A NOTE ON FINITELY GENERATED SECONDARY MODULES

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Abstract

Let $R$ be a commutative ring with identity. For a representable $R$-module $M$, the notion of attached prime submodules of $M$ is defined. It is shown that an attached prime submodule of a finitely generated representable module $M$ is of the form $PM$ for some $P \in \text{Att}_R(M)$, the set of attached prime ideals of $M$, and so, there is a lattice isomorphism between $\text{Att}_R(M)$ and $\text{Att}_P(M)$, the set of attached prime submodules of $M$.

1. Introduction

The theory of secondary representations is a sort of dual of the theory of primary decomposition in a module over a non-trivial commutative ring $R$. In fact, the set of attached prime ideals of a module contains a lot of information about the module itself. The purpose of this paper is to define the concept of attached prime submodules (see Definition 2.1) and to investigate their properties. We shortly summarize the content of the paper. For example, it is proved that, for a finitely generated
representable $R$-module $M$, $\text{Att}_p(M)$ is not empty. Also, if $N$ is a finitely generated representable submodule of $M$, then $|\text{Att}_p(M / N)| \leq |\text{Att}_p(M)| \leq |\text{Att}_p(N)| + |\text{Att}_p(M / N)|$. Finally, it is proved, in Theorem 2.17, if $M$ is a multiplication representable $R$-module, then $\text{Att}_p(M) = \text{Spec}(M)$.

Throughout this paper, $R$ will denote a commutative ring with identity and $M$, a unital module. Let $R$ be a commutative ring and let $M$ be an $R$-module. Given an element $r$ of $R$, we say that $r$ divides $M$, if $rM = M$, and we say that $r$ is nilpotent on $M$, if $r^nM = 0$, for some positive integer $n$. We say that $M$ is secondary, if it is non-zero and every $r$ in $R$ either divided $M$ or is nilpotent on $M$; in this case, the ideal $\text{nilrad}(M) = P$ is prime, and we also say that $M$ is $P$-secondary. A module $M$ is said to be representable, if it can be written as a sum $M_1 + \ldots + M_k$ of secondary modules; such a sum is called a secondary representation of $M$. If this representation is irredundant, we say that the attached primes of $M$ are $\text{Att}_R(M) = \{P_1, \ldots, P_k\}$, where $\text{nilrad}(M_i) = P_i$ [7].

A proper submodule $N$ of $M$ is prime, if for any $r \in R$ and $m \in M$ such that $rm \in N$, either $m \in N$ or $r \in (N : M) = \{a \in R : aM \subseteq N\}$. It is easy to show that, if $N$ is a prime submodule of $M$, then the annihilator $P$ of the module $M / N$ is a prime ideal of $R$, and $N$ is said to be $P$-prime submodule of $M$. The set of all prime submodules in an $R$-module $M$ is denoted $\text{Spec}(M)$. Let $P$ be a prime ideal of $R$ and $M$ be an $R$-module. We recall from [6], the subset $M(P)$ of $M$ defined by

$$M(P) = \{m \in M : Im \subseteq PM, \text{ for some ideal } I \nsubseteq P\}.$$ 

Then $M(P)$ is an $R$-submodule of $M$ and $PM \subseteq M(P)$. We will need the following lemma from [6].

**Lemma 1.1.** Let $M$ be an $R$-module. Then, the following hold:

(i) Let $I$ be an ideal of $R$. Then, there exists a proper submodule $N$ of $M$ such that $I = (N : M)$, if and only if $IM \neq M$ and $I = (IM : M)$.
(ii) For a prime ideal $P$ of $R$, let $N = M(P)$. Then $M = N$ or $N$ is a prime submodule of $M$ such that $(N : M) = P$.

2. Results

We begin the key definition of this paper.

**Definition 2.1.** Let $M$ be a representable module over a commutative ring $R$. We define the set attached prime submodules of $M$ as

$$\text{Att}_R(M) = \{M(P) : P \in \text{Att}_R(M)\},$$

where $\text{Att}_R(M)$ is the set of attached prime ideals of $M$.

For any representable $R$-module $M$, it is known that $\text{Att}_R(M) \neq 0$, but the following example shows that the condition “$M$ is finitely generated representable” in this paper is not superficial.

