REGULAR SEMIGROUPS WITH MULTIPLICATIVE
ORTHODOX TRANSVERSALS

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Abstract

In this paper, the concept of multiplicative orthodox transversals is first introduced, and some properties on multiplicative orthodox transversals are investigated. Then a construction of a regular semigroup with a multiplicative orthodox transversal is established. Finally, two examples are given to show a regular non-orthodox and an orthodox semigroup with a multiplicative orthodox transversal, respectively.

1. Introduction

The concept of an inverse transversal of a regular semigroup was first introduced by Blyth and McFadden in 1982 [1]. Afterwards, this class of regular semigroups attracted several author’s attention and a series of important results were obtained [1, 2, 7, 8]. If $S$ is a regular semigroup, then an inverse transversal of $S$ is an inverse subsemigroup $S^o$ such that $S^o$ meets $V(a)$, precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{x \in S : axa = a \text{ and } xax = x\}$ denotes the set of inverses.
of $a$. The intersection of $V(a)$ and $S^o$ is denoted by $V_{S^o}(a)$ and the unique element of $V_{S^o}(a)$ is denoted by $a^o$. It is well known that, the sets $I = \{e \in S : ee^o = e\}$ and $\Lambda = \{f \in S : f^o f = f\}$ are left regular and right regular bands, respectively, and play an important role in the study of regular semi groups with inverse transversals. Orthodox transversals were introduced by Chen [3] as a generalization of inverse transversals, and an excellent structure theorem for regular semi groups with quasi-ideal orthodox transversals was also given. Afterwards, Chen and Guo [4] considered the general case of orthodox transversals and investigated some properties concerning the sets $I$ and $\Lambda$ (defined below).

The second author [6] constructed regular semigroups with quasi-ideal orthodox transversals by a formal set. Analogous to the inverse transversal, the concept of a multiplicative orthodox transversal can be introduced. The main purpose of this paper is to introduce and study the class of regular semigroups with multiplicative orthodox transversals.

Let $S$ be a semigroup and $S^o$ be a subsemigroup of $S$. As usual, we shall write the set $S^o \cap V(x)$ as $V_{S^o}(x)$ for every $x \in S$, $x^o$ as an element of $V_{S^o}(x)$, and

$$I = \{a a^o : a \in S, a^o \in V_{S^o}(a)\}, \quad \Lambda = \{a^o a : a \in S, a^o \in V_{S^o}(a)\}.$$ 

Then $S^o$ is said to be an orthodox transversal of $S$, if the following conditions are satisfied.

(1.1) For all $a \in S$, $V_{S^o}(a) \neq \emptyset$;

(1.2) If $\forall a, b \in S$, and $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

Note that, if $S^o$ is an orthodox transversal of $S$, then $S$ is a regular semigroup by (1.1) and $S^o$ is an orthodox subsemigroup of $S$ by (1.2).

A subsemigroup $S^o$ of $S$ is said to be a quasi-ideal of $S$, if $S^o S S^o \subseteq S^o$. 

Finally, the following theorem will be frequently used without further mention.

(1.3) Let $e$ and $f$ be $D$-equivalent idempotents of a semigroup $S$. Then each element $a$ of $R_e \cap L_f$ has a unique inverse $a'$ in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$;

(1.4) Let $a, b$ be elements of a semigroup $S$. Then $ab \in R_a \cap L_b$, if and only if $L_a \cap R_b$ contains an idempotent.

We adopt the terminology and notation of [3, 6, 7].

2. Multiplicative Orthodox Transversals

**Definition 2.1.** Let $S$ be a regular semigroup with an orthodox transversal $S^\circ$. Then we shall say $S^\circ$ is multiplicative, if

$$(\forall x, y \in S)(\forall x^o \in V_{S^\circ}(x), y^o \in V_{S^\circ}(y)) x^o y y^o \in E(S^\circ),$$

where $E(S^\circ)$ is the band of idempotents of $S^\circ$.

