ON THE CONVERGENCE OF A JARRATT-TYPE METHOD USING RECURRENT FUNCTIONS

IOANNIS K. ARGYROS and SAÏD HILOUT

Department of Mathematics Sciences
Cameron University
Lawton, OK 73505
U.S.A.
e-mail: iargyros@cameron.edu

Laboratoire de Mathématiques et Applications
Poitiers University
Bd. Pierre et Marie Curie, Téléport 2, B.P. 30179
86962 Futuroscope Chasseneuil Cedex
France

Abstract

We use a fourth order Jarratt-type method to approximate a locally unique solution of a nonlinear equation in a Banach space setting. Using our new idea of recurrent functions, we provide new sufficient convergence conditions, and finer error bounds than before [1]-[10].

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution $x^*$ of equation

$$F(x) = 0, \quad (1.1)$$
where $F$ is a thrice differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modelled by difference of differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator $Q$, where $x$ is the state. Then, the equilibrium states are determined by solving Equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative, when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since, all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Many authors have developed high order methods for generating a sequence approximating $x^\ast$. A survey of such results can be found in [4], and the references there (see, also [1], [2], [3], [5]-[10]).

The natural generalization of the Newton method is to apply a multipoint scheme. Suppose that we know the analytic expressions of $F(x_n)$, $F'(x_n)$, and $F''(x_n)^{-1}$ at a recurrent step $x_n$, $(n \geq 0)$.

In order to increase the order of convergence, and to avoid the computation of the second Fréchet-derivative, we can add one more evaluation of $F(c_1x_n + c_2y_n)$, or $F'(c_1x_n + c_2y_n)$, where $c_1$ and $c_2$ are real constants, that are independent of $x_n$ and $y_n$, where as $y_n$ is generated by a Newton-step. A two-point scheme for functions of one variable was found, and developed by Ostrowski [10].
Following this idea, we provide a semilocal as well as a local convergence analysis for a fourth order inverse free Jarratt-type method (IFJTM) [1], [4], given by
\[
y_n = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0),
\]
\[
B_n = B(n, F) = F'(x_n)^{-1}\left( F'(x_n + \frac{2}{3}(y_n - x_n)) - F'(x_n) \right),
\]
\[
x_{n+1} = y_n - \frac{3}{4}B_n(I - \frac{3}{2}B_n)(y_n - x_n).
\]

The fourth order of (IFJTM) is the same as that of a two-step Newton’s method [4]. But, the computational cost is less than that of Newton’s method. In each step, we save one evaluation of the derivative, and the computation of one inverse.

Here, we use our new idea of recurrent functions in order to provide new sufficient convergence conditions, which can be weaker than before [1]. Using this approach, the error bounds and example on the distances are improved (see Remarks 2.2, and Theorem 3.3). This new idea can be used on other iterative methods [4].

2. Semilocal Convergence Analysis of (IFJTM)

We present our Theorem 2.1 in [1] in affine invariant form, since $F'(x_0)^{-1}F$ can be used for $F$ in the original proof of Theorem 2.1.

**Theorem 2.1.** Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be a thrice differentiable operator.

**Assume:**

there exist $x_0 \in \mathcal{D}$, $L \geq 0$, $M \geq 0$, $N \geq 0$, and $\eta \geq 0$, such that
\[
F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}),
\]
\[
\| F'(x_0)^{-1}F(x_0) \| \leq \eta,
\]
\[
\| F'(x_0)^{-1}F''(x) \| \leq M,
\]
\[ \| F'(x_0)^{-1} F^n(x) \| \leq N, \quad (2.4) \]
\[ \| F'(x_0)^{-1} (F^n(x) - F^n(y)) \| \leq L \| x - y \|, \quad (2.5) \]
for all \( x, y \in D, \)

\[ M \left( 1 + \frac{N}{6M^2} + \frac{13L}{36M^2} \right)^{\frac{3}{2}} \leq K, \quad (2.6) \]
\[ h = K \eta \leq 0.46568, \quad (2.7) \]

and

\[ \overline{U}(x_0, v^*) = \{ x \in X, \| x - x_0 \| \leq v^* \} \subseteq D, \quad (2.8) \]

where \( v^* \) and \( v^{**} \) are the zeros of functions

\[ g(t) = \frac{K}{2} t^2 - t + \eta, \quad (2.9) \]
given by

\[ v^* = \frac{1 - \sqrt{1 - 2h}}{h} \eta, \quad v^{**} = \frac{1 + \sqrt{1 - 2h}}{h} \eta. \quad (2.10) \]