**Example 2.2.** (1) Let $p$ be a fixed prime integer and $N_0 = Z^+ \cup \{0\}$.

Then,

$$E(p) = \{\alpha \in Q / Z : \alpha = r / p^n + Z, \text{ for some } r \in Z \text{ and } n \in N_0\}$$

is an Artinian and 0-secondary $Z$-module, but it is not a finitely generated $Z$-module. Moreover, by [5, p. 3745], $\text{Spec}(E(p)) = \text{Att}_R(E(p)) = 0$.

(2) Let $Q$ be the field of quotients of $Z$. Then $Q$ is 0-secondary, but it is not a finitely generated $Z$-module. On the other hand, by [5, Theorem 1], $\text{Spec}(Q) = \text{Att}_R(Q) = \{0\}$.

**Lemma 2.3.** Assume that $M$ is a finitely generated $R$-module, and let $P$ be a prime ideal of $R$ such that $P = (PM : M)$. Then $M(P)$ is a prime submodule of $M$.

**Proof.** By Lemma 1.1, it suffices to show that $M(P) \neq M$. Suppose not. There are elements $m_1, \ldots, m_s$ of $M$ with $M = \sum_{i=1}^s Rm_i$. For each $i, i = 1, \ldots, s$, there exists an ideal $I_i \subseteq P$ of $R$ such that $I_im_i \subseteq PM$. 

Set $I = \bigcap_{i=1}^{s} I_i$. Then $P$ prime gives $I \subsetneq P$. Since, by assumption $Im_i \subseteq I_i m_i \subseteq PM$, we must have $I \subseteq (PM : M) = P$, which is a contradiction.

**Theorem 2.4.** Let $M$ be a finitely generated representable $R$-module. Then $\text{Att}_P(M) \neq 0$.

**Proof.** Let $M = \sum_{i=1}^{k} M_i$ be a minimal secondary representation of $M$ with $\text{Att}_R(M) = \{P_1, \ldots, P_k\}$. Since $M$ is finitely generated, we must have a prime submodule $N$ of $M$. Therefore, without loss of generality, we can assume that $M_1 \not\subseteq N$, $P_1 = (N : M)$, and $M = M_1 + N$ by [2, Theorem 2.10]. It follows from Lemma 1.1 that $P_1 = (P_1 M : M)$; hence $M(P_1) \in \text{Att}_P(M)$ by Lemma 2.3, as required. 

**Lemma 2.5.** Let $M$ be a finitely generated $P$-secondary $R$-module. Then $P$ is a maximal ideal of $R$.

**Proof.** Let $Q$ be a maximal ideal containing $P$. Then by [7, Theorem 2.3], $(0 :_R M) \subseteq Q$. By usual determinant argument, we get $QM \neq M$. Let $q \in Q$. Then $qM \subseteq QM \neq M$, and hence $q \in P$. This gives $P = Q$.

**Proposition 2.6.** Let $M = \sum_{i=1}^{k} M_i$ be a minimal secondary representation of a finitely generated representable $R$-module $M$ with $\text{Att}_R(M) = \{P_1, \ldots, P_k\}$. Then, the following hold:

(i) $P_i M \neq M$ and $M(P_i) = P_i M$, for every $i = 1, 2, \ldots, k$.

(ii) $P_i M_i \neq M_i$ and $(P_i M_i : M_i) = P_i$, for every $i = 1, \ldots, k$.

**Proof.** (i) If $M = P_i M$ for some $i$, then there exists $p \in P_i$ such that $(1 - p)M = 0$ and so $1 - p \in (0 : M) \subseteq \bigcap_{j=1}^{k} (0 : M_j) \subseteq \bigcap_{j=1}^{k} P_j \subseteq P_i$ by [7, Theorem 2.3], which is a contradiction. Thus, $M \neq PM$ for every $P \in \text{Att}_R(M)$. Since by [7, Theorem 2.2], there is a non-zero quotient of
which is finitely generated and \( P_i \)-secondary \((i = 1, 2, ..., k)\), \( P_i \) is a maximal ideal of \( R \) for every \( i \) by Lemma 2.5. Since, the inclusion \( P_i M \subseteq M(P_i) \) is clear, we will prove the reverse inclusion. Let \( m \in M(P_i) \). Then, there exists an ideal \( I \subseteq P_i \) such that \( Im \subseteq P_i M \), so there is an element \( a \in I \) such that \( Ra + P_i = R \), so \( m = ram + pm \in P_i M \), for some \( r \in R \) and \( p \in P_i \), and so we have equality.