It is well-known that, the sets $I$ and $\Lambda$ play an important role in both the inverse transversal and the orthodox transversal. By $I$ and $\Lambda$, an orthodox transversal $S^\circ$ is said to be *multiplicative* and a *quasi-ideal* can be equivalently defined. An orthodox transversal $S^\circ$ is

1. multiplicative, if $\Lambda I \subseteq E(S^\circ)$.
2. a quasi-ideal, if $\Lambda I \subseteq S^\circ$.

By Lemma 2.1 in [7], we have

**Lemma 2.2.** Let $S$ be a regular semigroup with a multiplicative orthodox transversal $S^\circ$. Then $S^\circ$ is a quasi-ideal of $S$.

**Theorem 2.3.** Let $S$ be a regular semigroup with an orthodox transversal $S^\circ$. If $S^\circ$ is a band, then $S^\circ$ is multiplicative, if and only if $S^\circ$ is a quasi-ideal.
**Proof.** By Lemma 2.2, the condition is clearly necessary. Conversely, if \( S^o \) is a quasi-ideal, then \( AI \subseteq S^o \). So, \( AI \subseteq E(S^o) \) since we also have \( E(S^o) = S^o \) by \( S^o \) is a band. That is, \( S^o \) is multiplicative.

If \( S \) is a regular semigroup with an inverse transversal \( S^o \), Green’s relations \( \mathcal{L} \) and \( \mathcal{R} \) are given by

\[
(x, y) \in \mathcal{L} \iff x^o x = y^o y, \quad (x, y) \in \mathcal{R} \iff xx^o = yy^o.
\]

As for an orthodox transversal \( S^o \), we have to get a weaker result.

**Theorem 2.4.** Green’s relations \( \mathcal{L} \) and \( \mathcal{R} \) are given by

\[
(x, y) \in \mathcal{L} \iff V_{S^o}(x)x = V_{S^o}(y)y, \quad (x, y) \in \mathcal{R} \iff xV_{S^o}(x) = yV_{S^o}(y).
\]

**Proof.** If \( V_{S^o}(x)x = V_{S^o}(y)y \), then for any \( x^o \in V_{S^o}(x) \), there exists \( y^o \in V_{S^o}(y) \) such that \( x^o x = y^o y \). So \( xLx^o x = y^o yL \). If \( \mathcal{L} \), then for any \( x^o \in V_{S^o}(x) \) and \( y^o \in V_{S^o}(y) \), we have \( yRy^o \) and \( yLxLx^o x \). Since \( yy^o \) and \( x^o x \) are idempotents, it follows from (1.3) that there exists an inverse \( y' \) in \( L_{x^o} \cap R\L_{x} \) of \( y \) such that \( x^o x = y'y \). Since \( x^o R x^o x R y'y \), it following from Theorem 2.4 in [4] that \( y' \in S^o \). That is, to say for any \( x^o \in V_{S^o}(x) \), there exists \( y' \in V_{S^o}(y) \) such that \( x^o x = y'y \), and thus, \( V_{S^o}(x)x \subseteq V_{S^o}(y)y \). Similarly, we have \( V_{S^o}(y)y \subseteq V_{S^o}(x)x \). Therefore, \( V_{S^o}(x)x = V_{S^o}(y)y \). The dual result can be proved similarly.

**Theorem 2.5.** Let \( S \) be a regular semigroup with an orthodox transversal \( S^o \). Suppose that \( V_{S^o}(e) \subseteq E(S^o) \) for every \( e \in E(S) \). If \( x' \in V(x) \) for any \( x \in S \), then for any \( x^o \in V_{S^o}(x) \), there exists \( x'^o \in V_{S^o}(x^o) \), such that

\[
x'^o x^o \in V_{S^o}(xx') = V_{S^o}(xx^o), \quad x^o x'^o \in V_{S^o}(x'x) = V_{S^o}(x^o x),
\]
and
\[ x^{oo} \in V_{S^o}(x') = V_{S^o}(x'x^{oo}) = V_{S^o}(x^ox') = V_{S^o}(x^o). \]