Then, the following hold:

(i) Scalar sequences \( \{v_n\} \) and \( \{w_n\} \), given by:

\[ w_n = v_n - g'(v_n)^{-1} g(v_n), \quad (2.11) \]
\[ b_n = b(n, g) = g'(v_n)^{-1} \left( g'(v_n + \frac{2}{3} (w_n - v_n)) - g'(v_n) \right), \quad (2.12) \]
\[ v_{n+1} = w_n - \frac{3}{4} b_n (1 - \frac{3}{2} b_n) (w_n - v_n), \quad (2.13) \]

are non-decreasing and converge to their common limit \( v^* \), so that

\[ v_n \leq w_n \leq v_{n+1} \leq w_{n+1}. \quad (2.14) \]

(ii) Sequences \( \{x_n\} \) and \( \{y_n\} \), generated by (IFJTM) are well defined,
remain in \( \overline{U}(x_0, v^*) \) for all \( n \geq 0 \), and converge to a unique solution \( x^* \in \overline{U}(x_0, v^*) \) of equation \( F(x) = 0 \), which is the unique solution of equation \( F(x) = 0 \) in \( U(x_0, v^{**}) \).

Moreover, the following estimates hold for all \( n \geq 0 \):

\[
\|y_n - x_n\| \leq w_n - v_n, \quad (2.15)
\]
\[
\|x_{n+1} - y_n\| \leq w_{n+1} - v_n, \quad (2.16)
\]
\[
\|y_n - x^*\| \leq v^* - w_n, \quad (2.17)
\]
\[
\|x_n - x^*\| \leq v^* - v_n \leq \frac{(1 - \theta)^2 \eta (3\sqrt{5} \theta)^{4n-1}}{1 - \frac{1}{\sqrt{5}} (3\sqrt{5} \theta)^{4n}}, \quad (2.18)
\]

where

\[
\theta = \frac{v^*}{v^{**}}. \tag{2.19}
\]

**Remark 2.2.** The bounds of Theorem 2.1 can be improved under the same hypotheses, and computational cost in two cases:

**Case 1.** Define function \( g_0 \) by:

\[
g_0(t) = \frac{M_0}{2} t^2 - t + \eta. \tag{2.20}
\]

In view of (2.3), there exists \( M_0 \in [0, M] \), such that

\[
\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq M_0 \|x - x_0\|, \quad \text{for all } x \in D. \tag{2.21}
\]

We can find upper bounds on the norms \( \|F'(x)^{-1}F'(x_0)\| \) using \( M_0 \), which is actually needed and not \( K \) used in [1].

Note that

\[
M_0 \leq K, \tag{2.22}
\]
and \( \frac{K}{M_0} \) can be arbitrarily large [2]-[4].

Using (2.21), we have for \( x \in \mathcal{U}(x_0, v^*) \),

\[
\| F'(x_0)^{-1}(F'(x) - F'(x_0)) \| \leq M_0 \| x - x_0 \| \leq M \| x - x_0 \| \leq K v^* < 1.
\]  

(2.23)

It follows from (2.23), and the Banach lemma on invertible operators [4], that \( \| F'(x)^{-1}F'(x_0) \| \) exists, and

\[
\| F'(x)^{-1}F'(x_0) \| \leq \frac{1}{1 - M_0 \| x - x_0 \|}.
\]  

(2.24)

We can use (2.23) instead of the less precise used in [1]:

\[
\| F'(x)^{-1}F'(x_0) \| \leq \frac{1}{1 - K \| x - x_0 \|}.
\]  

(2.25)

This suggest that more precise scalar majorizing sequences \( \{\overline{v}_n\}, \{\overline{w}_n\} \) can be used, defined as follows for initial iterates \( \overline{v}_0 = 0, \overline{w}_1 = \eta \):

\[
\overline{w}_n = \overline{v}_n - \eta_0(\overline{v}_n)^{-1}g(\overline{v}_n),
\]  

