VISCOSITY APPROXIMATION METHODS FOR FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACE

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Abstract

In this paper, under the framework of Banach space with uniformly Gateaux differentiable norm and uniform normal structure, we use the existence theorem of fixed points of Li and Sims to investigate the convergence of the implicit iteration process and the explicit iteration process for asymptotically nonexpansive mappings. We get the convergence theorems.

1. Introduction

Let $E$ be a real Banach space, $E^*$ is the dual space of $E$, $K$ is a nonempty closed convex subset of $E$. Let $J : E \to 2^{E^*}$ denote the 2000 Mathematics Subject Classification: 47H09.

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The normalized duality mapping defined by \( J(x) := \{ f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|, x \in E \} \). A mapping \( T : K \rightarrow K \) is called a \( \text{contraction} \), if there exists a constant \( \alpha \in [0, 1) \) such that \( \|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in K \). The mapping \( T \) is called nonexpansive, if \( \|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K \), and asymptotically nonexpansive, if there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( \lim_{n \rightarrow \infty} k_n = 1 \), such that \( \|T^n x - T^n y\| \leq k_n \|x - y\| \) for all integers \( n \geq 0 \) and all \( x, y \in K \). It is clear that every contraction is nonexpansive, and every nonexpansive mapping is asymptotically nonexpansive. The converses do not hold. The asymptotically nonexpansive mappings are important generalizations of nonexpansive mappings.

In [16], Moudafi had proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. In [18], Xu studied the viscosity approximation methods proposed by Moudafi [16] for a \( \text{nonexpansive} \) mapping in a \( \text{uniformly smooth} \) Banach space.

Recently, Shahzad and Udomene [17] studied the convergence of the implicit iteration process and the explicit iteration process in a real Banach space with uniformly Gateaux differentiable norm and uniform normal structure.

In this paper, we give a new method to prove the theorem appeared in [17], and we get the following convergence theorems: Let \( E \) be a Banach space with a uniformly \( \text{Gateaux} \) differentiable norm and uniform normal structure. Let \( K \) be a nonempty bounded closed convex subset of \( E \) and \( T \) be an asymptotically nonexpansive mapping on \( K \). If \( f : K \rightarrow K \) is a contraction and a sequence \( \{t_n\} \subset (0, 1) \) satisfies \( \lim_{n \rightarrow \infty} t_n = 0 \). Then for any \( n \in N \), there exist an integer \( l(n) \) and a unique \( x_n \in K \), such that \( x_n = t_n f(x_n) + (1 - t_n)T^{l(n)} x_n \). Further, (1) if \( \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \), then \( \{x_n\} \) converges strongly to a fixed point \( p \in F(T) \), which is also the
unique solution to the variational inequality: $\langle (I-f)p, f(p-x^*) \rangle \leq 0$, \( \forall x^* \in F(T) \). (2) if \( \sum_{n=0}^{\infty} t_n(1-t_n) = \infty \), and for any \( y_0 \in K \), the explicit iteration process \( y_{n+1} := t_n f(y_n) + (1-t_n) P_{I_n} y_n, n \geq 1 \), satisfies \( \lim_n \| T y_n - y_n \| = 0 \), then \( \{y_n\} \) converges strongly to a fixed point \( p \).

Our main results extend the theorems in [18, 19] to the class of asymptotically nonexpansive mappings and remove some key conditions of iterative coefficients (cf. [4-7, 18, 19]).

2. Preliminaries

Let \( S := \{ x \in E : \| x \| = 1 \} \) denote the unit sphere of the Banach space \( E \). The space \( E \) is said to have a Gateaux differentiable norm, if the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}, \quad (*)$$

exists for each \( x, y \in S \); and \( E \) is said to have a uniformly Gateaux differentiable norm, if for each \( y \in S \) the limit \((*)\) is attained uniformly for \( x \in S \). Further, \( E \) is said to be uniformly smooth, if the limit \((*)\) exists uniformly for \( (x, y) \in S \times S \). It is well known that if \( E \) is smooth, then any duality mapping on \( E \) is single-valued, and if \( E \) has a uniformly Gateaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded sets.

