.1) by an additional state variable $x_0(t)$, which satisfies the state equation

$$2\dot{x}_0(t) = h[I(t) - \hat{I}]^2 + c[P(t) - \hat{P}(t)]^2. \tag{4.1}$$

Next, we introduce the two co-state variable z_s, $(s = 0, 1)$ and so the Hamiltonian function is given by:

$$H = z_0\dot{x}_0 + z_1\dot{I}. \tag{4.2}$$

Substituting from (2.1) and (4.1) into (4.2), we get

$$H = \frac{1}{2} z_0 \left\{h[I(t) - \hat{I}]^2 + c[P(t) - \hat{P}(t)]^2\right\} + z_1\left[P(t) - S(t) - \frac{f(t)}{1 - F(t)} I(t)\right]. \tag{4.3}$$

The Hamilton equations are:

$$\dot{z}_0 = -\frac{\partial H}{\partial x_0} = 0, \quad \dot{z}_1 = -\frac{\partial H}{\partial I}. \tag{4.4}$$

The other pair of Hamilton equations are just the state equations (2.1) and (3.2). Without loss of generality, we can choose $z_0 = -1$. Substituting (4.3) into (4.4), the co-state equations can be derived in the form:

$$\dot{z}_0 = 0, \quad \dot{z}_1 = -h[I(t) - \hat{I}] - \frac{f(t)}{1 - F(t)} z_1. \tag{4.5}$$

The optimal production rate can be determined from the condition $\partial H / \partial P = 0$. Hence, we get

$$P(t) = \frac{z_1(t)}{c} + \hat{P}. \tag{4.6}$$

Finally, substituting from (4.6) into (2.1) and (4.5), we get the nonlinear controlled state equations:

$$\left.\begin{aligned} \dot{I}(t) &= \frac{z_1(t)}{c} + \hat{P} - S(t) - \frac{f(t)}{1 - F(t)} I(t), \\ \dot{x}_0(t) &= \frac{1}{2} h[I(t) - \hat{I}]^2 + c[P(t) - \hat{P}(t)]^2, \\ \dot{z}_1 &= -h[I(t) - \hat{I}] - \frac{f(t)}{1 - F(t)} z_1(t). \end{aligned}\right\} \tag{4.7}$$

Together with the initial condition $I(0) = I_0$ and the terminal conditions $x_0(T) = J$ and $z(T) = 0,$ this a boundary value problem, the solution of this problem depends on the functions $S(t),$ which represents the demand rate and $f(t),$ which determines the deterioration rate.

5. Numerical Solution of Optimal Control

The problem as in Section 4 is a two boundary value problem, in which the initial and final states are known. For the numerical solution of the system (4.7), we will use Runge-Kutta numerical method with step size 0.01. In this section, we will discuss the numerical solution of (4.7) with three different cases of the demand rate. The numerical solutions will be displayed graphically.

Optimal production and replenishment policy to minimize the total cost per unit time may be obtained by using the methodology proposed in the preceeding sections. The following numerical example is illustrated the model.

5.1. Constant demand rate

In this subsection, we will present the numerical solution of the inventory system with deterioration (4.7) with constant demand. Different sets of values of the monetary and non-monetary parameters will be considered.

