MULTIPLE POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS INVOLVING HARDY TERMS AND CRITICAL SOBOLEV-HARDY EXPONENTS

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Abstract
In this paper, we study the multiplicity of positive solutions for semilinear elliptic equations with critical Sobolev-Hardy exponents and Hardy terms by variational methods and some analysis techniques.

1. Introduction and Main Results
In this paper, we study the multiplicity of positive solutions for the following semi-linear elliptic equation

\[
\begin{align*}
-\Delta u - \frac{u}{|x|^2} u &= \lambda |x|^{q-2} u + \frac{|x|^{2^*(s)-2}}{|x|^s} u \quad \text{in } \Omega \setminus \{0\}, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.1)

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where $0 \in \Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial \Omega$, $\lambda > 0$, $1 \leq q < 2$, $0 \leq \mu < \bar{\mu} = (N - 2)^2 / 4$, $\bar{\mu}$ is the best constant in the Hardy inequality, $0 \leq s < 2$ and $2^*(s) = 2(N - s) / (N - 2)$ is the critical Sobolev-Hardy exponent; note that $2^*(0) = 2N / (N - 2)$ is the critical Sobolev exponent.

When $\mu = 0$ and $s = 0$, (1.1) has been studied extensively for $2 < p \leq 2^*$ and various $q > 1$. See, e.g., [1, 2, 7] and the references therein. Ambrosetti et al. [1] proved that there exists $\Lambda^* > 0$ such that (1.1) has at least two positive solutions for $\lambda \in (0, \Lambda^*)$. Chen in [8] obtained the existence of two positive solutions by extending the problem in [1] to involve Hardy term under the condition of $\lambda \in (0, \Lambda^*)$, $0 \leq \mu < \bar{\mu} - 1$, and $s = 0$. Bouchekif and Matallah in [3] extended the paper [8] and proved that there exists $\Lambda^* > 0$ such that (1.1) admits at least two positive solutions for $\lambda \in (0, \Lambda^*)$ when $0 \leq \mu < \bar{\mu} - 1$ and $0 < s < 2$. In this paper, motivated by [3], we extend and improve the paper by Bouchekif and Matallah in [3]. We shall deal with the cases $0 \leq \mu < \bar{\mu}$, $0 \leq s < 2$ instead of $0 \leq \mu < \bar{\mu} - 1$, $0 \leq s < 2$.

Due to lack of compactness of the embeddings in $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$, $H^1_0(\Omega) \hookrightarrow L^2(\Omega, |x|^{-1} dx)$, and $H^1_0(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |x|^{-s} dx)$, we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais-Smale ($(PS)$ for short) condition in $H^1_0(\Omega)$. Our arguments are similar to those in [12] and [13], which are based on Ekeland’s variational principle [9].

On $H^1_0(\Omega)$, we use the norm $\|u\| = \left( \int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2} dx \right)^{1/2}$. Thanks to the Hardy inequality, the norm $\| \cdot \|$ is equivalent to the usual norm on $H^1_0(\Omega)$. The following minimization problem (also the Sobolev constant) will be useful in what follows:
\[ A_{\mu,s}(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{2^*(s)}}. \]

From Ghoussoub and Kang [10, Theorem 3.1], we know that \( A_{\mu,s}(\Omega) \) is independent of \( \Omega \), so we denote \( A_{\mu,s} \) instead of \( A_{\mu,s}(\Omega) \). Set

\[ \Lambda_1 = \left( \frac{2^*(s) - 2}{2^*(s) - q} \right)^{2^*(s)-2} \frac{2 - q}{2^*(s) - q} \frac{d_\Omega - sq}{2^*(s) d_\Omega} \frac{\Omega}{2^*(s)} A_{\mu,s} \left( \frac{2-q(N-s)}{2^{(2-s)\frac{q}{2}}} \right) > 0, \]

where \( d_\Omega = \sup_{x \in \Omega} |x| \) and \( |\Omega| \) is the Lebesgue measure of \( \Omega \).

Here are the main results of this paper:

**Theorem 1.1.** Suppose that \( N \geq 3, 0 \leq \mu < \bar{\mu}, 0 \leq s < 2 \). If \( \lambda \in (0, \Lambda_1) \), then (1.1) has at least one positive solution in \( H_0^1(\Omega) \).

**Theorem 1.2.** Suppose that \( N \geq 3, 0 \leq \mu < \bar{\mu}, 0 \leq s < 2 \). Then there exists \( \Lambda_2 > 0 \), such that for \( \lambda \in (0, \Lambda_2) \), (1.1) has at least two positive solutions in \( H_0^1(\Omega) \).

This paper is organized as follows. In Section 2, we give some properties of Nehari manifold. In Sections 3 and 4, we complete proofs of Theorems 1.1 and 1.2. Before ending this section, we explain some notations employed in this paper: \( B_r(x) \) is the ball centered at \( x \in \mathbb{R}^N \) with the radius \( r > 0 \), \( |\Omega| \) is the Lebesgue measure of \( \Omega \), \( d_\Omega = \sup_{x \in \Omega} |x| \) is the diameter of \( \Omega \), \( L^p(\Omega, |x|^{-s}) \) is the usual weighted \( L^p(\Omega) \) space with the weight \( |x|^{-s} \) for \( p \geq 1 \), \( H^{-1}(\Omega) \) denotes the dual space of \( H_0^1(\Omega) \), \( O(\varepsilon^t) \) is the quantity satisfying \( |O(\varepsilon^t)|/\varepsilon^t \leq C, o(\varepsilon^t) \) means \( |o(\varepsilon^t)|/\varepsilon^t \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) and \( o(1) \), a generic infinitesimal value. In the following argument, we always employ \( C, C_1, \ldots \), to denote various positive constants.
2. Nehari Manifold

Associated with (1.1), we consider the energy functional $J$ in $H^1_0(\Omega)$

$$J(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \frac{1}{2^{*}(s)} \int_\Omega \frac{|u|^{2^{*}(s)}}{|x|^{s}} \, dx \quad \text{for } u \in H^1_0(\Omega).$$

By the Hardy inequality, $J \in C^1(H^1_0(\Omega), \mathbb{R})$. It is well known that the solutions of (1.1) are the critical points of the energy functional $J$ on $H^1_0(\Omega)$.

As the energy functional $J$ is not bounded below on $H^1_0(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in H^1_0(\Omega) \setminus \{0\} : \langle J'(u), u \rangle = 0\}.$$ 

Thus, $u \in \mathcal{N}_\lambda$, if and only if

$$\langle J'(u), u \rangle = \|u\|^2 - \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \frac{1}{2^{*}(s)} \int_\Omega \frac{|u|^{2^{*}(s)}}{|x|^{s}} \, dx = 0. \quad (2.1)$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of (1.1). Moreover, we have the following results.

