SYMmetric Positive Semi-Definite Solutions of $AX = B$ and $XC = D$

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Abstract
In this paper, a sufficient and necessary condition for the matrix equations $AX = B$ and $XC = D$, where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times p}$, $C \in \mathbb{R}^{n\times p}$, and $D \in \mathbb{R}^{n\times p}$, to have a common symmetric positive semi-definite solution $X$ is established, and if it exists, a representation of the solution set $S_X$ is given. An optimal approximation between a given matrix $X \in \mathbb{R}^{n\times n}$ and the affine subspace $S_X$ is discussed, an explicit formula for the unique optimal approximation solution is presented, and a numerical example is provided.

1. Introduction

In this paper, we shall adopt the following notation. $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, $\text{SPR}^{n\times n}$ denotes the set of all symmetric positive semi-definite matrices in $\mathbb{R}^{n\times n}$. $I_n$ represents the identity matrix of size $n$. $A^T$, $A^+$, and $\|A\|$ stand for the transpose, the
Moore-Penrose generalized inverse, and the Frobenius norm of a real matrix $A$, respectively. For $A, B \in \mathbb{R}^{m \times n}$, we define an inner product in $\mathbb{R}^{m \times n} : \langle A, B \rangle = \text{trace}(B^T A)$, then $\mathbb{R}^{m \times n}$ is a Hilbert space. The matrix norm $\| \cdot \|$ induced by the inner product is the Frobenius norm. We write $A \succeq 0$, if $A$ is a real symmetric positive semi-definite matrix.

Matrix equation is one of the important study fields of linear algebra. The linear matrix equations

$$AX = B, \quad XC = D,$$

have been considered by many authors. In [6], Mitra gave the common solution of minimum possible rank based on the generalized matrix inverses and the matrix rank. Li [4] discussed the generalized reflexive solution of (1), a necessary and sufficient condition for the solvability and the expression of the general solution were obtained. Qiu [7] considered the constraint $PX = \pm XP$ solution of (1), where $P$ is a given Hermitian matrix satisfying $P^2 = I_n$. Qiu [8] further studied the least-squares solutions to the Equations (1) with some constraints: orthogonality, symmetric orthogonality, symmetric idempotent. Li [5] found a sufficient and necessary condition for the matrix Equations (1) to have symmetric and skew-antisymmetric solutions over the real quaternion and, for the consistent case, provided a representation of its general solution. Wang [9] considered the bisymmetric solutions of (1) over the real quaternion algebra. Recently, Dajić [2] studied the positive solutions to the Equations (1) for Hilbert space operators using generalized inverses, and a sufficient and necessary condition for its solvability, and a representation of its general solution was also established therein. In the present paper, we will consider symmetric positive semi-definite solutions of the matrix Equations (1), where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times p}$, and $D \in \mathbb{R}^{n \times p}$, and an associated optimal approximation problem:

$$\min_{\tilde{X} \in S_X} \|X - \tilde{X}\|,$$
where $\tilde{X}$ is a given matrix in $\mathbb{R}^{n \times n}$ and $S_X$ is the solution set of the matrix Equations (1). Clearly, when $\tilde{X} = 0$, the solution of (2) is the minimum norm solution of (1).

Using the singular value decomposition, we give a necessary and sufficient condition for the Equations (1) to have a solution $X \in \text{SPR}^{n \times n}$, and construct the solution set $S_X$ explicitly, when it is nonempty. We show that there exists a unique solution to the matrix optimal approximation problem (2), if the set $S_X$ is nonempty and present an explicit formula for the unique solution.

2. The Solution of the Matrix Equations (1)

To begin with, we introduce a lemma [10].

**Lemma 1.** Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times n}$, and the singular value decomposition of $A$ be

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T,$$

(3)

where $U = [U_1, U_2]$, $V = [V_1, V_2]$ are all orthogonal matrices and the partitions are compatible with the size of $\Sigma = \text{diag}\{\sigma_1, \cdots, \sigma_t\} > 0$, $t = \text{rank}(A)$. Then, the matrix equation

$$AX = B,$$

(4)

has a solution $X \in \text{SPR}^{n \times n}$, if and only if

$$BA^T \geq 0, \text{rank}(B) = \text{rank}(BA^T).$$

(5)

In which case, the general solution of the equation of (4) can be expressed as

$$X = X_0 + V_2 G V_2^T,$$

(6)

where
\[ X_0 = A^+B + (I_n - A^+A)(A^+B)^T + (I_n - A^+A)B^T(AB^T)^+B(I_n - A^+A), \]  

\[ (7) \]

**and G is an arbitrary symmetric positive semi-definite matrix.**

Inserting (6) into \( XC = D \), we get

\[ V_2GV_2^TC = D - X_0C. \]

\[ (8) \]