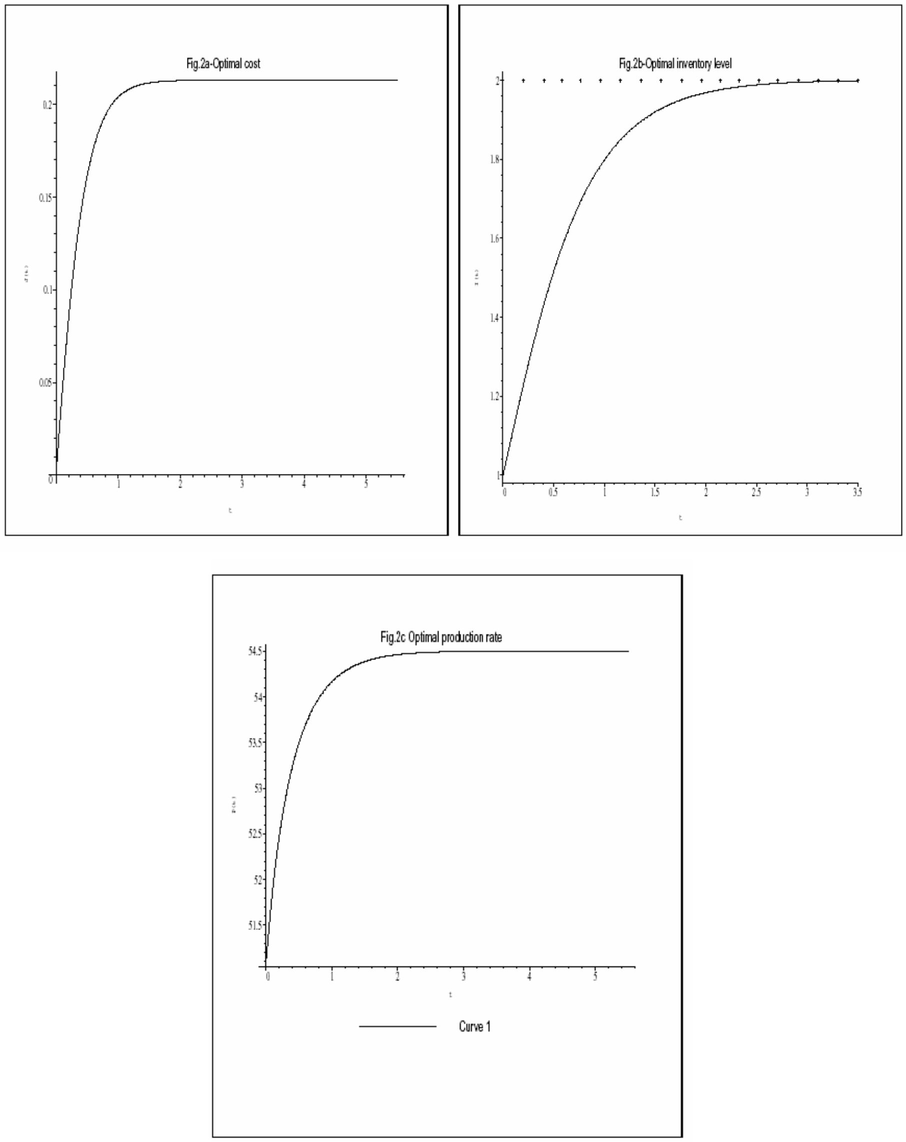

Figure 2. (a) The optimal cost function, (b) The optimal inventory level, (c) The optimal production rate, corresponding to the values of the system parameters are:

parameter	α	β	h	c	$\hat{I}$	I_0	T	$J(0)$	$S(t)$
value	0.5	2.1	0.5	0.7	2	1	5.5	0	50