(ii) Suppose that \( P_i M_i = M_i \) for some \( i \). Set \( N = M_1 + ... + M_{i-1} + M_{i+1} + ... + M_k \). The mapping \( v \) from \( M_i \) to \( M / N \) defined by \( v(a) = a + N \) is an epimorphism, and so \( M_i / \ker(v) \) is a finitely generated \( P_i \)-secondary \( R \)-module. We put \( \ker(v) = K \). Then, we must have \( P_i (M_i / K) = M_i / K \), so there is an element \( p_i \in P_i \) such that \( M_i = (1 - p_i)M_i \subseteq K \); hence \( M = N \), contradicting the choice of \( N \). Thus, \( M_i \neq P_i M_i \) for every \( i \).

Now, we show that for each \( 1 \leq i \leq k \), \((P_i M_i : M_i) = P_i\). Clearly, \( P_i \subseteq (P_i M_i : M_i) \). For the other containment, suppose that \( a \in (P_i M_i : M_i) \). If \( a \notin P_i \), then \( aM_i = M_i \subseteq P_i M_i \), a contradiction. This shows that \( P_i = (P_i M_i : M_i) \) for all \( i \).

**Theorem 2.7.** Let \( M \) be a finitely generated representable \( R \)-module with \( \text{Att}_R(M) = \{P_1, ..., P_k\} \). Then, \( \text{Att}_R(M) = \{P_1 M, ..., P_k M\} \). In particular, if \( M \) is a finitely generated \( P \)-secondary \( R \)-module, then \( \text{Att}_P(M) = \{M(P) = PM\} \).

**Proof.** Apply Proposition 2.6. \( \square \)

**Remark 2.8.** Let \( M \) be a finitely generated Artinian \( R \)-module. By [7, Theorem 5.2], \( M \) is representable, so if \( \text{Att}_R(M) = \{P_1, ..., P_k\} \), then \( \text{Att}_P(M) = \{P_1 M, ..., P_k M\} \) by Theorem 2.7.

**Lemma 2.9.** Let \( P \) be a prime ideal \( R \), \( M \) be an \( R \)-module, and \( N \) be an \( \mathcal{R} \)-submodule of \( M \). Then, the following hold:
(i) \((N + M(P))/N \subseteq (M/N)(P)\). In particular, if \((M/N)(P)\) is a prime submodule of \(M/N\), then \(M(P)\) is a prime submodule of \(M\).

(ii) If \(N \subseteq PM\), then \((M/N)(P) = M(P)/N\). In particular, \(M(P)\) is a prime submodule of \(M\), if and only if \((M/N)(P)\) is a prime submodule of \(M/N\).

**Proof.** (i) Let \(m + N \in (N + M(P))/N\), where \(m \in M(P)\). There is an ideal \(J\) of \(R\) such that \(Jm \subseteq PM\), so \(J(m + N) \subseteq P(M/N)\); hence \(m + N \in (M/N)(P)\). Finally, if \((M/N)(P)\) is prime, then \(M \neq M(P)\). Now, the assertion follows from Lemma 1.1.

(ii) The inclusion \(M(P)/N \subseteq (M/N)(P)\) follows from (i). For the other containment, let \(m + N \in (M/N)(P)\). Then, there exists an ideal \(I \subseteq P\) of \(R\) such that \(I(m + N) \subseteq P(M/N) = (PM)/N\), so \(Im \subseteq PM\); hence \(m \in M(P)\). Thus, \(m + N \in M(P)/N\), and so \(M/N(P) \subseteq M(P)/N\). The remaining part follows from the first part and Lemma 1.1. ■

**Proposition 2.10.** Let \(N\) be a submodule of a finitely generated representable \(R\)-module \(M\). Then, \(Att_p(M/N) \neq 0\) and \(|Att_p(M/N)| \leq |Att_p(M)|\).