It is easily seen that the equation of (8) is equivalent to

\[ V_1^T(D - X_0C) = 0, \]

\[ (9) \]

\[ GV_2^TC = V_2^T(D - X_0C). \]

\[ (10) \]

Since, \( V_1V_1^T = A^+A \) and \( AX_0 = B \), then the relation of (9) is equivalent to

\[ AD = BC. \]

\[ (11) \]

Let the singular value decomposition of \( V_2^TC \) be

\[ V_2^TC = P[\Omega \quad 0]Q^T, \]

\[ (12) \]

where \( P = [P_1, P_2] \), \( Q = [Q_1, Q_2] \) are all orthogonal matrices and the partitions are compatible with the size of \( \Omega = \text{diag}\{\omega_1, \ldots, \omega_s\} > 0, \)

\( s = \text{rank}(V_2^TC) \). It follows from Lemma 1 that the equation of (10) has a solution \( G \in \text{SPR}^{(n-t)\times(n-t)} \), if and only if

\[ C^TV_2V_2^T(D - X_0C) \geq 0, \]  

\[ \text{rank}((D - X_0C)^TV_2) = \text{rank}((D - X_0C)^TV_2V_2^TC). \]

\[ (13) \]

Notice that

\[ V_2V_2^T(D - X_0C) = (I_n - A^+A)(D - X_0C) = D - X_0C. \]

Therefore, the relations of (13) can be simplified as

\[ C^T(D - X_0C) \geq 0, \]  

\[ \text{rank}((D - X_0C)^T) = \text{rank}((D - X_0C)^TC). \]

\[ (14) \]
When the conditions (14) hold, the general solution of (10) can be written as
\[ G = G_0 + P_2 WP_2^T, \]
where
\[ G_0 = \tilde{D}\tilde{C}^+ + (\tilde{D}\tilde{C}^+)^T (I_{n-t} - \tilde{C}\tilde{C}^+) + (I_{n-t} - \tilde{C}\tilde{C}^+) \tilde{D}(\tilde{D}^T\tilde{C})^+ \tilde{D}^T (I_{n-t} - \tilde{C}\tilde{C}^+), \]
(15)
\[ \tilde{C} = V_2^T C, \tilde{D} = V_2^T (D - X_0 C), \text{and } W \geq 0 \text{ is an arbitrary matrix.} \]

As a summary of the above discussion, we have proved the following result.

**Theorem 1.** Given \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{n \times p}, \) and \( D \in \mathbb{R}^{n \times p}. \) Let the singular value decompositions of \( A \) and \( V_2^T C \) be given by (3) and (12), respectively. Then, the matrix Equations (1) have a solution \( X \in \text{SPR}^{n \times n}, \) if and only if the conditions (5), (11), and (14) are satisfied. In this case, the solution set of (1) can be expressed as
\[ S_X = \{ X \in \text{SPR}^{n \times n} | X = X_1 + V_2 P_2 WP_2^T V_2^T \}, \]
(16)
where \( X_1 = X_0 + V_2 G_0 V_2^T, \) and \( W \) is an arbitrary symmetric positive semi-definite matrix.

**3. The Solution of the Optimal Approximation Problem (2)**

In order to solve the optimal approximation problem (2), we need the following lemma [3].

**Lemma 2.** Let \( E \in \mathbb{R}^{n \times n}, E_1 = \frac{1}{2} (E + E^T), \) and let the polar decomposition of \( E_1 \) be \( E_1 = FH, \) where \( F \) is an orthogonal matrix and \( H \) is a symmetric positive semi-definite matrix. Let \( \hat{E} = \frac{1}{2} (E_1 + H), \) then
\[ \| \hat{E} - E \| = \min_{K \in \text{SPR}^{n \times n}} \| K - E \|. \]
Theorem 2. If the solution set $S_X$ is nonempty, then the optimal approximation problem (2) has a unique solution $\hat{X} \in S_X$. Furthermore, let $J = \frac{1}{2} P_2^T V_2^T (\tilde{X} + \tilde{X}^T) V_2 P_2 - P_2^T V_2^T X_1 V_2 P_2$ and the polar decomposition of $J$ be

$$J = LM,$$ 

(17)

where $L$ is an orthogonal matrix and $M \succeq 0$. Let $\hat{W} = \frac{1}{2} (J + M)$, then the unique solution $\hat{X}$ of (2) can be expressed as

$$\hat{X} = X_1 + V_2 P_2 W P_2^T V_2^T.$$ 

(18)