**Lemma 2.1.** The energy functional $J$ is coercive and bounded below on $\mathcal{N}_\lambda$.

**Proof.** If $u \in \mathcal{N}_\lambda$, then by (2.1), the Hölder inequality and the Sobolev-Hardy inequality

$$J(u) = \frac{2^{*}(s) - 2}{2^{*}(s)q} \|u\|^2 - \lambda \left(\frac{2^{*}(s) - q}{2^{*}(s)q}\right) \int_\Omega |u|^q \, dx$$

$$\geq \frac{2 - s}{2(N - s)} \|u\|^2 - \lambda \left(\frac{2^{*}(s) - q}{2^{*}(s)q}\right), \quad (2.2)$$
\[
\left( \int_{\Omega} |x|^{\frac{sq}{2^*(s) - q}} \, dx \right)^{\frac{2^*(s) - q}{2^*(s)}} \left( \int_{\Omega} \frac{|u|^q}{|x|^s} \, dx \right)^{\frac{q}{2^*(s)}} \\
\geq \frac{2 - s}{2(N - s)} \|u\|^2 - \lambda \left( \frac{2^*(s) - q}{2^*(s)} \right) \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \\
\geq \frac{2^*(s) - q}{2^*(s)} \lambda \left( \int_{\Omega} |x|^s \, dx \right)^{\frac{2^*(s) - q}{2^*(s)}} A_{\mu, s} \|u\|^2.
\] (2.3)

Thus, \( J \) is coercive and bounded below on \( N_\lambda \). \( \square \)

Define
\[
\psi(u) = \langle J'(u), u \rangle.
\]

Then for \( u \in N_\lambda \),
\[
\langle \psi(u), u \rangle = 2\|u\|^2 - \lambda q \int_{\Omega} \frac{|u|^q}{|x|^s} \, dx - 2^*(s) \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx
\]
\[
= (2 - q)\|u\|^2 - (2^*(s) - q) \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx
\]
\[
= \lambda (2^*(s) - q) \int_{\Omega} |x|^s \, dx - (2^*(s) - 2)\|u\|^2.
\] (2.4)

Similar to the method used in Tarantello [12], we split \( N_\lambda \) into three parts:

\[
N_\lambda^+ = \{ u \in N_\lambda : \langle \psi(u), u \rangle > 0 \},
\]
\[
N_\lambda^0 = \{ u \in N_\lambda : \langle \psi(u), u \rangle = 0 \},
\]
\[
N_\lambda^- = \{ u \in N_\lambda : \langle \psi(u), u \rangle < 0 \}.
\]

Then, we have the following results.

**Lemma 2.2.** Assume that \( u_\lambda \) is a local minimizer for \( J \) on \( N_\lambda \) and \( u_\lambda \notin N_\lambda^0 \). Then \( J'(u_\lambda) = 0 \) in \( H^{-1}(\Omega) \).
Proof. The proof is similar to the Theorem 2.3 of the paper [5]. □

Lemma 2.3. If \( \lambda \in (0, \Lambda_1) \), then \( \mathcal{N}_{\lambda}^0 = \emptyset \), where \( \Lambda_1 \) is the same as in (1.3).

Proof. Suppose otherwise, that is, there exists \( \lambda \in (0, \Lambda_1) \), such that \( \mathcal{N}_{\lambda}^0 \neq \emptyset \). Then by (2.4) and (2.5), for \( u \in \mathcal{N}_{\lambda}^0 \), we have

\[
\|u\|^2 = \frac{2^*(s) - q}{2 - q} \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx,
\]

and

\[
\|u\|^2 = \frac{\lambda^{2^*(s)} - q}{2^*(s) - 2} \int_{\Omega} |u|^q \, dx.
\]

Moreover, by the Hölder inequality and the Sobolev-Hardy inequality, we have

\[
\|u\| \geq \left( \frac{2 - q}{2^*(s) - q} A_{\mu, s}^{\frac{2^*(s)}{2}} \right)^{\frac{1}{2^*(s)-2}},
\]

and

\[
\|u\| \leq \left[ \lambda \frac{2^*(s) - q}{2^*(s) - 2} \int_{\Omega} \frac{|u|^{2^*(s) - q}}{|\Omega|^s} A_{\mu, s}^{\frac{q}{2}} \right]^{\frac{1}{2^*(s)-2}}.
\]

This implies

\[
\lambda \geq \left( \frac{2^*(s) - 2}{2^*(s) - q} \right)^{\frac{2 - q}{2^*(s) - q}} \frac{2^*(s) - q}{2^*(s) - 2} \int_{\Omega} \frac{|u|^{2^*(s) - q}}{|\Omega|^s} A_{\mu, s}^{\frac{(2-q)(N-s)}{2(2-s)}} \frac{q}{2} = \Lambda_1,
\]

which is a contradiction. Thus, we can conclude that if \( \lambda \in (0, \Lambda_1) \), we have \( \mathcal{N}_{\lambda}^0 = \emptyset \).

By Lemma 2.3, we write \( \mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^- \) and define
\[ \alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} J(u); \quad \alpha^+_\lambda = \inf_{u \in \mathcal{N}^+_\lambda} J(u); \quad \alpha^-_\lambda = \inf_{u \in \mathcal{N}^-_\lambda} J(u). \]

Then, we get the following result.

**Theorem 2.4.** (i) If \( \lambda \in (0, \Lambda_1) \), then we have \( \alpha_\lambda \leq \alpha^+_\lambda < 0 \);

(ii) If \( \lambda \in (0, \frac{q}{2} \Lambda_1) \), then \( \alpha^-\lambda > d_0 \) for some positive constant \( d_0 \) depending on \( \lambda, \mu, s, q, N, A_{\mu, s}, d_\Omega \), and \( |\Omega| \).

**Proof.** (i) Let \( u \in \mathcal{N}^+_\lambda \). By (2.4),

\[
\frac{2-q}{2^*(s)-q} \|u\|^2 > \int_\Omega \frac{|u|^{2^*(s)}}{|x|^q} dx,
\]

and so

\[
J(u) = \left( \frac{1}{2} - \frac{1}{q} \right) \|u\|^2 + \left( \frac{1}{q} - \frac{1}{2^*(s)} \right) \int_\Omega \frac{|u|^{2^*(s)}}{|x|^q} dx
\]

\[
< \left[ \frac{1}{2} - \frac{1}{q} \right] + \left( \frac{1}{q} - \frac{1}{2^*(s)} \right) \frac{2-q}{2^*(s)-q} \|u\|^2
\]

\[
= - \frac{(2-q)(2-s)}{2q(N-s)} \|u\|^2 < 0.
\]

Therefore, from the definitions of \( \alpha_\lambda, \alpha^+_\lambda \), we can deduce that \( \alpha_\lambda \leq \alpha^+_\lambda < 0 \).