(2.26)

\[
\overline{b}_n = b(n, g, \eta_0) = g_0'(\overline{v}_n)^{-1}\left(g'(\overline{v}_n + \frac{2}{3}(\overline{w}_n - \overline{v}_n)) - g'(\overline{v}_n)\right),
\]  

(2.27)

\[
\overline{v}_{n+1} = \overline{w}_n - \frac{3}{4} \overline{b}_n(1 - \frac{3}{2} \overline{b}_n)(\overline{w}_n - \overline{v}_n).
\]  

(2.28)

A simple induction argument, shows that if \( M_0 < K \), then,

\[
\overline{v}_n < v_n,
\]  

(2.29)

\[
\overline{w}_n < w_n,
\]  

(2.30)

\[
\overline{w}_n - \overline{v}_n < w_n - v_n,
\]  

(2.31)

\[
\overline{v}_{n+1} - \overline{w}_n < v_{n+1} - w_n,
\]  

(2.32)

and
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\( v^* \leq v^* \),

(2.33)

where

\[ v^* = \lim_{n \to \infty} v_n. \]

Note also that if \( M_0 = K \), then \( v_n = v_n, \overline{w}_n = w_n \).

**Case 2.** In view of the upper bound for \( \| F(x_{n+1}) \| \) obtained in Theorem 2.1 in [1], and (2.23), \( \{ t_n \}, \{ s_n \} \) given in (3.9), and (3.10) are also even more precise majorizing sequences for \( \{ x_n \} \) and \( \{ y_n \} \).

Therefore, if they converge under certain conditions (see Lemma 3.1), we can produce a new semilocal convergence theorem for (IFJTM), with sufficient convergence conditions or bounds that can be better than the ones of Theorem 2.1 (see, also Theorem 3.3 and Example 3.4).

Similar favorable comparisons (due to (2.22)) can be made with other iterative methods for fourth order.

### 3. Semilocal Convergence Analysis of (IFJTM)

We need to define some constants and sequences. Let \( L \geq 0, M_0 \geq 0, M \geq 0, N \geq 0, \) and \( \eta \geq 0 \) be given constants. Define constants \( a, b, c, d, \delta_0, \delta_1, \delta_2, \) and \( w_0 \) by:

\[ a = \frac{M^3}{3} + \frac{N M}{q} + \frac{13 L}{108}, \]

\( b = a + \frac{M^3}{8}, \)

\[ c = \frac{2 b}{M}, \]

\[ d = \frac{2 c}{M}, \]

\[ \delta_0 = M \eta. \]
\[ \delta_1 = \frac{2 M}{2 M_0 + M + \sqrt{(2 M_0 + M)^2 + 8 M_0 M}} < 1, \quad M_0 \neq 0 \text{ or } M \neq 0, \]

(3.6)

\[ \delta_2 = \frac{2 d}{M_0 + d + \sqrt{(M_0 + d)^2 + 4 M_0 d}} < 1, \quad M_0 \neq 0 \text{ or } d \neq 0, \]

(3.7)

\[ w_\infty = \frac{1}{1 + M_0 \eta}, \]

(3.8)

and scalar sequences \( \{t_n\}, \{s_n\} (n \geq 0) \) by:

\[ t_0 = 0, \quad s_0 = \eta, \]

\[ t_{n+1} = s_n + \frac{M (s_n - t_n)^2}{2 (1 - M_0 t_n)} \frac{1 - M s_n + (M - M_0) t_n}{1 - M_0 t_n}, \]

(3.9)

\[ s_{n+1} = t_{n+1} + \frac{\alpha_n}{1 - M_0 t_{n+1}}, \]

(3.10)

where

\[ \alpha_n = \frac{M}{2} (t_{n+1} - s_n)^2 + \frac{13 L (s_n - t_n)^4}{108} + \frac{N M (s_n - t_n)^4}{9 (1 - M_0 t_n)} + \frac{M^3 (s_n - t_n)^4}{3 (1 - M_0 t_n)^2}. \]

(3.11)

