ON THE DYNAMICS OF THE RECURSIVE
SEQUENCE
\[ x_{n+1} = \frac{b x_{n-1}}{A + B x_n^p x_{n-2}^q} \]

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Abstract
We investigate the global asymptotic behavior and the periodic character for the
rational difference equation
\[ x_{n+1} = \frac{b x_{n-1}}{A + B x_n^p x_{n-2}^q}, \quad n = 0, 1, 2, ... \]
where the parameters \( b, A, B, p, q \) are nonnegative real numbers and the initial
conditions \( x_{-2}, x_{-1}, x_0 \) are arbitrary nonnegative real numbers.

1. Introduction
We investigate the global asymptotic behavior and the periodic character of solutions of the third order rational difference equation.
where the parameters $b, A, B, p, q$ are nonnegative real numbers and the initial conditions $x_{-2}, x_{-1}, x_0$ are arbitrary nonnegative real numbers. The most general second order rational difference equation which is known to exhibit a similar trichotomy character is

\[ x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, 2, \ldots, \]

where the parameters $\alpha, \beta, \gamma, A$ are nonnegative real numbers and the initial conditions $x_{-1}, x_0$ are arbitrary nonnegative real numbers. See [6]. See also [1], [3] and [4].

Amleh et al. [2] investigated the third order rational difference equation

\[ x_{n+1} = \frac{a + bx_{n-1}}{A + Bx_{n-2}}, \quad n = 0, 1, 2, \ldots, \]

where $a, b, A, B$ are nonnegative real numbers and the initial conditions are nonnegative real numbers.

Let $I$ be an interval of real numbers and let $f: I \times I \times I \to I$ be a continuously differentiable function. Consider the difference equation

\[ x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \quad n = 0, 1, 2, \ldots \]

with $x_{-2}, x_{-1}, x_0 \in I$. Let $\bar{x}$ be the equilibrium point of Equation (1.3). The linearized equation of Equation (1.3) about $\bar{x}$ is

\[ x_{n+1} = c_1 x_n + c_2 x_{n-1} + c_3 x_{n-2}, \quad n = 0, 1, 2, \ldots, \]

where

\[ c_1 = \frac{\partial f}{\partial x_n}(\bar{x}, \bar{x}, \bar{x}), \quad c_2 = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \bar{x}, \bar{x}), \quad c_3 = \frac{\partial f}{\partial x_{n-2}}(\bar{x}, \bar{x}, \bar{x}). \]

The characteristic equation of Equation (1.4) is
\[ \lambda^3 - c_1\lambda^2 - c_2\lambda - c_3 = 0. \]  

\textbf{Definition 1.1.} Let \( \bar{x} \) be an equilibrium point of Equation (1.3).

(i) The equilibrium point \( \bar{x} \) of Equation (1.3) is called \textbf{locally stable} if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x_{-2}, x_{-1}, x_0 \in I \) with \( |x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta \), we have \( |x_n - \bar{x}| < \epsilon \) for all \( n \geq -2 \).

(ii) The equilibrium point \( \bar{x} \) of Equation (1.3) is called \textbf{locally asymptotically stable} if it is locally stable, and if there exists \( \gamma > 0 \) such that for all \( x_{-2}, x_{-1}, x_0 \in I \) with \( |x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma \), we have \( \lim_{n \to \infty} x_n = \bar{x} \).

(iii) The equilibrium point \( \bar{x} \) of Equation (1.3) is called \textbf{global attractor} if for every \( x_{-2}, x_{-1}, x_0 \in I \), we have \( \lim_{n \to \infty} x_n = \bar{x} \).

(iv) The equilibrium point \( \bar{x} \) of Equation (1.3) is called \textbf{globally asymptotically stable} if it is locally stable and global attractor.

(v) The equilibrium point \( \bar{x} \) of Equation (1.3) is called \textbf{unstable} if it is not stable.

(vi) The equilibrium point \( \bar{x} \) of Equation (1.3) is called \textbf{source or repeller} if there exists \( r > 0 \) such that for all \( x_{-2}, x_{-1}, x_0 \in I \) with \( 0 < |x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < r \), there exists \( N \geq 1 \) such that \( |x_N - \bar{x}| \geq r \). Clearly, a repeller is an unstable equilibrium.

\textbf{Theorem A} (Linearized Stability Theorem). \textit{The following statements are true.}

(1) If all roots of Equation (1.5) have modulus less than one, then the equilibrium point \( \bar{x} \) of Equation (1.3) is locally asymptotically stable.

(2) If at least one of the roots of Equation (1.5) has modulus greater than one, then the equilibrium point \( \bar{x} \) of Equation (1.3) is unstable.

(3) A necessary and sufficient conditions for all roots of Equation (1.5) to have modulus less than one are the following:
\[ |c_1 + c_3| < 1 - c_2, \quad |c_1 - 3c_3| < 3 + c_2, \quad c_3^2 - c_2 - c_1c_3 < 1. \]

In this case, \( \bar{x} \) is called a sink.