**Proof.** Since \(M/N\) is finitely generated, and it is representable by [7, Theorem 2.4], we have \(Att_p(M/N) \neq 0\) by Theorem 2.4. If \((M/N)(P) \in Att_p(M/N)\) (note that \(Att_R(M/N) \subseteq Att_R(M)\) by [7, Theorem 2.4]), then by Lemma 2.9, \(M(P) \in Att_p(M)\), as required. ■

**Theorem 2.11.** Let \(N\) be a finitely generated representable submodule of a finitely generated representable \(R\)-module \(M\). Then, \(|Att_p(M/N)| \leq |Att_p(M)| \leq |Att_p(N)| + |Att_p(M/N)|\).

**Proof.** The left-hand inequality was proved in Proposition 2.10. To prove the other inequality, let \(M(P) \in Att_p(M)\). It suffices to show that either \(N(P) \in Att_p(N)\) or \((M/N)(P) \in Att_p(M/N)\). Since \(P \in Att_R(M)\),
we must have a $P$-secondary quotient $M / K$ of $M$. If $M = N + K$, then $M / K \cong N / (N \cap K)$, so $(N / (N \cap K))(P)$ is a prime submodule of $N / (N \cap K)$ by Lemma 2.3; hence $N(P) \in \text{Att}_P(N)$ by Lemma 2.9. If on the other hand, $M \neq N + K$, then $M / (K + N)$ is a quotient $M / K$ and is therefore $P$-secondary; but it is also a quotient of $M / N$, so $(M / N)(P) \in \text{Att}_P(M / N)$ by Lemmas 2.3 and 2.9. \hfill \Box

**Lemma 2.12.** Assume that $P$ is a prime ideal of $R$ and let $M = M_1 \oplus \ldots \oplus M_k$. Then, the following hold:

(i) $M(P) = M_1(P) \oplus \ldots \oplus M_k(P)$.

(ii) $M(P)$ is a prime submodule of $M$, if and only if $M_i(P)$ is a prime submodule of $M_i$ for some $i$.

**Proof.** (i) Let $m = m_1 + \ldots + m_k \in M(P)$, where $m_i \in M_i$. There exists an ideal $I \not\subseteq P$ of $R$ such that $Im \subseteq PM$, so for each $i (1 \leq i \leq k)$, $Im_i \subseteq PM_i$; hence $m \in K = M_1(P) + \ldots + M_k(P)$. For the reverse inclusion, assume that $a = a_1 + \ldots + a_k \in K$. For each $i$, $i = 1, \ldots, k$, there exists an ideal $I_i \not\subseteq P$ of $R$ such that $I_i a_i \subseteq PM_i$. Set $I = \bigcap_{i=1}^{k} I_i$. Then $I \not\subseteq P$ and $Ia \subseteq PM$, so $a \in M(P)$ and so we have equality.

(ii) Apply (i) and Lemma 1.1. \hfill \Box

Let $M_1$ and $M_2$ be two finitely generated $P$-secondary $R$-modules, and let $M$ denote the direct sum $M_1 \oplus M_2$. Then, clearly, $M$ is a finitely generated $P$-secondary $R$-module, and so $\text{Att}_P(M) = \{M(P)\}$. This gives that $|\text{Att}_P(M)| = 1 < 2 = |\text{Att}_P(M_1)| + |\text{Att}_P(M_2)|$, but we have the following theorem:

**Theorem 2.13.** Assume that $M_1, \ldots, M_k$ are finitely generated $P_i$-secondary $R$-modules such that $P_i \neq P_j$ (for each $i \neq j$), and let $M = \bigoplus_{i=1}^{k} M_i$. Then $|\text{Att}_P(M)| = \sum_{i=1}^{k} |\text{Att}_P(M_i)|$. 
Proof. Since \( \text{Att}_R(M) = \bigcup_{i=1}^{k} \text{Att}_R(M_i) \), the result follows from Lemma 2.12.

Lemma 2.14. Let \( S \) be a multiplicatively closed subset of \( R \), \( M \) be an \( R \)-module, and \( P \) be a prime ideal of \( R \) with \( S \cap P = 0 \). Then, \( S^{-1}M(S^{-1}P) = S^{-1}M(P) \).