(ii) Let \( u \in \mathcal{N}^-\lambda \). By (2.4),

\[
\frac{2-q}{2^*(s)-q} \|u\|^2 < \int_\Omega \frac{|u|^{2^*(s)}}{|x|^q} dx.
\]

Moreover, by the Sobolev-Hardy inequality,

\[
\int_\Omega \frac{|u|^{2^*(s)}}{|x|^q} dx \leq A_{\mu, s} \frac{2^*(s)}{2} \|u|^{2^*(s)}.
\]
This implies
\[
\|u\| > \left( \frac{2 - q}{2^s(s) - q} \right) \frac{1}{2^s(s) - 2} \frac{N_{-s}}{A_{\mu, s}^{2(2-s)}} \text{ for all } u \in \mathcal{N}_{\lambda}^{-},
\]

By (2.3) in the proof of Lemma 2.1,
\[
J(u) \geq \|u\|^q \left[ \frac{2 - s}{2(N - s)} \|u\|^{2-q} - \mu \left( \frac{2^s(s) - q}{2^s(s)} \right) d_{\Omega}^{2s(s)} \frac{q}{2^s(s)q} \left( \frac{2 - q}{2(N - s)} \right) \right]
\]
\[
\geq \left( \frac{2 - q}{2^s(s) - q} \right)^{\frac{q}{2s(s)-2}} \frac{q}{2^s(s)} \frac{d_{\Omega}^{2s(s)}}{A_{\mu, s}^{2(2-s)}} \left[ \frac{2 - s}{2(N - s)} \left( \frac{2 - q}{2^s(s) - q} \right) \right]^{\frac{2-q}{2^s(s)-2}} A_{\mu, s}^{\frac{q}{2}}
\]
\[
- \mu \left( \frac{2^s(s) - q}{2^s(s)} \right) d_{\Omega}^{2s(s)} \frac{q}{2^s(s)q} \left( \frac{2 - q}{2(N - s)} \right) \right].
\]

Thus, if \( \lambda \in (0, \frac{q}{2} \Lambda_1) \), then
\[
J(u) > d_0 \text{ for all } u \in \mathcal{N}_{\lambda}^{-},
\]
for some positive constant \( d_0 = d_0(\lambda, \mu, s, q, N, A_{\mu, s}, d_{\Omega}, |\Omega|) \). This completes the proof.

\[\Box\]

**Lemma 2.5.** Let \( \lambda \in (0, \Lambda_1) \), then for every \( u \in H_0^1(\Omega), u \neq 0 \), there exist unique \( t^+ = t^+(u), t^- = t^-(u) > 0 \), such that \( t^+ u \in \mathcal{N}^+ \) and \( t^- u \in \mathcal{N}^- \). In particular, we have
\[
t^- > \left( \frac{(2 - q)\|u\|^2}{(2^s(s) - q) \int_{\Omega} \frac{\|u\|^{2^s(s)}}{|x|} dx} \right)^{\frac{1}{2^s(s)-2}} = t_{\text{max}} > t^+,
\]
\[
J(t^+ u) = \min_{0 \leq t \leq t_{\text{max}}} J(tu) \text{ and } J(t^- u) = \max_{t \geq 0} J(tu).
\]

**Proof.** The proof is similar to the Lemma 2.6 of the paper [6]. \( \Box \)
3. Proof of Theorem 1.1

First, we shall use the idea of Tarantello [12] to get the following results.

**Lemma 3.1.** For each \( u \in \mathcal{N}_\lambda \), there exist \( \epsilon > 0 \) and a differentiable function \( \xi : B(0; \epsilon) \subset H \to \mathbb{R}^+ \), such that \( \xi(0) = 1 \), the function \( \xi(u) \) \((u-v) \in \mathcal{N}_\lambda\) and

\[
\langle \xi'(0), v \rangle = \frac{2 \int_{\Omega} \nabla u \nabla vdx - \lambda q \int_{\Omega} |u|^{q-2} u v dx - 2^*(s) \int_{\Omega} \frac{|f|^{2^*(s)-2} u v dx}{|f|^{s}}}{(2-q)\|u\|^2 - (2^*(s)-q)\int_{\Omega} \frac{|f|^{2^*(s)} dx}{|f|^{s}}, \quad (3.1)
\]

for all \( v \in H \).

**Proof.** For \( u \in \mathcal{N}_\lambda \), define a function \( F : \mathbb{R} \times H \to \mathbb{R} \) by

\[
F_u(\xi, w) = \langle J'(\xi(u-w)), \xi(u-w) \rangle \\
= \xi^2 \int_{\Omega} |\nabla (u-w)|^2 dx - \xi q \int_{\Omega} |u-w|^q dx \\
- \xi 2^*(s) \int_{\Omega} \frac{|u-w|^{2^*(s)}}{|f|^{s}} dx.
\]

Then \( F_u(1, 0) = \langle J'(u), u \rangle = 0 \) and

\[
\frac{d}{d \xi} F_u(1, 0) = 2\|u\|^2 - \lambda q \int_{\Omega} |u|^q dx - 2^*(s) \int_{\Omega} \frac{|f|^{2^*(s)}}{|f|^{s}} dx \\
= (2-q)\|u\|^2 - (2^*(s)-q)\int_{\Omega} \frac{|f|^{2^*(s)}}{|f|^{s}} dx \neq 0.
\]

According to the implicit function theorem, there exist \( \epsilon > 0 \) and a differentiable function \( \xi : B(0; \epsilon) \subset H \to \mathbb{R} \), such that \( \xi(0) = 1 \),
\[
\langle \xi'(0), v \rangle = \frac{2 \int_{\Omega} \nabla u \nabla v dx - \lambda q \int_{\Omega} |u|^{q-2} u v dx - 2^* \left( 2 \int_{\Omega} \frac{|f|^2}{|f|^s} u v dx \right)}{(2-q)\|u\|^2 - (2^*(s)-q) \int_{\Omega} \frac{|f|^2}{|f|^s} dx},
\]

and

\[
F_u(\xi(v), v) = 0, \quad \text{for all } v \in B(0; \epsilon),
\]

which is equivalent to

\[
\langle J'(\xi(v)(u-v)), \xi(v)(u-v) \rangle = 0, \quad \text{for all } v \in B(0; \epsilon),
\]

that is, \( \xi(v)(u-v) \in \mathcal{N}_\lambda \).