Define constant \( \eta_0 \) by:

\[ \eta_0 = \frac{1}{M_0} \min \left\{ \frac{2 - \delta_0}{2 + \delta_0}, \frac{1 - \delta_1}{\delta_1}, \frac{1 - \delta_2}{\delta_2}, \frac{2 M_0}{M + 4 M_0} \right\}. \]

(3.12)

Then, if \( \eta \leq \eta_0 \), we have \( \delta_0 \in [0, 2] \), and the set:

\[ I = \left[ \max \{2 \delta_1, 2 \delta_2, \delta_0\}, 2 \omega_\infty \right] \neq \emptyset. \]

(3.13)

Choose

\[ \delta \in I. \]

(3.14)

In view of (3.9), and (3.10) for \( n = 0 \), there exists \( \eta_1 > 0 \), such that:
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\[ M_0 t_1 < 1, \]  

(3.15)

and

\[ s_1 - t_1 \leq \frac{\delta}{2} (s_0 - t_0), \]

for all \( \eta \in [0, \eta_1]. \)

Define constant \( \beta \) by:

\[ \beta = \min \{ \eta_0, \eta_1 \}. \]

(3.17)

Set

\[ K = \frac{1}{2\beta}, \]

(3.18)

provided that \( \beta \neq 0 \), and

\[ h_A = K \eta. \]

(3.19)

We can show the following result on majorizing sequences for (IFJTM).

**Lemma 3.1.** Assume:

\[ h_A \leq \frac{1}{2}. \]

(3.20)

Then, sequences \( \{t_n\}, \{s_n\} (n \geq 0) \) given by (3.9) and (3.10), are non-decreasing, bounded by:

\[ t^{**} = \left( 1 + \frac{2\delta}{2 - \delta} \right) \eta. \]

(3.21)

and converge to their unique least upper bound \( t^* \) satisfying

\[ t^* \in [0, t^{**}]. \]

(3.22)

Moreover, the following estimates hold for all \( n \geq 0: \)

\[ t_n \leq s_n \leq t_{n+1} \leq s_{n+1}, \]

(3.23)

and
Proof. We shall show using induction on $k$:

$$0 \leq t_{k+1} - s_k \leq \frac{\delta}{2} (s_k - t_k),$$

(3.25)

and

$$0 \leq s_{k+1} - t_{k+1} \leq \frac{\delta}{2} (s_k - t_k),$$

(3.26)

and

$$M_0 t_{k+1} < 1.$$  

(3.27)

Estimates (3.25)-(3.27) hold for $k = 0$ by (3.5), (3.15), (3.16), and the choice of $\delta$ given by (3.14).

Assume (3.24)-(3.27) hold for all $m \leq k$.

By the induction hypotheses, we have:

$$s_m \leq t_m + \frac{\delta}{2} (s_{m-1} - t_{m-1})$$

$$\leq s_{m-1} + \frac{\delta}{2} (s_{m-1} - t_{m-1}) + \frac{\delta}{2} (s_{m-1} - t_{m-1})$$

$$\leq \eta + 2 \left( \frac{\eta}{2} \eta + \left( \frac{\eta}{2} \right)^2 \eta + \cdots + \left( \frac{\eta}{2} \right)^m \eta \right)$$

(3.28)

$$= \eta + \frac{1 - \left( \frac{\eta}{2} \right)^m}{1 - \frac{\eta}{2}},$$

$$t_{m+1} \leq s_m + \frac{\delta}{2} (s_m - t_m)$$

$$= \eta + \frac{1 - \left( \frac{\eta}{2} \right)^m}{1 - \frac{\eta}{2}},$$

(3.29)

Instead of (3.25) and (3.27), we shall show:
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\[ \frac{M(s_m - t_m)^2}{2(1 - M_0 t_m)} \leq \frac{\delta}{2}(s_m - t_m), \quad (3.30) \]

since,

\[ \frac{1 - M s_m + (M - M_0)t_m}{1 - M_0 t_m} \leq 1, \quad (3.31) \]

or

\[ M(s_m - t_m) + \delta M_0 t_m - \delta \leq 0, \quad (3.32) \]

or

\[ M\left(\frac{\delta}{2}\right)^m \eta + \delta M_0 \left[1 + \frac{1 - \left(\frac{\delta}{2}\right)^{m-1}}{1 - \frac{\delta}{2}} \delta + \left(\frac{\delta}{2}\right)^m\right] \eta - \delta \leq 0. \quad (3.33) \]