Let \( K \) be a nonempty closed convex and bounded subset of the Banach space \( E \) and let the diameter of \( K \) be defined by \( d(K) := \sup \{ \| x - y \| : x, y \in K \} \). For each \( x \in K \), let \( r(x, K) := \sup \{ \| x - y \| : y \in K \} \) and let \( r(K) := \inf \{ r(x, K) : x \in K \} \) denote the Chebyshev radius of \( K \) relative to itself. The normal structure coefficient \( N(E) \) of \( E \) (cf. [3]) is defined by \( N(E) := \inf \{ d(K) / r(K) : K \) is a closed convex and bounded subset of \( E \) with \( d(K) > 0 \} \).
A space $E$ such that $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [1, 15]).

We shall let $LIM$ be a Banach limit. Recall that $LIM \in (\ell^\infty)^*$, such that $\|LIM\| = 1$, $\liminf_{n \to \infty} a_n \leq LIM a_n \leq \limsup_{n \to \infty} a_n$, and $LIM a_n = LIM a_{n+1}$ for all $\{a_n\} \in \ell^\infty$.

The following lemmas will be needed.

**Lemma 2.1** (Chidume [8], Lemma 2.1; Xu [18], Lemma 2.2). Let $E$ be an arbitrary real Banach space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle,$$

$\forall x, y \in E$, and $\forall j(x + y) \in J(x + y)$.

**Lemma 2.2** (Kim-Xu [13], Theorem 1). Let $E$ be a Banach space with uniform normal structure, $K$ be a nonempty closed convex and bounded subset of $E$, and $T : K \to K$ be an asymptotically nonexpansive mapping. Then $T$ has a fixed point.

**Lemma 2.3** (Goebel-Reich [11]). Let $E$ be a uniformly smooth Banach space, then any duality mapping on $E$ is single-valued, and the duality mapping is norm-to-weak uniformly continuous on bounded sets.

**Lemma 2.4** (Chidume [8], Lemma SR; Jung [12], Lemma 1). Let $K$ be a nonempty closed convex subset of a Banach space $E$ with a uniformly Gateaux differentiable norm and let $\{x_n\}$ be a bounded sequence in $K$. Let $g(x) = \lim_{n} \|x_n - x\|^2$, $x \in E$ be a Banach limit and $z \in C$. Then

$$g(z) = \min_{x \in K} g(x),$$

if and only if

$$\lim_{n} \langle y - z, J(x_n - z) \rangle \leq 0,$$

for all $\forall y \in K$. 
Lemma 2.5 (Xu [18], Chidume [9]). Assume \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that
\[
a_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\beta_n, \quad n = 0, 1, 2, \ldots,
\]
where \( \{\gamma_n\} \subset [0, 1], \quad \{\beta_n\} \subset [0, 1] \) such that \( \sum_{n=0}^{\infty} \gamma_n = \infty, \lim_{n \to \infty} \beta_n = 0. \)

Then \( \lim_{n \to \infty} a_n = 0. \)

Lemma 2.6 (Li-Sims [14], Theorem 2.1). Suppose \( X \) is a Banach space with uniform normal structure; \( C \) is a nonempty bounded subset of \( X \); and \( T : C \to C \) is an asymptotically nonexpansive type mapping, such that \( T \) is continuous on \( C \). Further, suppose that there exists a nonempty closed convex subset \( E \) of \( C \) with the following property (P):
\[
x \in E \text{ implies } \omega_{\omega}(x) \subset E,
\]
where \( \omega_{\omega}(x) \) is the weak \( \omega \)-limit set of \( T \) at \( x \), that is, the set
\[
\left\{ y \in X : y = \text{weak - limit of } T^{n_i}x \text{ for some } n_i \uparrow \infty \right\}.
\]

Then \( T \) has a fixed point in \( E \).