**Lemma 3.2.** For each \( u \in \mathcal{N}_\lambda, \) there exist \( \epsilon > 0 \) and a differentiable function \( \xi^- : B(0; \epsilon) \subset H \rightarrow \mathbb{R}^+ \), such that \( \xi^-(0) = 1 \), the function \( \xi^-(v)(u-v) \in \mathcal{N}_\lambda \) and

\[
\langle (\xi^-)'(0), v \rangle = \frac{2 \int_{\Omega} \nabla u \nabla v dx - \lambda q \int_{\Omega} |u|^{q-2} u v dx - 2^* \left( 2 \int_{\Omega} \frac{|f|^2}{|f|^s} u v dx \right)}{(2-q)\|u\|^2 - (2^*(s)-q) \int_{\Omega} \frac{|f|^2}{|f|^s} dx},
\]

for all \( v \in H \).

**Proof.** Similar to the argument in Lemma 3.1, there exist \( \epsilon > 0 \) and a differentiable function \( \xi^- : B(0; \epsilon) \subset H \rightarrow \mathbb{R} \), such that \( \xi^-(0) = 1 \) and \( \xi^-(v)(u-v) \in \mathcal{N}_\lambda \) for all \( v \in B(0; \epsilon) \). Since

\[
\langle \psi'(u), u \rangle = (2-q)\|u\|^2 - (2^*(s)-q) \int_{\Omega} \frac{|f|^2}{|f|^s} dx < 0.
\]

Thus, by the continuity of the function \( \xi^- \), we have

\[
\langle \psi'(\xi^-(v)(u-v)), \xi^-(v)(u-v) \rangle = (2-q)\|\xi^-(v)(u-v)\|^2 - (2^*(s)-q) \int_{\Omega} |\xi^-(v)(u-v)|^2 dx < 0,
\]
if $\epsilon$ is sufficiently small, this implies that $\xi^-(v)(u - v) \in \mathcal{N}_\lambda^-$. 

We give some definitions about the $(PS)$-sequence for $J$.

**Definition 3.3.** Let $c \in \mathbb{R}$ and $J \in C^1(H^1_0(\Omega), \mathbb{R})$.

(i) $\{u_n\}$ is a $(PS)_c$-sequence in $H^1_0(\Omega)$ for $J$, if $J(u_n) = c + o(1)$ and $J'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.

(ii) We say that $J$ satisfies the $(PS)_c$-condition, if any $(PS)_c$-sequence $\{u_n\}$ in $H^1_0(\Omega)$ for $J$ has a convergent subsequence.

**Proposition 3.4.** (i) If $\lambda \in (0, \Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda$ in $H^1_0(\Omega)$ for $J$;

(ii) If $\lambda \in (0, \frac{q}{2} \Lambda_1)$, then there exists a $(PS)_{-\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in $H^1_0(\Omega)$ for $J$.

**Proof.** (i) By Lemma 3.1 and the Ekeland variational principle [9], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$, such that

$$J(u_n) < \alpha_\lambda + \frac{1}{n},$$

$$J(u_n) < J(w) + \frac{1}{n} \|w - u_n\| \text{ for each } w \in \mathcal{N}_\lambda.$$  

(3.2)

By $\alpha_\lambda < 0$ and taking $n$ large, we have

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{2^*(s)}\right) \lambda \int_{\Omega} |u_n|^q dx$$

$$< \alpha_\lambda + \frac{1}{n} < \frac{\alpha_\lambda}{2} < 0.$$  

(3.3)

From (2.3) and (3.3), $\alpha_\lambda < 0$, the Hölder inequality, and the Sobolev-Hardy inequality, we deduce that
\[\frac{sq}{\lambda d_{\Omega} \left(\frac{2^*(s)-q}{2^*(s)}\right) A_{\mu,s}} \|u_n\|^q \geq \lambda \int_{\Omega} |u_n|^q \, dx > \frac{-2^*(s)q}{2(2^*(s)-q)} \alpha_\lambda > 0. \quad (3.4)\]

Consequently, \( u_n \neq 0 \) and putting together (3.3), (3.4), the Hölder inequality, and the Sobolev-Hardy inequality, we obtain

\[
\|u_n\| > \left[ \frac{-2^*(s)q}{2\lambda(2^*(s)-q)} \alpha_\lambda d_{\Omega} \left(\frac{2^*(s)-q}{2^*(s)}\right) A_{\mu,s}^{\frac{q}{2}} \right]^{1/q},
\]

\[
\|u_n\| < \left[ \frac{2(2^*(s)-q)}{q(2^*(s)-2)} \lambda d_{\Omega} \left(\frac{2^*(s)-q}{2^*(s)}\right) A_{\mu,s}^{\frac{q}{2}} \right]^{1/(2-q)}. \quad (3.5)
\]

Now, we show that

\[\|J'(u_n)\|_{H^{-1}(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.\]

Applying Lemma 3.1 with \( u_n \) to obtain the functions \( \xi_n : B(0; \epsilon_n) \to \mathbb{R}^+ \) for some \( \epsilon_n > 0 \), such that \( \xi_n(w)(u_n - w) \in \mathcal{N}_\lambda \). Choose \( 0 < \rho < \epsilon_n \). Let \( u \in H \) with \( u \neq 0 \) and let \( w_\rho = \frac{\rho u}{\|u\|} \). We set \( \eta_\rho = \xi_n(w_\rho)(u_n - w_\rho) \).

Since \( \eta_\rho \in \mathcal{N}_\lambda \), we deduce from (3.2) that

\[J(\eta_\rho) - J(u_n) \geq -\frac{1}{n} \|\eta_\rho - u_n\|,\]

and by the mean value theorem, we have

\[\langle J'(u_n), \eta_\rho - u_n \rangle + o(\|\eta_\rho - u_n\|) \geq -\frac{1}{n} \|\eta_\rho - u_n\|.\]

Thus,

\[\langle J'(u_n), -w_\rho \rangle + (\xi_n(w_\rho) - 1)\langle J'(u_n), (u_n - w_\rho) \rangle \geq -\frac{1}{n} \|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \quad (3.6)\]

Since, \( \xi_n(w_\rho)(u_n - w_\rho) \in \mathcal{N}_\lambda \) and (3.6), it follows that
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- \( \rho \langle J'(u_n), \frac{u}{\|u\|} \rangle + (\xi_n(w_\rho) - 1)\langle J'(u_n) - J'(\eta_\rho), (u_n - w_\rho) \rangle \)

\[ \geq -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \]

Thus,

\[ \langle J'(u_n), \frac{u}{\|u\|} \rangle \leq \frac{\|\eta_\rho - u_n\|}{n\rho} + o(\|\eta_\rho - u_n\|) \]

\[ + \left( \frac{\xi_n(w_\rho) - 1}{\rho} \right) \langle J'(u_n) - J'(\eta_\rho), (u_n - w_\rho) \rangle. \quad (3.7) \]

Since, \( \|\eta_\rho - u_n\| \leq \rho\xi_n(w_\rho) + |\xi_n(w_\rho) - 1|\|u_n\| \) and

\[ \lim_{\rho \to 0} \left| \frac{\xi_n(w_\rho) - 1}{\rho} \right| \leq \|\xi_n'(0)\|, \]

if we let \( \rho \to 0 \) in (3.7) for a fixed \( n \), then by (3.5), we can find a constant \( C > 0 \), independent of \( \rho \), such that

\[ \langle J'(u_n), \frac{u}{\|u\|} \rangle \leq \frac{C}{n}(1 + \|\xi_n'(0)\|). \]

The proof will be complete once we show that \( \|\xi_n'(0)\| \) is uniformly bounded in \( n \). By (3.1), (3.5), the Hölder inequality, and the Sobolev-Hardy inequality, we have

\[ \langle \xi_n'(0), v \rangle \leq \frac{b\|v\|}{\| (2 - q)^{\frac{q}{2}} - (2^*(s) - q) \frac{\|u_n\|^{2^{*}(s)}}{\|\xi_n'(0)\|} dx \] for some \( b > 0 \).