2. The Special Cases \( bABpq = 0 \)

In this section, we examine the character of solutions of Equation (1.1) when one or more of the parameters of Equation (1.1) are zero.

There are five such equations namely

\[
x_{n+1} = \frac{bx_{n-1}}{Bx_n x_{n-2}^q}, \quad n = 0, 1, 2, \ldots \tag{2.1}
\]

\[
x_{n+1} = \frac{b}{A} x_{n-1}, \quad n = 0, 1, 2, \ldots \tag{2.2}
\]

\[
x_{n+1} = 0, \quad n = 0, 1, 2, \ldots \tag{2.3}
\]

\[
x_{n+1} = \frac{bx_{n-1}}{A + Bx_n^q}, \quad n = 0, 1, 2, \ldots \tag{2.4}
\]

\[
x_{n+1} = \frac{bx_{n-1}}{A + Bx_n^p}, \quad n = 0, 1, 2, \ldots \tag{2.5}
\]

Equation (2.4) was investigated in [3] and Equation (2.5) was investigated in [5]. In each of the remaining cases, it is assumed that all parameters in these equations are positive. Equation (2.3) is trivial, Equation (2.2) is linear and Equation (2.1) is a non-linear third order, the change of variables \( x_n = e^{y_n} \) reduces it to a third order linear difference equation.

3. A General Oscillation Result

The change of variables \( x_n = (A/B)^{\frac{1}{p+q}} y_n \) reduces Equation (1.1) to the difference equation

\[
y_{n+1} = \frac{r y_{n-1}}{1 + y_n^p y_{n-2}^q}, \quad n = 0, 1, 2, \ldots, \tag{3.1}
\]

where \( r = \frac{b}{A} > 0. \)
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Note that $\gamma_1 = 0$ is always an equilibrium point. When $r > 1$, Equation (3.1) also possesses the unique positive equilibrium $
abla_2 = (r - 1)^{\frac{1}{p+q}}$.

**Theorem B** [1]. Assume that $F \in C([0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty))$ is such that $F(u, v, w)$ is non-increasing in $u$ and $w$, and non-decreasing in $v$. Let $\bar{x}$ be an equilibrium point of the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}), \quad n = 0, 1, \ldots$$

(3.2)

and let $\{x_n\}_{n=-2}^\infty$ be a solution of Equation (3.2) such that either

$$x_{-2}, x_0 \geq \bar{x} \quad \text{and} \quad x_{-1} < \bar{x},$$

or

$$x_{-2}, x_0 < \bar{x} \quad \text{and} \quad x_{-1} \geq \bar{x}.$$

Then $\{x_n\}_{n=-2}^\infty$ oscillates about $\bar{x}$ with semicycles of length one.

**Corollary 3.1.** Assume that $r > 1$, let $\{y_n\}_{n=-2}^\infty$ be a solution of Equation (3.1) such that either

$$y_{-2}, y_0 \geq \nabla_2 = (r - 1)^{\frac{1}{p+q}} \quad \text{and} \quad y_{-1} < \nabla_2 = (r - 1)^{\frac{1}{p+q}},$$

or

$$y_{-2}, y_0 < \nabla_2 = (r - 1)^{\frac{1}{p+q}} \quad \text{and} \quad y_{-1} \geq \nabla_2 = (r - 1)^{\frac{1}{p+q}}.$$

Then $\{y_n\}_{n=-2}^\infty$ oscillates about the positive equilibrium point $\nabla_2 = (r - 1)^{\frac{1}{p+q}}$ with semi-cycles of length one.

**Proof.** The proof follows immediately from Theorem B.

**4. The Dynamics of Equation (3.1)**

In this section, we investigate the dynamics of Equation (3.1) with $r > 0$ and with nonnegative initial conditions.
Theorem 4.1. For Equation (3.1), we have the following results

(i) Assume that \( r < 1 \), then the zero equilibrium point is locally asymptotically stable.

(ii) Assume that \( r > 1 \), then the zero equilibrium point is saddle point.

(iii) The positive equilibrium point \( \bar{y}_2 \) is unstable.

Proof. The linearized equation associated with Equation (3.1) about \( \bar{y}_1 = 0 \) has the form

\[
z_{n+1} - rz_{n-1} = 0, \quad n = 0, 1, 2, ...
\]

so, the characteristic equation of Equation (3.1) about \( \bar{y}_1 = 0 \), is

\[
\lambda^3 - r\lambda = 0,
\]

then the proof of (i), (ii) follows immediately from Theorem A.