**Proof.** Let \( x \in S \) and \( x' \in V(x) \). Since \( xx' \mathcal{R} x \), it follows from \( xx' \cdot V_{S^o}(xx') = x \cdot V_{S^o}(x) \) that there exists \( (xx')^0 \in V_{S^o}(xx') \) such that \( xx' \cdot (xx')^0 = x^{oo} \). By the assumption, \( (xx')^0 \in E(S^o) \). So that
\[ xx^0(xx')^0 = xx'(xx')^0(xx')^0 = xx'(xx')^0 = xx^0, \]
and
\[ (xx')^0xx^0 = (xx')^0xx'(xx')^0 = (xx')^0, \]
which shows that \( xx^0 \mathcal{L} (xx')^0 \). So \( x^0 \mathcal{L} (xx')^0 \) since \( x^0 \mathcal{L} xx^0 \). Since \( (xx')^0 \in E(S^o) \), so for \( (xx')^0 \in V_{S^o}((xx')^0) \), there exists \( x^{oo} \in V_{S^o}(x^0) \) such that \( x^{oo}x^0 = (xx')^0(xx')^0 = (xx')^0 \), that is, \( x^{oo}x^0 \in V_{S^o}(xx') \). And since, we also have \( x^{oo}x^0 \in V_{S^o}(xx') \), so \( x^{oo}x^0 \in V_{S^o}(x^0) = V_{S^o}(x^0) \). Similarly, there exists \( x^0 \in V_{S^o}(x^0) \) such that \( x^0x^0 \in V_{S^o}(x^0x) = V_{S^o}(x^0x) \). Since, we also have \( x^0x^{oo} \in V_{S^o}(x^0x) \), so \( x^0x^{oo} \in V_{S^o}(x^0x) = V_{S^o}(x^0x) \).

By using the above facts, we have
\[
\begin{align*}
x^{oo}x'x^{oo} &= x^{oo}x'x^{oo}x^0x^{oo} = x^{oo}x'x^0\cdot(xx')^0xx^0 \\
&= x^{oo}x'x^{oo}x^0 = x^{oo}x^0x^0x^0x^0x^0x^0 \\
&= x^{oo}x^0x^{oo} = x^{oo},
\end{align*}
\]
and
\[
\begin{align*}
x'x^{oo}x' &= x'x'x^{oo}x^0x^{oo}x' = x'x'(xx')^0x^{oo}x' \\
&= x'x^{oo}x^{oo}x' = x'x(x')^0x' = x'.
\end{align*}
\]
Consequently, \( x^{oo} \in V_{S^o}(x') \). Also, we have
\[
x'xx^o \cdot x^{oo} \cdot x'x^o = x'x(x'x)^o x'x \cdot x^o = x'x^{oo},
\]
and
\[
x^{oo} \cdot x'x^o \cdot x^{oo} = x^{oo} \cdot xx^{oo} \cdot x'x \cdot x^o = x^{oo} \cdot x^o \cdot x^{oo} = x^{oo}.
\]
That is to say \( x^{oo} \in V_{S^o}(x'x^o) \). Similarly, \( x^{oo} \in V_{S^o}(x^oxx') \). Therefore,
\[
x^{oo} \in V_{S^o}(x') = V_{S^o}(x'x^o) = V_{S^o}(x^oxx') = V_{S^o}(x^o).
\]

**Theorem 2.6.** Let \( S \) be a regular semigroup with an orthodox transversal \( S^o \). Suppose that \( V_{S^o}(e) \subseteq E(S^o) \) for every \( e \in E(S) \). Then for any \( x, y \in S \), \( V(x) \cap V(y) \neq \emptyset \), if and only if \( V_{S^o}(x) = V_{S^o}(y) \).

**Proof.** Take \( a' \in V(x) \cap V(y) \). Then by Theorem 2.5, for any \( x^o \in V_{S^o}(x), y^o \in V_{S^o}(y) \), there exist \( x^{oo} \in V_{S^o}(x^o) \), \( y^{oo} \in V_{S^o}(y^o) \), such that \( x^{oo} \in V_{S^o}(a') \), \( y^{oo} \in V_{S^o}(a') \). Thus, \( V_{S^o}(x^o) = V_{S^o}(a') = V_{S^o}(y^o) \).