We only need to show that

\[ |(2 - q)^{\frac{q}{2}} - (2^*(s) - q)\frac{\|u_n\|^{2^*(s)}}{\|\xi_n'(0)\|} dx | > C, \]

for some \( C > 0 \) and \( n \) large enough. We argue by contradiction. Assume that there exists a subsequence \( \{u_n\} \), such that
\[(2 - q)\|u_n\|^2 - (2^* (s) - q) \int_{\Omega} \frac{|u_n|^{2^*(s)}}{|f|} \, dx = o(1). \tag{3.8}\]

By (3.8) and the fact that \( u_n \in N_{\lambda} \), we get
\[
\|u_n\|^2 = \frac{2^*(s) - q}{2 - q} \int_{\Omega} \frac{|u_n|^{2^*(s)}}{|f|} \, dx + o(1),
\]

and
\[
\|u_n\|^2 = \lambda \frac{2^*(s) - q}{2^*(s) - 2} \int_{\Omega} |u_n|^q \, dx + o(1).
\]

Moreover, by the Hölder inequality and the Sobolev-Hardy inequality, we have
\[
\|u_n\| \geq \left[ \frac{2 - q}{2^*(s) - q} A_{\mu}^{\frac{\nu^* (s)}{2^*(s) - 2}} \right]^{\frac{1}{2-q}} + o(1),
\]

and
\[
\|u_n\| \leq \left[ \lambda \frac{2^*(s) - q}{2^*(s) - 2} d_{\Omega}^{\frac{s q}{2^*(s) - 2}} \right] \frac{2^*(s) - q}{2^*(s) - q} A_{\mu, s}^{\frac{q}{2}} + o(1).
\]

This implies \( \lambda \geq \Lambda_1 \), which is a contradiction. We obtain
\[
\langle J'(u_n), \frac{u}{\|u\|} \rangle \leq \frac{C}{n}.
\]

This completes the proof of (i).

(ii) Similarly, by applying Lemma 3.2, we can prove (ii). We will omit detailed proof here.

Now, we establish the existence of a local minimum for \( J \) on \( N_{\lambda}^+ \).

**Theorem 3.5.** If \( \lambda \in (0, \Lambda_1) \), then \( J \) has a minimizer \( u_{\lambda} \) in \( N_{\lambda}^+ \) and it satisfies
(i) \( J(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+; \)

(ii) \( u_\lambda \) is a positive solution of (1.1).

**Proof.** By Proposition 3.4 (i), there exists a minimizing sequence \( \{u_n\} \) for \( J \) on \( N_\lambda \), such that

\[
J(u_n) = \alpha_\lambda + o(1) \quad \text{and} \quad J'(u_n) = o(1) \quad \text{in} \quad H^{-1}(\Omega). \tag{3.9}
\]

Since \( J \) is coercive on \( N_\lambda \) (see Lemma 2.1), we get that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Going if necessary to a subsequence, we can assume that there exists \( u_\lambda \in H^1_0(\Omega) \), such that

\[
\begin{aligned}
&u_n \rightharpoonup u_\lambda \quad \text{weakly in} \quad H^1_0(\Omega), \\
&u_n \rightharpoonup u_\lambda \quad \text{weakly in} \quad L^{2^*(s)}(\Omega, |\cdot|^s), \\
&u_n \rightarrow u_\lambda \quad \text{almost everywhere in} \quad \Omega, \\
&u_n \rightarrow u_\lambda \quad \text{strongly in} \quad L^r(\Omega) \quad \text{for all} \quad 1 \leq r < 2^*.
\end{aligned} \tag{3.10}
\]

First, we claim that \( u_\lambda \) is a nontrivial solution of (1.1). By (3.9) and (3.10), it is easy to see that \( u_\lambda \) is a solution of (1.1). From \( u_n \in N_\lambda \) and (2.2), we deduce that

\[
\lambda \int_\Omega |u_n|^q \, dx = \frac{q(2^*(s) - 2)}{2(2^*(s) - q)} \|u_n\|^2 - \frac{2^*(s)q}{2^*(s) - q} J(u_n). \tag{3.11}
\]

Let \( n \to \infty \) in (3.11), by (3.9), (3.10), and \( \alpha_\lambda < 0 \), we get

\[
\lambda \int_\Omega |u_\lambda|^q \, dx \geq - \frac{2^*(s)q}{2^*(s) - q} \alpha_\lambda > 0.
\]

Thus, \( u_\lambda \in N_\lambda \) is a nontrivial solution of (1.1). Now, we prove that \( u_n \to u_\lambda \) strongly in \( H^1_0(\Omega) \) and \( J(u_\lambda) = \alpha_\lambda \). By (3.11), if \( u \in N_\lambda \), then

\[
J(u) = \frac{2 - s}{2(N - s)} \|u\|^2 - \frac{2^*(s) - q}{2^*(s)q} \lambda \int_\Omega |u|^q \, dx. \tag{3.12}
\]
In order to prove that \( J(u_\lambda) = \alpha_\lambda \), it suffices to recall that \( u_\lambda \in \mathcal{N}_\lambda \), by (3.12) and applying Fatou’s lemma to get

\[
\alpha_\lambda \leq J(u_\lambda) = \frac{2-s}{2(N-s)} \|u_\lambda\|^2 \left(\frac{2^*(s) - q}{2^*(s)q}\right) \int_\Omega |u_\lambda|^q\,dx
\]

\[
\leq \liminf_{n \to \infty} \left(\frac{2-s}{2(N-s)} \|u_n\|^2 - \frac{2^*(s) - q}{2^*(s)q}\lambda \int_\Omega |u_n|^q\,dx\right)
\]

\[
\leq \liminf_{n \to \infty} J(u_n) = \alpha_\lambda.
\]

This implies that \( J(u_\lambda) = \alpha_\lambda \) and \( \lim_{n \to \infty} \|u_n\|^2 = \|u_\lambda\|^2 \). Let \( v_n = u_n - u_\lambda \), then by Brézis-Lieb lemma [4] implies that

\[
\|v_n\|^2 = \|u_n\|^2 - \|u_\lambda\|^2 + o(1).
\]

Therefore, \( u_n \to u_\lambda \) strongly in \( H^1_0(\Omega) \). Moreover, we have \( u_\lambda \in \mathcal{N}_\lambda^+ \).