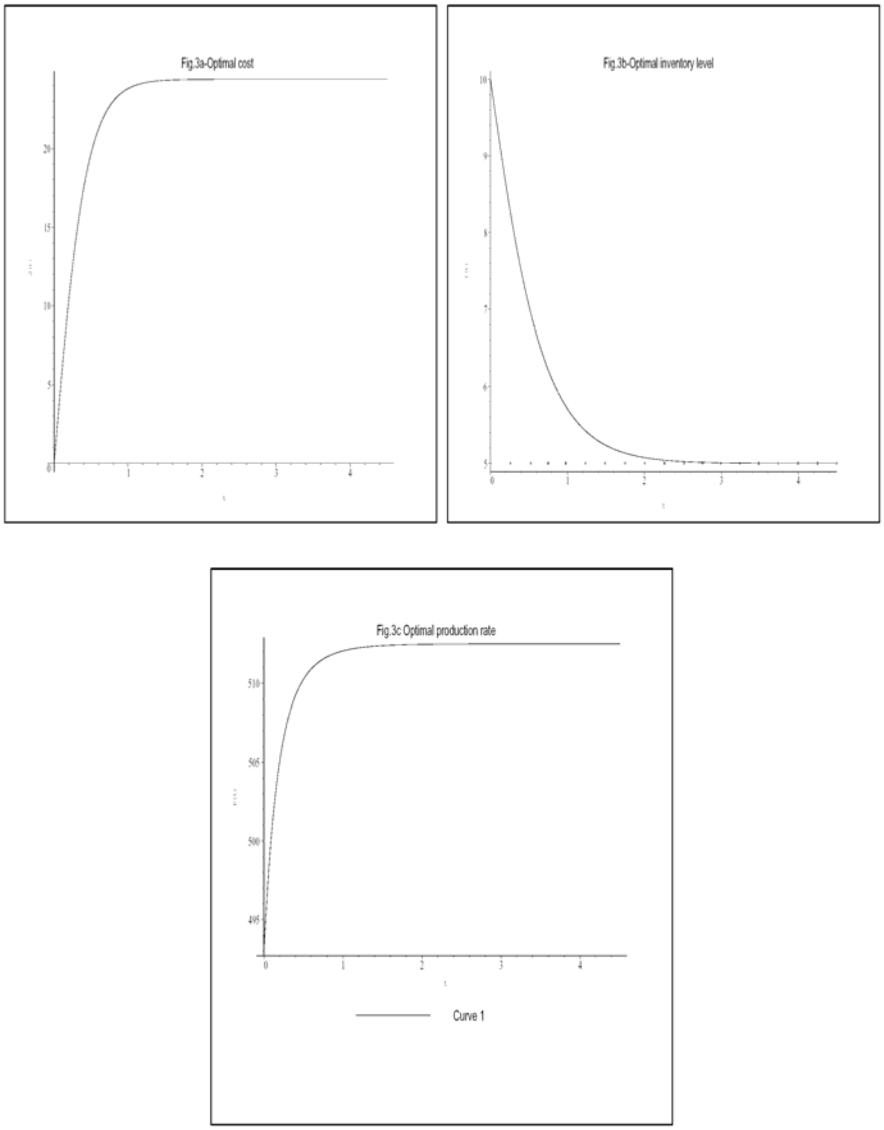

Figure 3. (a) The optimal cost function, (b) The optimal inventory level, (c) The optimal production rate, corresponding to the values of the system parameters are:

parameter	α	β	h	c	$\hat{I}$	I_0	T	$J(0)$	$S(t)$
value	2.5	2.1	0.7	1.25	5	10	4.5	0	500

Figures 2 and 3 are the graphical representation of the total cost function, inventory level, and production rate for two different examples of the constant demand rates.

From numerical integration, we conclude that, in the case of constant demand rate, we find both of inventory level and demand rate tend to target value exponentially.

5.2. Logistic demand rate

In this subsection, we will present the numerical solution of the inventory system with deterioration (4.7) with logistic demand rate $S(t) = t(k - t)$. Different sets of values of the monetary and non-monetary parameters will be considered.

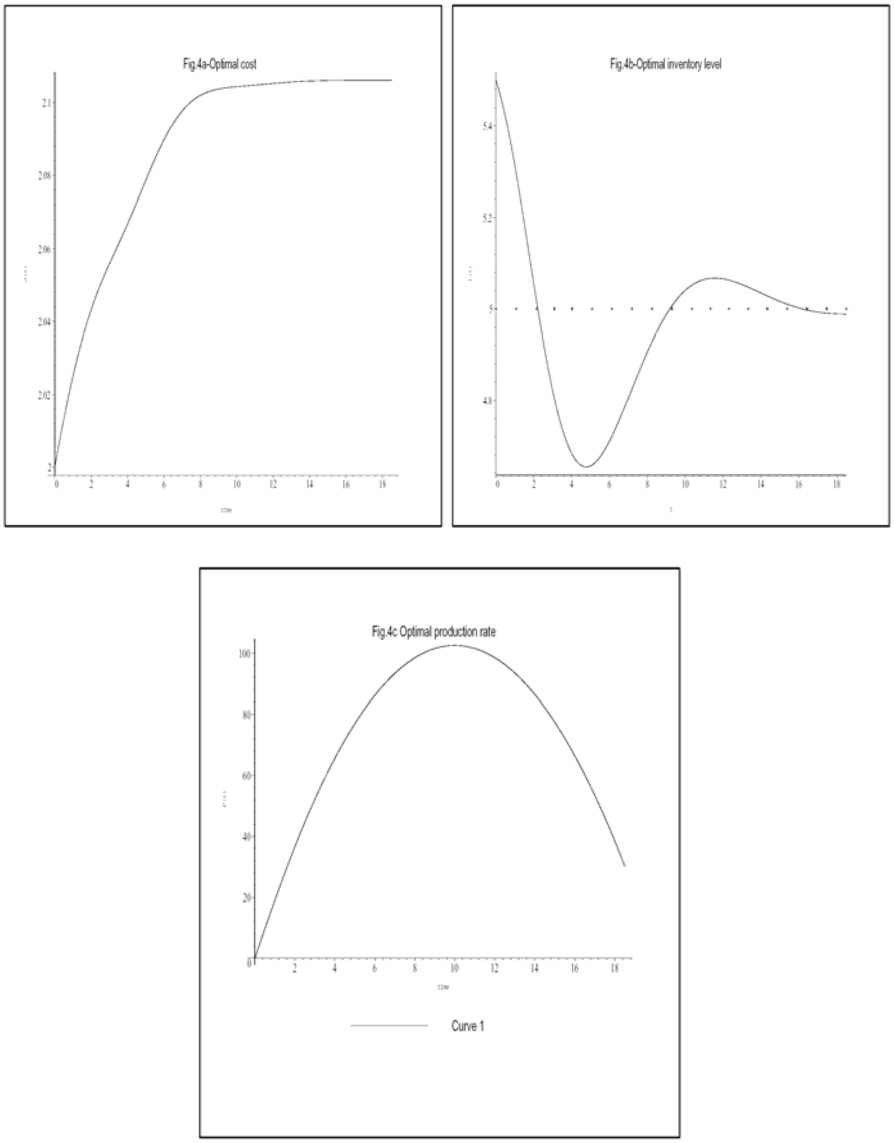

Figure 4. (a) The optimal cost function, (b) The optimal inventory level, (c) The optimal production rate, corresponding to the values of the system parameters are:

parameter	α	β	h	c	$\hat{I}$	I_0	T	$J(0)$	k
value	0.5	5.1	0.5	0.7	5	2	18.5	0	20

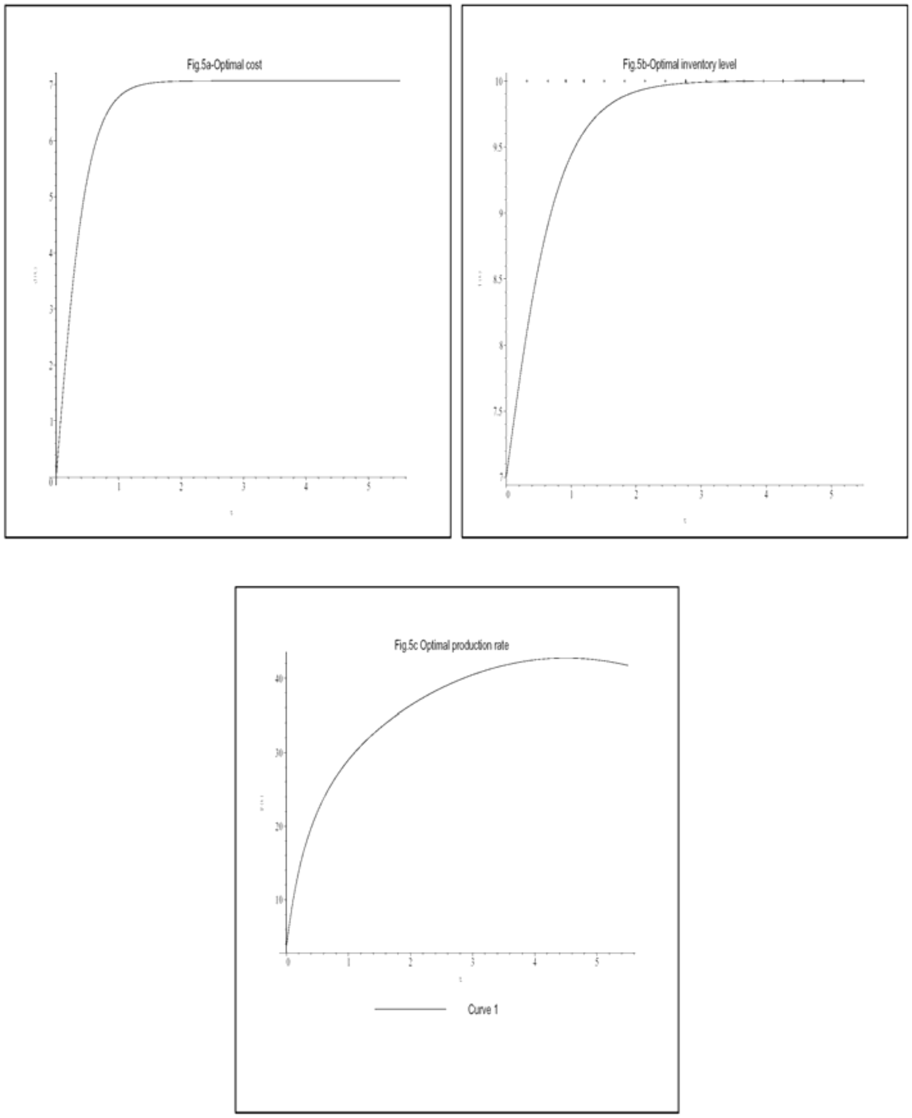

Figure 5. (a) The optimal cost function, (b) The optimal inventory level, (c) The optimal production rate, corresponding to the values of the system parameters are:

parameter	α	β	h	c	$\hat{I}$	I_0	T	$J(0)$	k
value	2.25	2.1	0.7	1.2	10	7	5.5	0	9

Figures 4 and 5 are the graphical representation of the total cost function, inventory level, and production rate for two different examples of the logistic demand rates.