The linearized equation of Equation (3.1) about \( \bar{y}_2 = (r - 1)^{\frac{1}{p+q}} \) is

\[
z_{n+1} + p(1 - \frac{1}{r})z_n - z_{n+1} + q(1 - \frac{1}{r})z_{n-2} = 0, \quad n = 0, 1, 2, ...
\]

so, the characteristic equation of Equation (3.1) about \( \bar{y}_2 = (r - 1)^{\frac{1}{p+q}} \) is

\[
\lambda^3 + p(1 - \frac{1}{r})\lambda^2 - \lambda + q(1 - \frac{1}{r}) = 0.
\]

Set

\[
f(\lambda) = \lambda^3 + p(1 - \frac{1}{r})\lambda^2 - \lambda + q(1 - \frac{1}{r}),
\]

then \( f(-1) = \frac{(p + q)(r - 1)}{r} > 0 \) and \( \lim_{\lambda \to -\infty} f(\lambda) = -\infty \), so \( f(\lambda) \) has at least a root in \( (-\infty, -1) \). Then the proof of (iii) follows.

Theorem 4.2. Assume \( r < 1 \), then the zero equilibrium point of Equation (3.1) is globally asymptotically stable.

Proof. We know by Theorem 4.1 that \( \bar{y}_1 = 0 \) is locally asymptotically stable equilibrium point of Equation (3.1), and so it suffices to show that \( \bar{y}_1 = 0 \) is a global attractor of Equation (3.1).
\[ 0 \leq y_{n+1} = \frac{ry_{n-1}}{1 + y_n^p y_{n-2}^q} \leq ry_{n-1}, \]
since \( r < 1 \), then
\[ \lim_{n \to \infty} y_n = 0. \]

The next Theorem shows that Equation (3.1) has a prime period two solutions when \( r = 1 \).

**Theorem 4.3.** Assume that \( r = 1 \), then Equation (3.1) possesses the prime period two solutions
\[ ..., \phi, 0, \phi, 0, \phi, ... \] (4.1)
with \( \phi > 0 \). Furthermore every solution of Equation (3.1) converges to a period (not necessarily prime) two solutions (4.1) with \( \phi \geq 0 \).

**Proof.** Let
\[ ..., \phi, \psi, \phi, \psi, ... \]
be a period two solution of Equation (3.1). Then
\[ \phi = \frac{r\phi}{1 + \psi^{p+q}} \quad \text{and} \quad \psi = \frac{r\psi}{1 + \phi^{p+q}}. \]

If \( \phi \neq 0, \psi \neq 0 \), then \( \phi = \psi = (r - 1)\frac{1}{\psi^{p+q}} \), which is impossible. Hence \( \psi = 0 \), which implies that \( (r - 1)\phi = 0 \), so \( r = 1 \).

To complete the proof, Assume that \( r = 1 \) and let \( \{y_n\}_{n=2}^{\infty} \) be a solution of Equation (4.1), then
\[ y_{n+1} - y_{n-1} = -\frac{y_n^p y_{n-1}^q}{1 + y_n^p y_{n-2}^q} \leq 0. \]

So, the even terms of this solution decrease to a limit (say \( \Phi \geq 0 \)), and the odd terms decrease to a limit (say \( \Psi \geq 0 \)). Thus
which implies that

\[ \Phi^p \psi^q = 0 \quad \text{and} \quad \psi^p \Phi^q = 0. \]

This completes the proof. \(\square\)

The next Theorem shows that when \( r > 1 \), Equation (3.1) possesses unbounded solutions.

**Theorem 4.4.** Assume \( r > 1 \). Then Equation (3.1) possesses unbounded solutions. In particular, every solution of Equation (3.1) which oscillates about the equilibrium \( \bar{y}_2 = (r - 1)_{p+q} \) with semicycles of length one is unbounded.

**Proof.** We will prove that every solution \( \{y_n\}_{n=-2}^{\infty} \) of Equation (3.1) which oscillates with semicycles of length one is unbounded (see corollary 3.1).

Assume that \( \{y_n\}_{n=-2}^{\infty} \) be a solution of Equation (3.1) such that

\[ y_{2n+1} < \bar{y}_2 = (r - 1)_{p+q} \quad \text{and} \quad y_{2n} > \bar{y}_2 = (r - 1)_{p+q}, \quad n \geq -1. \]

Then

\[ y_{2n+2} = \frac{ry_{2n}}{1 + y_{2n+1}^p y_{2n-1}^q} > \frac{ry_{2n}}{1 + \bar{y}_2^p} = y_{2n}, \]

and

\[ y_{2n+3} = \frac{ry_{2n+1}}{1 + y_{2n+2}^p y_{2n}^q} < \frac{ry_{2n+1}}{1 + \bar{y}_2^p} = y_{2n+1}, \]

from which it follows that

\[ \lim_{n \to \infty} y_{2n} = \infty \quad \text{and} \quad \lim_{n \to \infty} y_{2n+1} = 0, \]

which completes the proof.
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References

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