On the contrary, if \( u_\lambda \in \mathcal{N}_\lambda^- \), then by Lemma 2.5, there are unique \( t_0^+ \) and \( t_0^- \) such that \( t_0^+ u_\lambda \in \mathcal{N}_\lambda^+ \) and \( t_0^- u_\lambda \in \mathcal{N}_\lambda^- \). In particular, we have \( 0 < t_0^+ < t_0^- = 1 \). Since

\[
\frac{d}{dt} J(t_0^+ u_\lambda) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J(t_0^+ u_\lambda) > 0,
\]

there exists \( t_0^+ < \bar{t} \leq t_0^- \) such that \( J(t_0^+ u_\lambda) < J(\bar{t} u_\lambda) \). By Lemma 2.5,

\[
J(t_0^+ u_\lambda) < J(\bar{t} u_\lambda) \leq J(t_0^- u_\lambda) = J(u_\lambda),
\]

which is a contradiction. Since \( J(u_\lambda) = J(|u_\lambda|) \) and \( |u_\lambda| \in \mathcal{N}_\lambda^+ \), by Lemma 2.2, we may assume that \( u_\lambda \) is a nontrivial nonnegative solution of (1.1). From the strong maximum principle, we deduce that \( u_\lambda \) is a positive solution of (1.1).

Now, we begin the proof of Theorem 1.1: By Theorem 3.5, we obtain (1.1) has a positive solution \( u_\lambda \) in \( H^1_0(\Omega) \).
Next, we shall establish the existence of the second positive solution of (1.1) by proving that $J$ satisfies the $(PS)_{c_S}$ condition.

**Lemma 4.1.** If $\{u_n\}$ is a $(PS)_{c_S}$ -sequence for $J$ with $u_n - u$ in $H^1_0(\Omega)$, then $J'(u) = 0$, and there exists a constant $C_0$ depending on $\mu$, $s$, $q$, $N$, $A_{\mu,s}$, $d_\Omega$, and $|\Omega|$, such that $J(u) \geq -C_0 \lambda^{\frac{2}{2-q}}$.

**Proof.** If $\{u_n\}$ is a $(PS)_{c_S}$ -sequence for $J$ with $u_n - u$ in $H^1_0(\Omega)$, it is easy to see that $J'(u) = 0$. This implies that $\langle J'(u), u \rangle = 0$, and

$$\int_{\Omega} \frac{|u|^p}{|x|^s} \, dx = |u|^2 - \lambda \int_{\Omega} |u|^q \, dx.$$ 

Consequently,

$$J(u) = \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \|u\|^2 - \left( \frac{1}{q} - \frac{1}{2^*(s)} \right) \lambda \int_{\Omega} |u|^q \, dx.$$ 

Using the Hölder inequality, the Young inequality, and the Sobolev-Hardy inequality, we have

$$J(u) = \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \|u\|^2 - \left( \frac{1}{q} - \frac{1}{2^*(s)} \right) \lambda \int_{\Omega} |u|^q \, dx$$

$$\geq \frac{2 - s}{2(N - s)} \|u\|^2 - \frac{2^*(s) - q}{2^*(s)q} \lambda \|u\|_{L^{2^*(s)}(\Omega)} \|u\|_{L^{2^*(s)}(\Omega)} \|u\|_{L^{2^*(s)}(\Omega)} \|u\|_{L^{2^*(s)}(\Omega)} \lambda$$

$$\geq \frac{2 - s}{2(N - s)} \|u\|^2 - \frac{2^*(s) - q}{2^*(s)q} \lambda \|u\|_{L^{2^*(s)}(\Omega)} \|u\|_{L^{2^*(s)}(\Omega)} \lambda$$

$$\geq \frac{2 - s}{2(N - s)} \|u\|^2 - \frac{2 - s}{2(N - s)} \|u\|^2 - C_0 \lambda^{\frac{2}{2-q}} = -C_0 \lambda^{\frac{2}{2-q}},$$

where $C_0$ is a positive constant depending on $\mu$, $s$, $q$, $N$, $A_{\mu,s}$, $d_\Omega$, and $|\Omega|$. \qed
Lemma 4.2. The functional $J$ satisfies the $(PS)_c$-condition for all

$$c \in \left( -\infty, \frac{2-s}{2(N-s)} A^{\frac{N-s}{s}} - C_0 \lambda^{\frac{2}{2-q}} \right),$$

where $C_0$ is the positive constant given in Lemma 4.1.

Proof. Let $\{u_n\} \subset H^1_0(\Omega)$ be a $(PS)_c$-sequence, which satisfies $J(u_n) = c + o(1)$ and $J'(u_n) = o(1)$. Using standard arguments, it follows that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Thus, there exists a subsequence still denoted by $\{u_n\}$ and a function $u \in H^1_0(\Omega)$, such that

$$\begin{align*}
&u_n - u \text{ weakly in } H^1_0(\Omega), \\
&u_n - u \text{ weakly in } L^{2s}(\Omega, \lvert \cdot \rvert^{-s}), \\
&u_n \to u \text{ almost everywhere in } \Omega, \\
&u_n \to u \text{ strongly in } L^r(\Omega) \text{ for all } 1 \leq r < 2^*. 
\end{align*}$$

By Lemma 4.1, we have that $J'(u) = 0$ and

$$\lambda \int_\Omega |u_n|^q \, dx = \lambda \int_\Omega |u|^q \, dx + o(1). \quad (4.1)$$

Let $v_n = u_n - u$. Then from the Brézis-Lieb lemma (see [4]), we obtain

$$\|v_n\|^q \leq \|u_n\|^q - \|u\|^q + o(1), \quad (4.2)$$

$$\int_\Omega \frac{|v_n|^{2s}(s) \rvert \cdot \rvert^{-s}}{\rvert x \rvert^{s}} \, dx = \int_\Omega \frac{|u_n|^{2s}(s) \rvert \cdot \rvert^{-s}}{\rvert x \rvert^{s}} \, dx - \int_\Omega \frac{|u|^{2s}(s) \rvert \cdot \rvert^{-s}}{\rvert x \rvert^{s}} \, dx + o(1). \quad (4.3)$$