Take \( x^{oo} \in V_{S^o}(x^o) = V_{S^o}(y^o) \). Then
\[
x^o \in V_{S^o}(x) \cap V_{S^o}(x^{oo}) \text{ implies } V_{S^o}(x) = V_{S^o}(x^{oo}),
\]
\[
y^o \in V_{S^o}(y) \cap V_{S^o}(x^{oo}) \text{ implies } V_{S^o}(y) = V_{S^o}(x^{oo}),
\]
and therefore, \( V_{S^o}(x) = V_{S^o}(y) \).

The converse is obvious.

**Theorem 2.7.** Let \( S \) be a regular semigroup with an orthodox transversal \( S^o \). Then \( S^o \) is multiplicative, if and only if \( S^o \) is a quasi-ideal of \( S \) and \( V_{S^o}(e) \subseteq E(S^o) \) for every \( e \in E(S) \).
Proof. If $S^o$ is multiplicative, then by Lemma 2.2, $S^o$ is a quasi-ideal of $S$ and for any $e \in E(S)$, $e^0 \in V_{S^o}(e)$, we have $e^0 = e^0ee^0 = e^0e = e^0 \in E(S^o)$. Conversely, let $e \in \Lambda$, $f \in I$, and let $x \in V_{S^o}(ef)$. Then $(fx)^2 = fxe$ and $ef \in V(fxe) \cap S^o$, so by assumption, we have that $ef \in E(S^o)$. Thus $\Lambda I \subseteq E(S^o)$, and consequently, $S^o$ is multiplicative.

Let $S$ be a regular semigroup with a multiplicative inverse transversal $S^o$, and let $\langle E(S) \rangle$ be the idempotent-generated regular subsemigroup of $S$. In [1], it has been shown that $\langle E(S) \rangle = \{x \in S : x^0 \in E(S^o)\}$. As for the orthodox transversal, we have a similar result.

**Theorem 2.8.** Let $S$ be a regular semigroup with a multiplicative orthodox transversal $S^o$. Then $\langle E(S) \rangle = \{x \in S : V_{S^o}(x) \subseteq E(S^o)\}$.

**Proof.** For any $x \in S$, if $V_{S^o}(x) \subseteq E(S^o)$, take $x^0 \in V_{S^o}(x)$, then we have

$$x = xx^0 = xx^0 \cdot x^0 x \in \langle E(S) \rangle.$$ 

For the converse inclusion, it suffices to prove that $V_{S^o}(xy) \subseteq E(S^o)$ for all $x, y \in E(S)$. Let $x^0 \in V_{S^o}(x)$ and $y^0 \in V_{S^o}(y)$, then by Theorem 2.7, $x^0, y^0 \in E(S^o)$. Since $S^o$ is a multiplicative orthodox transversal of $S$, we have $x^0y^0 \in E(S^o)$, and so $y^0x^0x^0y^0x^0 \in E(S^o)$. But $y^0x^0x^0y^0x^0 \in V(xy)$, so we have that $V_{S^o}(xy)$ contains an idempotent, so by Corollary 2.1 of [4], it consists of idempotents only. Now, the proof of this theorem is completed.

**3. The Structure Theorem**

In 1999, Chen [3] gave a structure theorem for regular semigroups with quasi-ideal orthodox transversals. There a regular mapping $*$ means
a mapping \( \Lambda \times I \to S^0 \), denoted by \((\lambda, i) \to \lambda \ast i\), which satisfies three conditions:

(a) \( (\forall e, f \in E^0) \) \( e(\lambda \ast i)f = (e\lambda) \ast (if) \).

(b) If \( \lambda \in E^0 \) or \( i \in E^0 \), then \( \lambda \ast i = \lambda i \).

(c) If \( \lambda^+, i^* \in E^0 \) and \( \lambda^+ R \lambda \subseteq L \lambda^+ \) and \( i^* R i \), then \( (i^* i') \cdot V_{S^0}(\lambda' \ast i') = (\lambda^+ \lambda^+) \subseteq V_{S^0}(\lambda \ast i) \).