Estimate (3.33) motivates us to define functions \( f_m \) on \([0, +\infty)(m > 1)\) for \( w = \frac{\delta}{2} \):

\[ f_1(w) = M \eta + 2 M_0 (2 + w) \eta - 2, \]

\[ f_m(w) = M w^{m-1} \eta + 2 M_0 \left(1 + 2 w (1 + w + \cdots + w^{m-2}) + w^m\right) \eta - 2. \quad (3.34) \]

We need a relationship between two consecutive \( f_m \):

\[ f_{m+1}(w) = M w^m \eta + 2 M_0 \left(1 + 2 w (1 + w + \cdots + w^{m-1}) + w^{m+1}\right) \eta - 2 \]

\[ = M w^m \eta + M w^{m-1} \eta - M w^{m-1} \eta \]

\[ + 2 M_0 \left(1 + 2 w (1 + w + \cdots + w^{m-2}) + w^m + w^m + w^{m+1}\right) \eta - 2 \]

\[ = f_m(w) + g_1(w) w^{m-1} \eta, \quad (3.35) \]

where

\[ g_1(w) = 2 M_0 w^2 + (2 M_0 + M) w - M. \quad (3.36) \]

Note that \( \delta_1 \) given by (3.6) is the unique positive zero of function \( g_1 \).
We have:
\[ f_1(0) = (M + 4 M_0) \eta - 2 < 0, \quad (3.37) \]
\[ f_m(0) = 2 ((M + M_0 \eta - 1) < 0, \quad (m > 1). \quad (3.38) \]
and for sufficiently large \( w > 0 \):
\[ f_m(w) > 0, \quad (w > 0), \quad (m \geq 1). \quad (3.39) \]
It then follows from (3.37)-(3.39), and the intermediate value theorem that there exists a positive zero \( w_m \), for each function \( f_m(m \geq 1) \). The zero’s \( w_m \) are unique, since:
\[ f'_m(w) > 0, \quad (w > 0), \quad (m \geq 1). \quad (3.40) \]
Estimate (3.33) certainly holds, if
\[ f_m(w) \leq 0, \quad \text{for all } w \in [0, w_m], \quad (m \geq 1). \quad (3.41) \]
If there exists \( m \geq 0 \), such that \( w_{m+1} \geq \delta_1 \), then using (3.35), we get:
\[ f_{m+1}(w_{m+1}) = f_m(w_{m+1}) + g_1(w_{m+1}) w_{m+1}^{m-1} \eta, \]
or
\[ f_m(w_{m+1}) \leq 0, \quad (3.42) \]
since \( f_{m+1}(w_{m+1}) = 0 \), and \( g_1(w_{m+1}) w_{m+1}^{m-1} \eta \geq 0 \), which imply:
\[ w_{m+1} \leq w_m. \quad (3.43) \]
We can certainly choose the last of the \( w_m \)'s denoted by \( w_\infty \) (given in (3.8), and obtained from (3.33) by letting \( k \to \infty \), to be \( w_{m+1} \).

It then follows, sequence \( \{w_m\} \) is non-increasing, bounded below by zero, and as such it converges to its unique maximum lowest bound \( w^* \) satisfying \( w^* \geq w_\infty \).

Then, estimate (3.41) certainly holds, if
\[ \delta_1 \leq w_\infty, \quad (3.44) \]
which is true by the choice of \( \eta_0 \).
That completes the induction for (3.25) and (3.27).