Proof. By Theorem 1, we know that if the conditions (5), (11), and (14) are satisfied, then the solution set $S_X$ is nonempty. It is easy to verify that $S_X$ is a closed convex set in Hilbert space $\mathbb{S}^{n \times n}$. Therefore, for a given matrix $\hat{X} \in \mathbb{R}^{n \times n}$, it follows from the best approximation theorem (see Aubin [1]), that there exists a unique solution $\tilde{X}$ in $S_X$ such that $\|\tilde{X} - \tilde{X}\| = \min_{\hat{X} \in S_X} \|X - \tilde{X}\|$. For any matrix $X \in S_X$, we have

$$\|X - \tilde{X}\|^2 = \|V_2 P_2 WP_2^T V_2^T - (\tilde{X} - X_1)\|^2$$

$$= \|V^T V_2 P_2 WP_2^T V_2^T V - V^T (\tilde{X} - X_1) V\|^2$$

$$= \xi + \|P_2 WP_2^T - V_2^T (\tilde{X} - X_1) V_2\|^2$$

$$= \xi + \|P_2 WP_2^T P - P_2^T V_2^T (\tilde{X} - X_1) V_2 P\|^2$$

$$= \pi + \|W - P_2^T V_2^T (\tilde{X} - X_1) V_2 P_2\|^2,$$

where

$$\xi = \|V_1^T (\tilde{X} - X_1)\|^2 + \|V_2^T (\tilde{X} - X_1) V_1\|^2;$$

$$\pi = \xi + \|P_1^T V_2^T (\tilde{X} - X_1) V_2\|^2 + \|P_2^T V_2^T (\tilde{X} - X_1) V_2 P_1\|^2.$$
Therefore, \( \|X - \tilde{X}\| = \min \), if and only if

\[
\|W - P_2^T V_2^T (\tilde{X} - X_1) V_2 P_2\| = \min, \text{ s. t. } W \succeq 0.
\] (19)

By Lemma 2, we conclude that the solution of the minimization problem (19) is \( W = \hat{W} \). Substitution \( W = \hat{W} \) into (16) yields (18).

4. A Numerical Example

Based on Theorems 1 and 2, we can state the following algorithm.

**Algorithm 1.** (An algorithm for solving the optimal approximation problem (2))

1. Input \( A, B, C, D \), and \( \tilde{X} \).

2. If the condition (5) is satisfied, then we continue. Otherwise, we stop.

3. Find the singular value decompositions of \( A \) and \( V_2^T C \) according to (3) and (12), respectively.

4. If the conditions (11) and (14) are satisfied, then the solution set \( S_X \) is nonempty and we continue. Otherwise, we stop.

5. Compute \( X_0 \) and \( G_0 \) by (7) and (15), respectively.

6. Compute \( X_1 = X_0 + V_2 G_0 V_2^T \).

7. Compute the polar decomposition of the matrix \( J = \frac{1}{2} P_2^T V_2^T \)
\( (\tilde{X} + \tilde{X}^T) V_2 P_2 - P_2^T V_2^T X_1 V_2 P_2 \) by (17).

8. Compute \( \hat{W} = \frac{1}{2} (J + M) \).

9. Compute \( \hat{X} \) according to (18).
Let \( m = 2, n = 6, p = 3 \). Given

\[
A = \begin{bmatrix}
1.7597 & 2.7333 & 3.357 & 0.5351 & 2.4288 & 1.4819 \\
3.7335 & 0.8502 & 2.5151 & 0.8285 & 2.5196 & 2.3006 
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
305.68 & 370.07 & 406.15 & 250.34 & 406.47 & 278.38 \\
304.91 & 355.94 & 403.06 & 252.24 & 392.31 & 269.31 
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
2.7085 & 4.0987 & 0.0981 \\
0.2634 & 0.5571 & 1.1404 \\
0.1631 & 0.212 & 3.5215 \\
1.8761 & 3.6744 & 0.3455 \\
0.0772 & 3.6512 & 2.2054 \\
2.3038 & 0.0946 & 3.7887 
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
158.78 & 301.26 & 250.36 \\
176.54 & 345.8 & 304.57 \\
195.4 & 346.73 & 380.76 \\
131.38 & 243.96 & 210.78 \\
189.08 & 356.9 & 357.92 \\
125.92 & 228.65 & 260.85 
\end{bmatrix},
\]

and

\[
X = \begin{bmatrix}
33.729 & 16.204 & 32.725 & 14.565 & 4.931 & 0.54244 \\
8.2055 & 0.65675 & 26.206 & 31.724 & 7.198 & 26.511 \\
27.055 & 28.114 & 32.55 & 0.35003 & 7.0581 & 14.862 
\end{bmatrix},
\]

It is easy to verify that the conditions (5), (11), and (14) hold. Using Algorithm 1, we obtain the optimal approximation solution of (2) as follows.

\[
\hat{X} = \begin{bmatrix}
25.879 & 26.119 & 44.946 & 23.629 & 34.786 & 27.803 \\
26.531 & 36.213 & 34.786 & 20.803 & 38.826 & 26.043 \\
\end{bmatrix},
\]

It is easily seen that \( \hat{X} \) is a symmetric positive semi-definite matrix. Furthermore, we can figure out

\[
\|A\hat{X} - B\| + \|\hat{X}C - D\| = 1.225e - 012.
\]
References