3. Main Results

Theorem 3.1. Let \( E \) be a Banach space with a uniformly Gateaux differentiable norm and uniform normal structure. Let \( K \) be a nonempty bounded closed convex subset of \( E \) and \( T \) be an asymptotically nonexpansive mapping on \( K \). If \( f : K \to K \) is a contraction and a sequence \( \{t_n\} \subset (0, 1) \) satisfies \( \lim_{n \to \infty} t_n = 0. \)

Then

(i) for any \( n \in N \), there exists an integer \( l(n) \) and a unique \( x_n \in K \), such that
\[
x_n = t_nf(x_n) + (1 - t_n)T^{l(n)}x_n.
\]
(ii) if \( \lim_{n \to \infty} \| x_n - Tx_n \| = 0 \), then \( \{ x_n \} \) converges strongly to a fixed point \( p \in F(T) \), which is also the unique solution to the variational inequality:

\[
\left\langle (I - f)p, j(p - x^*) \right\rangle \leq 0, \ \forall x^* \in F(T).
\] (3.2)

**Proof.** (1) For each \( t_n \in (0, 1) \), \( \lim_{t \to 1} k_t = 1 \), then there exists an integer \( l(n) > 0 \), such that \( \frac{k_{l(n)} - 1}{t_n^2} \leq \lim_{n \to \infty} \frac{k_{l(n)} - 1}{t_n} \leq \lim_{n \to \infty} \frac{t_n^2}{t_n} = 0 \), this implies that

\[
k_{l(n)} - 1 = o(t_n)(n \to \infty).
\]

By the conditions on \( \{ t_n \} \), for each integer \( n \geq 0 \), the mapping \( S_n : K \to K \) defined for each \( x \in K \) by \( S_n(x) := t_n f(x) + (1 - t_n) T^{l(n)} x \) is a contraction, when \( n \) is large enough.

In fact, \( \forall x, y \in K \),

\[
\| S_n(x) - S_n(y) \| \leq t_n \| f(x) - f(y) \| + (1 - t_n) \| T^{l(n)} x - T^{l(n)} y \|
\leq [\alpha \cdot t_n + (1 - t_n) k_{l(n)}] \cdot \| x - y \|.
\]

The condition \( k_{l(n)} - 1 = o(t_n)(n \to \infty) \) imply \( \frac{k_{l(n)} - 1}{t_n} < 1 - \alpha < \frac{1 - \alpha}{1 - t_n} \), for sufficient large \( n \geq 0 \), i.e., \( \alpha \cdot t_n + (1 - t_n) k_{l(n)} < 1 \). By Banach’s contraction principle, for each \( n \in N \), we have a unique point \( x_n \in K \) satisfying Equation (3.1).

(2) First, we show the uniqueness of solutions of the variational inequality (3.2). In fact, supposing \( p, q \in F(T) \) satisfy (3.2), we get that

\[
\left\langle (I - f)p, j(p - q) \right\rangle \leq 0,
\]

(3.3)

\[
\left\langle (I - f)q, j(q - p) \right\rangle \leq 0.
\]

(3.4)

Adding up (3.3) and (3.4), we have that

\[
(1 - \alpha) \| p - q \|^2 \leq \left\langle (I - f)p - (I - f)q, j(p - q) \right\rangle \leq 0.
\]

We must have \( p = q \), and the uniqueness is proved.
Define the mapping \( g : K \to \mathbb{R} \) by \( g(x) := \text{LIM}_n \| x_n - x \|^2, \forall x \in K \). Then, since the function \( g \) on \( K \) is convex and continuous, \( g(x) \to \infty \) as \( \| x \| \to \infty \), and \( E \) is reflexive, there exists \( z \in K \) with \( g(z) = \min \{ g(x) : x \in K \} \) (cf. [2, p.79]). We have that the set \( M = \{ z \in K : g(z) = \inf_{x \in K} g(x) \} \neq \emptyset \), and \( M \) is a nonempty closed convex subset of \( K \).