In order for us to show (3.26) and (3.27), we need the estimates:

\[
13 \frac{L}{108} (s_m - t_m)^4 + \frac{N M (s_m - t_m)^4}{9 (1 - M_0 t_m)} + \frac{M^3 (s_m - t_m)^4}{3 (1 - M_0 t_m)^2}
\]

\[
= \left(13 \frac{L}{108} (1 - M_0 t_m)^2 + \frac{N M (1 - M_0 t_m)}{9} + \frac{M^3}{3} \right) \frac{(s_m - t_m)^4}{(1 - M_0 t_m)^2}
\]

\[
\leq \left(\frac{13 L}{108} + \frac{N M}{9} + \frac{M^3}{3} \right) \frac{(s_m - t_m)^4}{(1 - M_0 t_m)^2}
\]

\[
= \frac{a (s_m - t_m)^4}{(1 - M_0 t_m)^2},
\]

(3.45)

and

\[
\frac{M}{2} (t_{m+1} - s_m)^2 \leq \frac{M^3}{3} \frac{(s_m - t_m)^4}{(1 - M_0 t_m)^2}.
\]

(3.46)

In view of (3.45) and (3.46), we have:

\[
s_{m+1} - t_{m+1} \leq \frac{b (s_m - t_m)^4}{(1 - M_0 t_{m+1})(1 - M_0 t_m)^2}
\]

\[
= \frac{2 b M}{M} \left(\frac{M}{2} \frac{(s_m - t_m)^2}{1 - M_0 t_m} \right) \frac{(s_m - t_m)^2}{(1 - M_0 t_{m+1})}
\]

\[
\leq \frac{2 c}{M} \left(\frac{\delta}{M} \right) \frac{M}{2} \frac{(s_m - t_m)^3}{1 - M_0 t_{m+1}} \frac{1 - M_0 t_m}{1 - M_0 t_m}
\]

(3.47)

\[
\leq \frac{d \left(\frac{\delta}{2}\right)^2 (s_m - t_m)^2 (1 - M_0 t_m)}{1 - M_0 t_{m+1}}
\]

\[
\leq \frac{d \left(\frac{\delta}{2}\right)^2 (s_m - t_m)^2}{1 - M_0 t_{m+1}}.
\]
That is, by (3.26) and (3.47), we can show:

\[ \frac{d \left( \frac{\delta}{2} \right)^2 (s_m - t_m)^2}{1 - M_0 t_{m+1}} \leq \frac{\delta}{2} (s_m - t_m), \quad (3.48) \]

or

\[ d \frac{\delta}{2} (s_m - t_m) + M_0 t_{m+1} - 1 \leq 0, \quad (3.49) \]

or

\[ d \left( \frac{\delta}{2} \right)^m \eta + M_0 \left[ 1 + \frac{1 - \left( \frac{\delta}{2} \right)^m}{1 - \frac{\delta}{2}} \delta + \left( \frac{\delta}{2} \right)^{m+1} \right] \eta - 1 \leq 0. \quad (3.50) \]

We introduce recurrent functions \( p_m \) on \( [0, +\infty) \) \((m \geq 1)\) for \( w = \frac{\delta}{2} \) by:

\[ p_m(w) = dw^{m+1} \eta + 2 M_0 \left( 1 + 2 w (1 + w + \cdots + w^{m-1}) + w^{m+1} \right) \eta - 2. \quad (3.51) \]

We need a relationship between two consecutive \( p_m \):

\[ p_{m+1}(w) = dw^{m+1} \eta - dw^m \eta + M_0 (w^{m+1} + w^{m+2}) \eta + p_m(w) \]

\[ = p_m(w) + g_2(w) w^m \eta, \quad (3.52) \]

where

\[ g_2(w) = M_0 w^2 + (M_0 + d) w - d. \quad (3.53) \]

Note that \( \delta_2 \) given by (3.7) is the unique positive zero of function \( g_2 \).

We have:

\[ p_m(0) = M_0 \eta - 1 < 0, \quad (m \geq 1), \quad (3.54) \]

and

\[ p_m(w) > 0, \quad (w > 0), \quad (m \geq 1), \quad (3.55) \]

for sufficiently large \( w > 0 \).
It follows from (3.54), (3.55), and the intermediate value theorem that there exists a positive zero \( w_m^1 \) for each function \( p_m (m \geq 1) \). The zero's \( w_m^1 \) are unique, since:

\[
p_m'(w) > 0, \quad (w > 0), \quad (m \geq 1).
\]  

(3.56)

If there exists \( m \geq 0 \), such that \( w_{m+1}^1 \geq \delta_2 \), then using (3.52), we get:

\[
p_{m+1}(w_{m+1}^1) = p_m(w_{m+1}^1) + g_2(w_{m+1}^1)w_{m+1}^{m-1} \eta,
\]

or

\[
p_m(w_{m+1}^1) \leq 0,
\]  