If the mapping \( \ast \) is \( \Lambda \times I \to E^0 \) instead of \( \Lambda \times I \to S^0 \), then we can construct a regular semigroup with a multiplicative orthodox transversal, and the condition (c) can be removed. Since, this construction is a modification of that Chen gave in [3], the proofs have a good deal in common. So, we only give the outline of the proof, for more details, see [3].

In what following, \( S^0 \) denotes an orthodox semigroup, \( E^0 \) denotes the set of idempotents of \( S^0 \), and \( \sigma \) denotes the minimum inverse semigroup congruence on \( S^0 \). For \( a \in S^0 \), the idempotents in \( L_a(R_a) \) are denoted by \( a_+, a_{++}, \ldots (a_+, a_{++}, \ldots) \), the \( \sigma \)-class containing \( a \) will be denoted by \( T(a) \). Let \( I \) and \( \Lambda \) are bands such that \( I \cap \Lambda = E^0 \).

The triple \((I, S^0, \Lambda)\) is said to be permissible, if \( E^0 \) is a quasi-ideal orthodox transversal of both \( I \) and \( \Lambda \), and \( L(I) = I, R(\Lambda) = \Lambda \). Denote the elements of \( I \) by \( i, i_1, i', \ldots \), and the \( R \)-class of \( I \) containing \( i \) by \( R_i \). Denote the elements of \( \Lambda \) by \( \lambda, \lambda_1, \lambda', \ldots \), and the \( L \)-class of \( \Lambda \) containing \( \lambda \) by \( L_\lambda \).

**Theorem 3.1.** Suppose that \((I, S^0, \Lambda)\) is a permissible triple with a mapping \( \ast : \Lambda \ast I \to E^0 \), denoted by \((\lambda, i) \to \lambda \ast i\), which satisfies the following:

\[(3.1) (\forall e, f \in E^0) e(\lambda \ast i)f = (e\lambda) \ast (if),\]
We define a subset of $I / \mathcal{R} \times S^0 / \sigma \times \lambda / \mathcal{L}$ as follows:

$$
\Gamma = \{(R_i, T(x), L_\lambda) : (\exists i^*, \lambda^+ \in E^0) i L_i^* \mathcal{R} x, \lambda^+ \mathcal{L} \mathcal{L} x\},
$$

where $x \in S^0$. Define multiplication in $\Gamma$ by

$$(R_i, T(x), L_\lambda) (R_i, T(x), L_\lambda) = (R_{ia}, T(a), L_{a \lambda_1}),$$

where $a = x(\lambda \ast i_1) x_1$. Then $\Gamma$ is a regular semigroup containing a multiplicative orthodox transversal isomorphic to $S^0$.

Conversely, every regular semigroup with a multiplicative orthodox transversal can be constructed in this way.

**Proof.** Since condition (c) is crucial to the proof of that the multiplication in $\Gamma$ is well-defined [3], we first proved that (c) is valid under our assumption.

If $\lambda^+, \lambda^* \in E^0$ and $\lambda^+ \mathcal{R} \lambda \mathcal{L} \lambda'$ and $i^* \mathcal{L} i \mathcal{R} i'$, take $s \in V_{S^0}(\lambda' \ast i')$. Then, we have

\[
(\lambda \ast i) (i^* i' s \lambda \ast \lambda^+) (\lambda \ast i) = (\lambda \ast i i^* i') s (\lambda \lambda^+ \lambda \ast i)
\]

\[
= (\lambda \ast i') s (\lambda' \ast i)
\]

\[
= (\lambda \lambda' \ast i') s (\lambda' \ast i')
\]

\[
= \lambda (\lambda' \ast i') i
\]

\[
= \lambda \ast i,
\]

and

\[
(i^* i' s \lambda \lambda^+) (\lambda \ast i) (i^* i' s \lambda \lambda^+) = i^* i' s (\lambda \lambda^+ \lambda \ast i i^* i') s \lambda \lambda^+
\]

\[
= i^* i' s (\lambda' \ast i') s \lambda \lambda^+
\]

\[
= i^* i' s \lambda \lambda^+.
\]
Thus
\[ (i^*i') \cdot V_{S^0}(\lambda' \ast i') \cdot (\lambda' \lambda^+ \ast \lambda) \subseteq V_{S^0}(\lambda \ast i). \]