Now, we prove that there exists \( p \in M \) such that \( p \in F(T) \). \( \forall z \in M, K \) is bounded, and so are \( \{ T^n z \} \). It is known that every space with a uniform normal structure is reflexive, so \( E \) is reflexive, \( \{ T^n z \} \) admits a subsequence \( \{ T^{n_i} z \} \) converging weakly to some \( z_0 \in E \). For any one fixed integer \( s > 0 \), we have

\[
\| x_n - T^s z \| \leq \| x_n - T^s x_n \| + k_s \| x_n - z \|.
\]  \hspace{1cm} (3.5)

Using \( \lim_{n \to \infty} \| x_n - T x_n \| = 0 \), we obtain \( \lim_{n \to \infty} \| x_n - T^s x_n \| = 0 \).

Since,

\[
g(T^s z) = \text{LIM}_n \| x_n - T^s z \| \leq \text{LIM}_n \| x_n - T^s x_n \| + \text{LIM}_n k_s \| x_n - z \| \leq k_s g(z).
\]

Hence \( g(T^{n_i} z) \leq k_{n_i} g(z) \),

\[
g(z_0) = \lim_{i \to \infty} g(T^{n_i} z) \leq \lim_{i \to \infty} k_{n_i} g(z) = g(z).
\]

Now, we have \( z_0 \in M \).

That is, \( \forall z \in M, W_u(z) \subset M \) from above. By Lemma 2.6, we conclude that there exists \( p \in M \) such that \( p \in F(T) \).

Now, we show that \( \{ x_n \} \) is relatively sequentially compact.

Indeed, let \( N_1 \) be an infinite subset of \( N \), then \( \{ y_n : n \in N_1 \} \) is a subsequence of \( \{ x_n : n \in N \} \). Define the mapping \( g : K \to \mathbb{R} \) by \( \tilde{g}(x) := \text{LIM}_n \| y_n - x \|^2, \forall x \in K \), we have \( \tilde{M} = \{ z : \tilde{g}(z) = \min_{x \in K} \tilde{g}(x) \} \neq \emptyset \),
and $\tilde{M} \cap F(T) \neq \emptyset$. \( \forall p \in \tilde{M} \cap F(T) \), from the iterative process (3.1), we estimate as follows:

\[
\langle y_n - f(y_n), j(y_n - p) \rangle = \frac{1 - t_{\ell(n)}}{t_{\ell(n)}} \langle T_{\ell(n)} y_n - y_n, j(y_n - p) \rangle
\]

\[
= \frac{1 - t_{\ell(n)}}{t_{\ell(n)}} \langle (T_{\ell(n)} y_n - p) - (y_n - p), j(y_n - p) \rangle
\]

\[
\leq \frac{1 - t_{\ell(n)}}{t_{\ell(n)}} (k_{\ell(n)} - 1) \|y_n - p\|^2.
\]

So that

\[
\lim_n \langle y_n - f(y_n), j(y_n - p) \rangle \leq 0, \quad (3.6)
\]

by \( p \in M \) and Lemma 2.4, we have

\[
\lim_n \langle y - p, j(y_n - p) \rangle \leq 0, \ \forall y \in K. \quad (3.7)
\]

In (3.7), let \( y = f(p) \) and add up to (3.6), then

\[
\lim_n \langle y_n - f(y_n) + f(p) - p, j(y_n - p) \rangle \leq 0.
\]

We have

\[
\lim_n \|y_n - p\|^2 \leq \lim_n \langle f(y_n) - f(p), j(y_n - p) \rangle \leq \alpha \lim_n \|y_n - p\|^2,
\]

i.e., \( \lim_n \|y_n - p\|^2 = 0 \), hence, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \), which strongly converges to \( p \).

Finally, we show that \( x_n \to p \).

Let a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) strongly converges to \( q \), we have

\[
\lim_{n_i \to \infty} \|x_{n_i} - T_{x_{n_i}}\| = 0 \quad \text{because of} \quad \lim_{n \to \infty} \|x_n - T_{x_n}\| = 0, \quad \text{so that} \quad \|q - Tq\| = 0, \quad \text{i.e.,} \quad q \in F(T).
\]

From (3.6) and Lemma 2.3, we get

\[
\langle q - f(q), j(q - p) \rangle \leq 0, \quad \forall p \in F(T).
\]
This $q \in F(T)$ is the solution of the variational inequality (3.2); hence, $p = q$ by uniqueness. In summary, we have proved that $\{x_n\}$ is relatively sequentially compact and each cluster point of $\{x_n\}$ (as $n \to \infty$) equals $q$. Therefore, $x_n \to q$ as $n \to \infty$. The proof is completed.