(3.57)

since \( p_{m+1}(w_{m+1}^1) = 0 \), and \( g_2(w_{m+1}^1)w_{m+1}^{m-1} \eta \geq 0 \), which imply:

\[
w_{m+1}^1 \leq w_m^1, \quad (m \geq 0).
\]  

(3.58)

Estimate (3.50) certainly holds, if we show:

\[
p_m(w) \leq 0, \quad \text{for all } w \in \left[0, w_m^1\right], \quad (m \geq 1).
\]  

(3.59)

We can certainly choose the last of the \( w_m^1 \)'s denoted by \( w_\infty^1 \) (obtained from (3.50) by letting \( k \to \infty \)), to be \( w_{m+1}^1 \). Note that \( w_\infty = w_\infty^1 \).

It then follows, sequence \( \{w_m^1\} \) is non-increasing, bounded below by zero, and as such it converges to its unique maximum lowest bound \( w^* \), satisfying \( w^* \geq w_\infty \).

Then, estimate (3.59) certainly holds, if

\[
\delta_2 \leq w_\infty,
\]  

(3.60)

which is true by the choice of \( \eta_0 \).

The induction is now completed.
It follows that sequences \( \{ t_n \} \), \( \{ s_n \} \) are non-decreasing, bounded above by \( t^{**} \), and as such they converge to their common, and unique least upper bound \( t^* \), satisfying (3.22).

That completes the proof of Lemma 3.1.

We next show that the order of convergence of scalar iteration \( \{ t_n \} \) is four.

**Proposition 3.2.** Under the hypotheses of Lemma 3.1, further assume:

\[
\frac{3}{b} \eta < 1. \tag{3.61}
\]

Fix:

\[
q \in \left( \frac{3}{b}, \frac{1}{\eta} \right), \quad \eta \neq 0. \tag{3.62}
\]

Define parameters \( p_0, p \) by:

\[
p_0 = \frac{1}{M_0} \left( 1 - \frac{\sqrt{3b}}{q} \right), \tag{3.63}
\]

\[
p = \frac{M q}{2 \sqrt{3b}}, \quad b \neq 0, \tag{3.64}
\]

and

function \( g_3 \) on \( [0, \frac{1}{q}] \) by:

\[
g_3(t) = t + \frac{1}{q} + \frac{p}{q^2} \left( \frac{(q t)^2}{1 - (q t)^2} + t^2 \right). \tag{3.65}
\]

Moreover, assume:

\[
\min\{ t_1, g_3(\eta) \} \leq p_0. \tag{3.66}
\]

Then, the following estimates hold for all \( k \geq 0 \):
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\[ t_{k+1} - s_k \leq \frac{p}{q^2} (q \eta)^{k+1}, \]  

(3.67)

and

\[ s_k - t_k \leq \frac{1}{q} (q \eta)^{k^2}. \]  

(3.68)

**Proof.** We shall show:

\[ s_{m+1} - t_{m+1} \leq q^3 (s_m - t_m)^4. \]  

(3.69)

If estimate (3.69) holds, then

\[ q (s_{m+1} - t_{m+1}) \leq (q (s_m - t_m))^4 \]

\[ \leq (q \eta)^{m+1}, \]  

(3.70)

which implies (3.68).

We have the estimate

\[ \frac{M}{2} (t_{m+1} - s_m)^2 + \frac{13L}{108} (s_m - t_m)^4 \leq \frac{N M}{9 (1 - M_0 t_m)} (s_m - t_m)^4 \]

\[ + \frac{M^3}{3 (1 - M_0 t_m)^2} (s_m - t_m)^4 \]

\[ \leq \frac{M}{2} \left( \frac{M (s_m - t_m)^2}{2 (1 - M_0 t_m)} \right)^2 + \frac{13L}{108} (s_m - t_m)^4 \]

\[ + \frac{N M}{9 (1 - M_0 t_m)} (s_m - t_m)^4 + \frac{M^3}{3 (1 - M_0 t_m)^2} (s_m - t_m)^4 \]