Up to now, all the conditions of Theorem 3.4 in [3] are satisfied. Therefore, as the proofs in [3], \( \Gamma \) is a regular semigroup containing a quasi-ideal orthodox transversal \( W \), where
\[ W = \{(R_i, T(x), L_{\lambda}) : (\exists i \in T(x), i' \in R_i \cap E^0, \lambda' \in L_{\lambda}' \cap E^0) \}. \]

In the following, we only need to prove that \( W \) is a multiplicative orthodox transversal of \( \Gamma \).

Let \( (R_i, T(x), L_{\lambda}) \in W \), by Lemma 3.7 in [3], we can assume that \( i, \lambda \in E^0 \). Then \( (R_i, T(x), L_{\lambda}) \in E(W) \), if and only if \( x = x \lambda \lambda i \lambda \). In fact,
\[ (R_i, T(x), L_{\lambda})^2 = (R_{i(x \lambda \lambda i \lambda)}, T(x \lambda \lambda i \lambda), L_{(x \lambda \lambda i \lambda) \lambda}). \]

If \( x = x \lambda \lambda i \lambda \), then \( T(x) = T(x \lambda \lambda i \lambda) \), and \( R_{i(x \lambda \lambda i \lambda)} = R_{ix} = R_{ix} = R_0 \), since \( i \mathcal{R} x \mathcal{R} x_0 \). Similarly, \( L_{(x \lambda \lambda i \lambda) \lambda} = L_{\lambda} \). That is, \( (R_i, T(x), L_{\lambda}) \in E(W) \).

Conversely, if \( (R_i, T(x), L_{\lambda}) \in E(W) \), then \( T(x) = T(x \lambda \lambda i \lambda) \). Let \( x' \) be the common inverse of \( x \) and \( x \lambda \lambda i \lambda \). Then \( x' \lambda \lambda i x x' = x' \lambda \lambda i x x' = x \lambda \lambda i x \), and consequently, \( x \lambda \lambda i x = x \).

Take
\[ k = (R_i, T(x), L_{\lambda}), l = (R_j, T(y), L_{\mu}) \in \Gamma, \]
and
\[ (R_i, T(x'), L_{\lambda'}) \in V_W(k), (R_j, T(y'), L_{\mu'}) \in V_W(l), \]
where \( T(x'), T(y') \) are the inverses of \( T(x), T(y) \) in \( S^0 / \mathcal{S} \), respectively, and \( V_W(a) \) as described in Lemma 3.8 in [3]. We can assume that \( i', \lambda', j', \mu' \in E^0 \). Then
\[ \left( R_{i'}, T(x'), L_{\lambda'} \right) (R_i, T(x), L_{\lambda}) (R_j, T(y), L_{\mu}) (R_{j'}, T(y'), L_{\mu'}) \]
\[ (R_{i\mu}, T(a), L_{\mu} \mu', T(b), L_{\mu} \mu') \]
\[ = (R_{i\mu}, T(c), L_{c \mu} \mu') \in W, \]
where \( a = x' \lambda \ast i x = x' \lambda' x, \)
\( b = y(\mu \ast j)y' = y \mu' y', \) \( c = a(\ast \lambda \ast j b) \)
\( b' = a(\ast j)b. \)

Also, we have
\( c \ast c \ast b \ast \mu' \ast i' a \ast e \ast c \ast c \)
\[ = c' \mu' a' e \ast c \]
\[ = a(\ast j)b \ast b \ast \mu' \ast i' a' \ast a \ast j \ast j \ast b \]
\[ = x' \lambda' x(\ast j) y \mu' y' \ast i' x' \lambda' x(\ast j) y \mu' y' \]
\[ = x' \lambda' x(\ast j) y \mu' y' \ast x' \lambda' x(\ast j) y \mu' y' \]
\[ = c \ast c. \quad (\text{since } u', i' \in E^0 \text{ and } i' \ast R x', \mu' \ast L y') \]

Thus \( x' \lambda' x \subseteq x' E^0 x \subseteq E^0, \) since \( x, x' \in S^0 \) and \( S^0 \) is orthodox.