**Theorem 3.2.** Let $E$ be a Banach space with a uniformly Gateaux differentiable norm and uniform normal structure. Let $K$ be a nonempty bounded closed convex subset of $E$ and $T$ be an asymptotically nonexpansive mapping on $K$. If $f : K \to K$ is a contraction and a sequence $\{t_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n(1 - t_n) = \infty$. Then

(i) for any $n \in N$, there exists an integer $l(n)$ and a unique $x_n \in K$, such that

$$x_n = t_n f(x_n) + (1 - t_n)T^{l(n)}x_n.$$ \hspace{1cm} (3.8)

(ii) if $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$ and for any $y_0 \in K$, the explicit iteration process

$$y_{n+1} := t_n f(y_n) + (1 - t_n)T^{l(n)}y_n, \hspace{1cm} n \geq 1,$$

satisfies $\lim_{n \to \infty} \|T^n y_n - y_n\| = 0$, then $\{y_n\}$ converges strongly to a fixed point $p \in F(T)$, which is also the unique solution to the variational inequality (3.2).

**Proof.** Part (i) has already been proved in Theorem 3.1. Assume that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, and $\lim_{n \to \infty} \|T^n y_n - y_n\| = 0$. We proceed to prove part (ii).

Let $n > m$. Then, from (3.1) and Lemma 2.1, we have

$$\|x_m - y_n\|^2 \leq \|T^{l(m)}x_m - y_n\|^2 + 2t_m \langle f(x_m) - T^{l(m)}x_m, j(x_m - y_n) \rangle.$$ 

So that

$$\langle f(x_m) - T^{l(m)}x_m, j(y_n - x_m) \rangle$$
\[
\leq \frac{1}{2t_m}\left(\left\|T^{(m)}x_m - y_n - y_n\right\|^2 - \left\|x_m - y_n\right\|^2\right)
\]
\[
\leq \frac{1}{2t_m}\left(\left\|T^{(m)}x_m - T^{(m)}y_n\right\| + \left\|T^{(m)}y_n - y_n\right\|^2 - \left\|x_m - y_n\right\|^2\right)
\]
\[
\leq \frac{1}{2t_m}\left((k^{(m)}_{m})^2 - 1\right)\left\|x_m - y_n\right\|^2 + \left\|T^{(m)}y_n - y_n\right\|^2 
+ \left\|T^{(m)}y_n - y_n\right\|.\right)
\]

Since \(K\) is bounded, for some constant \(M > 0\) such that
\[
\max\left\{2k_{m\left(m\right)}\left\|x_m - y_n\right\| + \left\|T^{(m)}y_n - y_n\right\|, \left\|x_m - y_n\right\|^2\right\} \leq M,
\]
it follows that
\[
\langle f(x_m) - T^{(m)}x_m, j(y_n - x_m) \rangle \leq \frac{k^{2\left(m\right)}_{m\left(m\right)} - 1}{2t_m} M + \frac{1}{2t_m} \left\|T^{(m)}y_n - y_n\right\| M.
\]
So that
\[
\lim_{n \to \infty}\sup\langle f(x_m) - T^{(m)}x_m, j(y_n - x_m) \rangle \leq \frac{k^{2\left(m\right)}_{m\left(m\right)} - 1}{2t_m} M.
\]

By Theorem 3.1, \(x_m \to p \in F(T)\), which solves the variational inequality (3.2). Since, \(j\) is norm-to-weak* continuous on bounded sets, in the limit as \(m \to \infty\), we obtain that
\[
\lim_{n \to \infty}\sup\langle f(p) - p, j(y_n - p) \rangle \leq 0.
\]
(3.9)

From (3.9), we observe that, there exists a sequence \(\{e_n\}, e_n \geq 0\) for all \(n \geq 0\) such that \(\langle f(p) - p, j(y_{n+1} - p) \rangle \leq e_n\) with \(e_n \to 0\) as \(n \to \infty\).