5.3. Solenoidal demand rate

In this subsection, we will present the numerical solution of the inventory system with deterioration (4.7) with logistic demand rate $S(t) = k + \sin \omega t$. Different sets of values of the monetary and non-monetary parameters will be considered.

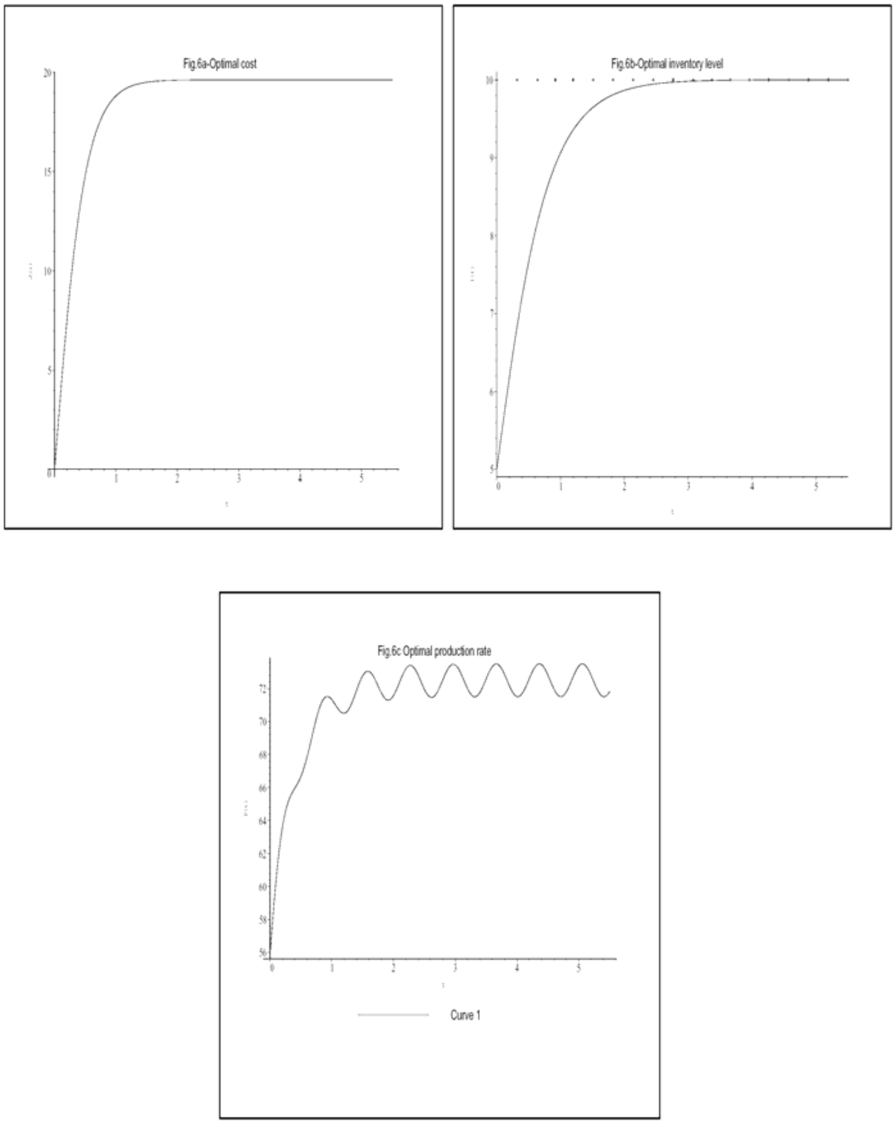

Figure 6. (a) The optimal cost function, (b) The optimal inventory level, (c) The optimal production rate, corresponding to the values of the system parameters are:

parameter	α	β	h	c	$\hat{I}$	I_0	T	$J(0)$	k	ω
value	2.5	2.1	0.7	2.1	1.2	20	5.5	0	50	9

Figure 6 is the graphical representation of the total cost function, inventory level, and production rate for fixed example of the solenoidal demand rates.

From these numerical examples, we can observe that, the total cost for the model starting with lower value, it grows exponentially to the final value of the total cost. This observation agrees with known, results from literature concerning the finite horizon inventory models (Teng et al. [20], Skouri and Papachristos, [17]) [17, 20].

6. Conclusion

In this paper, an inventory system with generalized exponentially distributed deterioration items has been studied. The optimal inventory level and optimal production rate are discussed. The numerical solution of the controlled inventory system is discussed. The total cost of the inventory-production system is discussed. From the above discussion, we conclude that the total cost of the system increased exponentially with time. Both of optimal inventory level and optimal production rate depend on the choosing of the demand rate. Therefore, dynamic optimal control of time-varying inventory systems remains an open and active research area.

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