Proof. Let \( m/s \in S^{-1}M(S^{-1}P) \), where \( m \in M, s \in S \). Then, there exists an ideal \( S^{-1}I \subseteq S^{-1}P \) of \( S^{-1}R \) (so \( I \not\subseteq P \)) such that \( (S^{-1}I)m/s \subseteq (S^{-1}P)(S^{-1}M) \). Let \( r \in I \). Then, there are elements \( x \in PM \) and \( t \in S \) with \( (rm)/s = x/t \), so \( rutm = ux \in PM \) for some \( u \in S \), and so \( m/s = (mtu)/stu \in S^{-1}M(P) \). Thus, \( S^{-1}M(S^{-1}P) \subseteq S^{-1}M(P) \). For the other containment, let \( a/t \in S^{-1}M(P) \). Then, there is an ideal \( J \not\subseteq P \) of \( R \) (so \( S^{-1}J \not\subseteq S^{-1}P \)) such that \( Ja \subseteq PM \). Let \( c/w \in S^{-1}J \), where \( c \in J \) and \( w \in S \). Then \( (ca)/(wt) \in S^{-1}(Ja) \subseteq S^{-1}(PM) \subseteq (S^{-1}P) \), as needed.

Let \( S \) be a multiplicatively closed subset of \( R \), \( M \) be a representable \( R \)-module, and \( P \) be a prime ideal of \( R \). We put \( S^{-1}(\text{Att}_P(M)) = \{S^{-1}(M(P)) : P \in \text{Att}_R(M)\} \).

Theorem 2.15. Let \( M \) be a finitely generated representable \( R \)-module. Then \( \text{Att}_{S^{-1}P}(S^{-1}M) \subseteq S^{-1}(\text{Att}_P(M)) \).

Proof. Let \( S^{-1}M(S^{-1}P) \in \text{Att}_{S^{-1}P}(S^{-1}M) \), where \( S^{-1}P \in \text{Att}_{S^{-1}R}(S^{-1}M) \subseteq S^{-1}(\text{Att}_R(M)) \) by [7, Theorem 4.7]. So, there exists \( P' \in \text{Att}_R(M) \) such that \( S^{-1}P = S^{-1}P' \). By Lemma 2.14, we must have \( S^{-1}M(S^{-1}P) = S^{-1}M(S^{-1}P') = S^{-1}M(P') \in S^{-1}(\text{Att}_P(M)) \).

Recall that the sets of associated and supported prime ideals of a given \( R \)-module \( M \) are defined, respectively, as:
Lemma 2.16. Let $M$ be a finitely generated representable $R$-module. Then $\text{Att}_R(M) \subseteq \text{Supp}_R(M)$.

Proof. It suffices to show that $(0 : M) \subseteq P$ for every $P \in \text{Att}_R(M)$, since $M$ is finitely generated. Let $M = \sum_{i=1}^k M_i$ be a minimal secondary representation of $M$ with $\text{Att}(M) = \{P_1, \ldots, P_k\}$. Then By [7, Theorem 2.3], for each $j$, we obtain that $(0 : M) = \subseteq \cap_{i=1}^k P_i \subseteq P_j$, as needed. \hfill \Box

Theorem 2.17. Let $M$ be a multiplication representable $R$-module with $\text{Att}_R(M) = \{P_1, \ldots, P_k\}$. Then $\text{Att}_P(M) = \text{Spec}(M) = \{M(P_1), \ldots, M(P_k)\}$.

Proof. By [2, Theorem 2.2], $M$ is finitely generated, $PM \neq M$ and $(PM : M) = P$ for every $P \in \text{Att}_R(M)$, so $M(P) \neq M$ is a prime submodule of $M$ for every $P \in \text{Att}_R(M)$. Now, the assertion follows from [2, Theorem 2.6] and Lemma 1.1. \hfill \Box

We recall from [3], the set of associated and of supported prime submodules of $M$ as

$$\text{Ass}_P(M) = \{M(P) : P \in \text{Ass}_R(M)\},$$

and

$$\text{Supp}_P(M) = \{M(P) : p \in \text{Supp}_R(M)\}.$$

Theorem 2.18. Let $R$ be a commutative ring and let $M$ be a multiplication representable $R$-module with $\text{Att}_R(M) = \{P_1, \ldots, P_k\}$. Then, $\text{Att}_P(M) = \text{Ass}_P(M) = \text{Supp}_P(M) = \{M(P_1), \ldots, M(P_k)\}$.

Proof. Apply Theorem 2.17 and [3, Lemma 2.2]. \hfill \Box
References