Now, from the iterative process (3.8) and Lemma 2.1, we estimate as follows: \(\forall p \in F(T)\)
\[
\left\|y_{n+1} - p\right\|^2 \leq \left\|T^{(n)}y_n - p\right\|^2 + 2t_n\left\langle f(y_n) - T^{(n)}y_n, j(y_{n+1} - p) \right\rangle
\]
\[
= \left\|T^{(n)}y_n - p\right\|^2 + 2t_n\left\langle f(y_n) - f(p) + f(p) - p + p - y_{n+1} \right\rangle
\]
$$\begin{align*} &+ y_{n+1} - T^{(n)}(y_n, j(y_n - p)) \\
&\leq \left\| T^{(n)}(y_n - p) \right\|^2 + 2t_n \left[ d \left\| y_n - p \right\| \cdot \left\| y_{n+1} - p \right\| + \left\| y_{n+1} - p \right\|^2 \right] \\
&+ 2t_n \left[ y_{n+1} - T^{(n)}(y_n) \cdot \left\| y_{n+1} - p \right\| + \epsilon_n \right]. \end{align*}$$

Since $K$ is bounded, for some constant $d > 0$ such that $\left\| f(x_n) - T^{(n)}(y_n) \right\| \leq d$, accordingly to (3.8), we have

$$\left\| y_{n+1} - T^{(n)}(y_n) \right\| = t_n \left\| f(x_n) - T^{(n)}(y_n) \right\| \leq t_n d,$$

so that

$$\begin{align*}
\left\| y_{n+1} - p \right\|^2 &\leq k_{(n)}^2 \left\| y_n - p \right\|^2 + 2t_n \left[ d \left\| y_n - p \right\| \cdot \left\| y_{n+1} - p \right\| \right] \\
&- \left\| y_{n+1} - p \right\|^2 + t_n d \left\| y_{n+1} - p \right\| + \epsilon_n \right] \\
&\leq k_{(n)}^2 \left\| y_n - p \right\|^2 + t_n d \left( \left\| y_n - p \right\|^2 + t_n \left\| y_{n+1} - p \right\|^2 \right) - 2t_n d \left( 1 + \left\| y_{n+1} - p \right\|^2 \right) + 2t_n d \left[ + \left\| y_{n+1} - p \right\|^2 \right] + 2\epsilon_n \right].
\end{align*}$$

Let

$$\begin{align*}
\left\| y_n - p \right\|^2 &= a_n, \quad \lambda_n = \frac{1 - 2\alpha t_n + 2t_n - t_n^2 d - k_{(n)}^2}{1 - \alpha t_n + 2t_n - t_n^2 d}, \quad \gamma_n = \frac{2t_n \epsilon_n + t_n^2 d}{1 - \alpha t_n + 2t_n - t_n^2 d}.
\end{align*}$$

We get

$$a_{n+1} \leq (1 - \lambda_n) a_n + \gamma_n.$$

It is clear that $\lim_{n \to \infty} \frac{\lambda_n}{t_n} = 2 - 2\alpha > 0$, then $\lim_{n \to \infty} \frac{\lambda_n}{t_n (1 - t_n)} = 2 - 2\alpha > 0$, it

$$\text{can easily be shown that } \sum_{n=0}^{\infty} \lambda_n = \infty \text{ by } \sum_{n=0}^{\infty} t_n (1 - t_n) = \infty. \text{ Furthermore, }$$

$$\lim_{n \to \infty} \frac{\gamma_n}{\lambda_n} = 0. \text{ Hence, it follows from Lemma 2.5 that } \lim_{n \to \infty} a_n = 0, \text{ i.e.,}$$
\[
\lim_{n \to \infty} \|y_n - p\| = 0,
\]
then \(\{y_n\}\) converges strongly to a fixed point \(p \in F(T)\), which is also the unique solution to the variational inequality (3.2). This completes the proof.

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