Since, $J(u_n) = c + o(1)$, $J'(u_n) = o(1)$, and (4.1)-(4.3), we can deduce that

$$\frac{1}{2} \int_\Omega |v_n|^{2s}(s) \rvert \cdot \rvert^{-s} \, dx = c - J(u) + o(1), \quad (4.4)$$

and
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\[ \|v_n\|^2 \rightarrow l, \quad \int_{\Omega} \frac{|v_n|^{2^*(s)}}{|x|^s} dx \rightarrow l. \]  

(4.5)

By (1.2), we deduce that

\[ \|v_n\|^2 \geq A_{\mu,s} \left( \int_{\Omega} \frac{|v_n|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}, \]

then \( l \geq A_{\mu,s}^{\frac{N-s}{2^*(s)}}. \) Either \( l = 0 \) or \( l \geq A_{\mu,s}^{\frac{N-s}{2^*(s)}}. \) If \( l = 0 \), this completes the proof. Assume that \( l \geq A_{\mu,s}^{\frac{N-s}{2^*(s)}} \), from Lemma 4.1, (4.4) and (4.5), we get

\[ c \geq \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) l + J(u) \geq \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2^*(s)}} - C_0 \epsilon^{2^*(s)-q}, \]

which is a contradiction. Therefore, \( l = 0 \) and we conclude that \( u_n \rightarrow u \)

in \( H_0^1(\Omega) \).

From the discussion above, \( J \) satisfies \( (PS)_c \)-condition.

Let \( R_0 \) be a positive constants, such that \( B_{2R_0}(0) \subset \Omega \). We take a cut-off function \( \eta(x) \in C_0^\infty(\Omega) \), such that \( \eta(x) = 1 \) for \( |x| \leq R_0 \), \( \eta(x) = 0 \) for \( |x| > 2R_0 \), \( 0 \leq \eta \leq 1 \). From Lemma 2.2 in [11], we know that \( A_{\mu,s} \) is attained, when \( \Omega = \mathbb{R}^N \) by the functions

\[ y_\epsilon(x) = \left( \frac{2\epsilon(N-s)\sqrt{\mu - \mu}}{\sqrt{\mu}} \right)^{\frac{2^*(s)}{2^*(s) - s}} \left( \frac{\sqrt{\mu - \mu} - \epsilon}{\sqrt{\mu - \mu} - \epsilon} \right)^{\frac{N-s}{2^*(s) - s}}, \]

for all \( \epsilon > 0 \). Moreover, the function \( y_\epsilon(x) \) solve the equation
\[- \Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u \quad \text{in } \mathbb{R}^N \setminus \{0\}.\]

Let \( C_\varepsilon = \left( \frac{2\varepsilon (N - s) \sqrt{\mu - \mu}}{\sqrt{\mu}} \right)^{\frac{2-s}{2}} \), \( \gamma_1 = \sqrt{\mu} - \sqrt{\mu} - \mu \), \( \gamma_2 = \sqrt{\mu} + \sqrt{\mu} - \mu \),

\[ U_\varepsilon(x) = \frac{y_\varepsilon(x)}{C_\varepsilon}. \text{ Set} \]

\[ u_\varepsilon(x) = \eta(x) U_\varepsilon(x) = \frac{\eta(x)}{\varepsilon \left[ \varepsilon |x|^{\frac{3-\gamma_1}{2}} \right] + \left| x \right|^{\frac{2-s}{2}} \left[ \frac{\gamma_1}{\gamma_2} \right]^{\frac{2}{2-s}}}, \]

\[ v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\int_{\Omega} \frac{\left| u_\varepsilon(x)^{2^*(s)} \right|}{|x|^s} \, dx}, \]

so that \( \|v_\varepsilon\| L^{2^*(s)}(\Omega, |x|^{-s}) \) = \( \int_{\Omega} |v_\varepsilon|^{2^*(s)} |x|^{-s} \, dx = 1 \). Then, we can get the following results by the methods used in [10];

\[ \|v_\varepsilon\|^2 = A_{\mu, s} + O(\varepsilon^{\frac{N-2}{2-s}}). \tag{4.6} \]

**Lemma 4.3.** Suppose that \( N \geq 3 \), \( 0 \leq \mu < \overline{\mu} \), \( 0 \leq s < 2 \). Then there exist \( v_0 \in H_0^1(\Omega) \setminus \{0\} \) and \( \Lambda^* > 0 \), such that for \( \lambda \in (0, \Lambda^*) \), we have

\[ \sup_{t \geq 0} J(tv_0) < \frac{2-s}{2(N-s)} A_{\mu, s}^{\frac{N-s}{2-s}} - C_0 \lambda^{\frac{2}{2-s}}, \]

where \( C_0 \) is the positive constant given in Lemma 4.1.

In particular, \( \alpha_{\lambda, \varepsilon} < \frac{2-s}{2(N-s)} A_{\mu, s}^{\frac{N-s}{2-s}} - C_0 \lambda^{\frac{2}{2-s}} \) for all \( \lambda \in (0, \Lambda^*) \).

**Proof.** We consider the function

\[ g(t) = \frac{t^2}{2} \|u_t\|^2 - \frac{t^{2^*(s)}}{2^*(s)} \int_{\Omega} \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} \, dx. \]

By (4.6) and the fact,
\[
\max_{t \geq 0} \left( \frac{t^2}{2} - \frac{t^{2s}(s)}{2^s(s)} \right) = \frac{2 - s}{2(N - s)} \left( \frac{a}{b^{2/s}(s)} \right)^{N-2 \over 2-s} \quad \text{for any } a, b > 0,
\]

we can deduce that
\[
\max_{t \geq 0} g(t) = \frac{2 - s}{2(N - s)} \|u_e\|^{2(N-s)\over 2-s} \geq \frac{2 - s}{2(N - s)} A_{\mu, s}^{N-s} + O(\varepsilon^{N-2 \over 2-s}).
\quad \text{(4.7)}
\]

**Step 1.** Let \( \varepsilon = \lambda^{(2-q)(N-2)} \). We claim that there exists \( \Lambda^* > 0 \), such that
\[
\sup_{t \geq 0} J(tu_e) < \frac{2 - s}{2(N - s)} A_{\mu, s}^{N-s} - C_0 \lambda^{2-q} \quad \text{for all } \lambda \in (0, \Lambda^*).
\]

Let \( \delta_1 > 0 \) be such that
\[
\frac{2 - s}{2(N - s)} A_{\mu, s}^{N-s} - C_0 \lambda^{2-q} > 0, \quad \text{for all } \lambda \in (0, \delta_1).
\]

Using the definitions of \( J, u_e \), we get
\[
J(tu_e) \leq \frac{t^2}{2} \|u_e\|^2, \quad \text{for all } t \geq 0, \quad \text{for all } \lambda > 0,
\]

which implies that there exists \( t_0 \in (0, 1) \) satisfying
\[
\sup_{0 \leq t \leq t_0} J(tu_e) < \frac{2 - s}{2(N - s)} A_{\mu, s}^{N-s} - C_0 \lambda^{2-q}, \quad \text{for all } \lambda \in (0, \delta_1).
\]