Similarly, \( y \mu' y' \in E^0. \) Consequently, \( c = x' \lambda' x \ast (\ast j) \cdot y \mu' y' \in E^0 \cdot E^0 \cdot E^0 \subseteq E^0 \) since \( S^0 \) is orthodox. Therefore, \( V_W(k) k l V_W(l) \subseteq E(W), \) and so \( W \) is a multiplicative orthodox transversal of \( \Gamma. \)

Conversely, let \( S \) be a regular semigroup and \( S^0 \) be a multiplicative orthodox transversal of \( S. \) For every \( (\lambda, i) \in \Lambda \ast I, \) put \( \lambda \ast i = \lambda i. \) Then, \( \lambda \ast i \in E^0 \) since \( S^0 \) is a multiplicative orthodox transversal of \( S. \) All the other conditions in Theorem 3.1 are satisfied as in [3].

In the following, we will give two examples, both of which are based on \( 2 \times 2 \) matrices. In Example 2, we take use of the semigroup \( \text{Sing}_{2 \times 2} R \) of singular real \( 2 \times 2 \) matrices. We let \( \text{Sing}_{2 \times 2}^* R \) be the subset of those matrices, whose leading element (i.e., that in the \((1, 1)\)-position) is non-zero. Example 1 is to show a regular non-orthodox semigroups with a
multiplicative orthodox transversal, and the orthodox transversal is not an inverse transversal. Example 2 is to show an orthodox semigroup with a multiplicative orthodox transversal, and the orthodox transversal is neither the orthodox semigroup itself, nor the inverse transversal.

**Example 1** [non-orthodox]. Let $F$ be a field of characteristic 2 and consider the subset $S$ of $\text{Mat}_{2 \times 2} F$ given by

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \{a, b, c, d, e\}.$$

The only non-idempotent of $S$ is $a$, and clearly, $d \in V(a)$. Hence $S$ is regular. That $S$ is not orthodox follows from the equality $bc = a$. It is readily verified that

$$V(a) = \{d\}, V(b) = \{d, b\}, V(c) = \{d, c\}, V(d) = \{d, a, b, c\}, V(e) = \{e\}.$$

Let $S^o = \{b, d, e\}$. Then $S^o$ is an orthodox subsemigroup of $S$. It is easy to check that $S^o$ is an orthodox (in fact, band) transversal of $S$. Simple calculation reveal that $I = \{b, d\}$ and $\Lambda = \{b, c, d, e\}$. It follows that $\Lambda I = \{b, d, e\}$ and so $S^o$ is multiplicative.

**Example 2** [orthodox]. Let $S_1$ be the subset of $\text{Sing}_{2 \times 2} R$ described by

$$S_1 = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in R, \ x \neq 0 \right\}.$$

Then $S_1$ is a semigroup. It is readily seen that $\begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x^{-1} & x^{-1} \\ 0 & 0 \end{bmatrix} \in V \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$, and so $S_1$ is regular. The idempotents are the matrices of the form $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$, and consequently $S_1$ is orthodox. Let

$$S_1^0 = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}, x \neq 0 \right\}.$$
Then $S^0_1$ is an orthodox subsemigroup of $S$ and

$$V \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \cap S^0_1 = \begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^{-1} & x^{-1} \\ 0 & 0 \end{bmatrix}. $$

It is easy to check that $S^0_1$ is an orthodox transversal of $S$. Again as can readily be verified, we have

$$I = \{ XX^0; X \in S_1 \} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = E(S^0_1),$$

$$\Lambda = \{ X^0 X; X \in S_1 \} = \begin{bmatrix} 1 & z \\ 0 & 0 \end{bmatrix}; z \in R = E(S_1),$$

from which it follows that $\Lambda I = E(S^0_1)$, and therefore, $S^0_1$ is multiplicative.

References