Using the definitions of \( J, u_e \), and by (4.7), we have
\[
\sup_{t \geq t_0} J(tu_e) = \sup_{t \geq t_0} \left( g(t) - \frac{t^q}{q} \lambda \int_{\Omega} |u_e|^q \, dx \right)
\leq \frac{2 - s}{2(N - s)} A_{\mu, s}^{N-s} + O(\varepsilon^{N-2 \over 2-s}) - \frac{t_0^q}{q} \lambda \int_{B_{R_0}(0)} |u_e|^q \, dx.
\quad \text{(4.8)}
\]

Let \( 0 < \varepsilon \leq R_0^{2-q} \), we have
\[
\int_{B_{R_0}(0)} |u|_q^q \, dx = \int_{B_{R_0}(0)} \frac{1}{[\varepsilon |x|^{\frac{\gamma_1(2-s)}{2q_\mu}} + |x|^{\frac{\gamma_2(2-s)}{2q_\mu}}]^{\frac{2-s}{2-s}}} \, dx \\
\geq \int_{B_{R_0}(0)} \frac{1}{2R_0^{\frac{\gamma_2(2-s)}{2-s}}} \, dx \\
= C_1(\mu, s, q, N, R_0).
\]

Combining with (4.8) and (4.9), for all \( \varepsilon = \lambda^{\frac{(2-s)}{(2-s)N_0}} \in (0, R_0^{\frac{(2-s)(\gamma_2-\eta_1)}{2\sqrt{\eta_1}}}) \), we get

\[
\sup_{t \geq t_0} J(tu_\varepsilon) \leq \frac{2-s}{2(N-s)} A_{\frac{2-s}{\mu}, s}^{\frac{N-s}{2-s}} + O(\lambda^{\frac{2}{2-s}}) - \frac{t_0^q}{q} C_1 \lambda.
\]

Hence, we can choose \( \delta_2 > 0 \), such that

\[
O(\lambda^{\frac{2}{2-s}}) - \frac{t_0^q}{q} C_1 \lambda < -C_0 \lambda^{\frac{2}{2-s}}, \text{ for all } \lambda \in (0, \delta_2).
\]

If we set \( \Lambda^* = \min\{\delta_1, (R_0^{(2-s)}\sqrt{|\mu-\mu|}, \delta_2) > 0 \), then for \( \lambda \in (0, \Lambda^*) \) and \( \varepsilon = \lambda^{\frac{(2-s)}{(2-s)N_0}} \), we have

\[
\sup_{t \geq t_0} J(tu_\varepsilon) < \frac{2-s}{2(N-s)} A_{\frac{2-s}{\mu}, s}^{\frac{N-s}{2-s}} - C_0 \lambda^{\frac{2}{2-s}}.
\]

**Step 2.** Prove that \( \alpha_\lambda < \frac{2-s}{2(N-s)} A_{\frac{2-s}{\mu}, s}^{\frac{N-s}{2-s}} - C_0 \lambda^{\frac{2}{2-s}}, \text{ for all } \lambda \in (0, \Lambda^*) \). By the definition of \( u_\varepsilon \), we have

\[
\int_{\Omega} |u_\varepsilon|_q^q \, dx > 0 \text{ and } \int_{\Omega} |u_\varepsilon|^{2s} \, dx > 0.
\]

Combining this with Lemma 2.5, from the definition of \( \alpha_\lambda \), and the results in **Step 1**, we obtain that there exists \( t_\varepsilon > 0 \), such that \( t_\varepsilon u_\varepsilon \in N_\lambda^- \), and
\[ \alpha_\lambda^- \leq J(t_k u_\lambda) \leq \sup_{t \geq 0} J(t u_\lambda) < \frac{2 - s}{2(N - s)} A_{\mu,s}^{\frac{N-s}{2}} - C_0 \lambda^{\frac{2}{2 - q}}, \]

for all \( \lambda \in (0, \Lambda^*) \).

Now, we establish the existence of a local minimum of \( J \) on \( \mathcal{N}_\lambda^- \).

**Theorem 4.4.** There exists \( \Lambda_2 > 0 \), such that for \( \lambda \in (0, \Lambda_2) \) the functional \( J \) has a minimizer \( U_\lambda^- \) in \( \mathcal{N}_\lambda^- \) and satisfies

(i) \( J(U_\lambda^-) = \alpha_\lambda^- \);

(ii) \( U_\lambda^- \) is a positive solution of (1.1) in \( H_0^1(\Omega) \) for some \( \alpha \in (0, 1) \),

where \( \Lambda_2 = \min\{\Lambda^*, \frac{q}{2} \Lambda_1\} \), \( \Lambda^* \) is defined as in Lemma 4.3 and \( \Lambda_1 \) is defined as in (1.3).

**Proof.** By Proposition 3.4 (ii), for all \( \lambda \in (0, \frac{q}{2} \Lambda_1) \), there exists a \((PS)_{\alpha_\lambda^-}\) -sequence \( \{u_n\} \subset \mathcal{N}_\lambda^- \) in \( H_0^1(\Omega) \) for \( J \). From Lemmas 4.2, 4.3, and Theorem 2.4 (ii), for \( \lambda \in (0, \Lambda^*) \), \( J \) satisfies \((PS)_{\alpha_\lambda^-}\) condition and \( \alpha_\lambda^- > 0 \). Since \( J \) is coercive on \( \mathcal{N}_\lambda^- \) (see Lemma 2.1), we get that \( \{u_n\} \) is bounded in \( H_0^1(\Omega) \). Therefore, there exist a subsequence still denoted by \( \{u_n\} \) and \( U_\lambda^- \in \mathcal{N}_\lambda^- \) such that \( u_n \rightarrow U_\lambda^- \) strongly in \( H_0^1(\Omega) \) and \( J(U_\lambda^-) = \alpha_\lambda^- > 0 \), for all \( \lambda \in (0, \Lambda_2) \). Finally, by using the same arguments as in the proof of Theorem 3.5, for all \( \lambda \in (0, \Lambda_2) \), we have that \( U_\lambda^- \) is a positive solution of (1.1).

Now, we complete the proof of **Theorem 1.2**: By Theorems 3.5, 4.4, we obtain (1.1) has two positive solutions \( u_\lambda^- \) and \( U_\lambda^- \), such that \( u_\lambda^- \in \mathcal{N}_\lambda^+ \), \( U_\lambda^- \in \mathcal{N}_\lambda^- \). Since, \( \mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset \), this implies that \( u_\lambda^- \) and \( U_\lambda^- \) are distinct.
References