\[ \leq \frac{M^3}{8} \frac{(s_m - t_m)^4}{(1 - M_0 t_m)^2} + \frac{13L}{108} \frac{(s_m - t_m)^4}{(1 - M_0 t_m)^2} (1 - M_0 t_m)^2 \]

\[ + \frac{N M (s_m - t_m)^4}{9 (1 - M_0 t_m)^2} (1 - M_0 t_m) + \frac{M^3}{3} \frac{(s_m - t_m)^4}{(1 - M_0 t_m)^2} \]
\[ \leq \frac{b (s_m - t_m)^4}{(1 - M_0 t_m)^2}, \]

that is, we have

\[ s_{m+1} - t_{m+1} \leq \frac{b (s_m - t_m)^4}{(1 - M_0 t_{m+1})(1 - M_0 t_m)}. \]

Instead of showing (3.69), we can show:

\[ \frac{b (s_m - t_m)^4}{(1 - M_0 t_m)^2(1 - M_0 t_{m+1})} \leq q^3 (s_m - t_m)^4, \quad (3.71) \]

or

\[ \frac{b}{(1 - M_0 t_{m+1})^3} \leq q^3, \quad (3.72) \]

or

\[ t_{m+1} \leq p_0. \quad (3.73) \]

By hypothesis (3.66), we have:

\[ t_1 \leq p_0. \quad (3.74) \]

Assume:

\[ t_m \leq p_0. \quad (3.75) \]

We also have:

\[ t_{m+1} - s_m = \frac{M (s_m - t_m)^2}{2(1 - M_0 t_m)} \leq \frac{M q}{2^{3/2} b} (s_m - t_m)^2 = p (s_m - t_m)^2. \quad (3.76) \]

We get in turn:

\[ t_{m+1} \leq (s_m - t_m) + (t_m - s_{m-1}) + \cdots + (t_1 - s_0) + s_0 + p (s_m - t_m)^2 \leq \eta + \frac{1}{q}(q \eta)^m + p \left( (s_m - t_m)^2 + (s_{m-1} - t_{m-1})^2 + \cdots + (s_0 - t_0)^2 \right) \]
\[
\eta + \frac{1}{q} (q \eta)^4 + \frac{p}{q^2} \left( (q \eta)^4 + (q \eta)^3 + \cdots + \eta^2 \right)
\]
\[
= \eta + \frac{1}{q} (q \eta)^4 + \frac{p}{q^2} \left( (q \eta)^4 + (q \eta)^3 + \cdots + \eta^2 \right)
\]
\[
+ \left( (q \eta)^4 + \eta^2 \right)
\]
\[
\leq \eta + \frac{1}{q} + \frac{p}{q^2} \left( (q \eta)^2 + (q \eta)^2 + \cdots + \eta^2 \right)
\]
\[
\leq \eta + \frac{1}{q} \left( \frac{(q \eta)^2}{1 - (q \eta)^2} + \eta^2 \right) = g_3(\eta) \leq p_0.
\]  
(3.77)

which completes the induction for (3.73), and the proof of Proposition 3.2.

\[\Box\]

We state the main semilocal convergence result for (IFJTM). The proof simply uses Lemma 3.1 and Proposition 3.2, instead of (2.6) and (2.7). Hence, it is omitted (see also Remark 2.2).

**Theorem 3.3.** Under hypotheses (2.1)-(2.5) and (2.23), further assume: hypotheses of Lemma 3.1 hold, and

\[\bar{U}(x, t^*) \subseteq D.\]  
(3.78)

Then, sequences \{x_n\}, \{y_n\} generated by (IFJTM) are well defined, remain in \(\bar{U}(x, t^*)\) for all \(n \geq 0\), and converge to a unique solution \(x^*\) of equation \(F(x) = 0\) in \(\bar{U}(x, t^*)\).

Moreover, the following estimates hold:

\[\|y_n - x_n\| \leq s_n - t_n,\]
\[\|x_{n+1} - y_n\| \leq t_{n+1} - s_n,\]
\[\|x_n - x^*\| \leq t^* - t_n,\]
\[\|y_n - x^*\| \leq t^* - s_n.\]
Furthermore, under hypotheses of Proposition 3.2, estimates (3.67), and (3.68) also hold.

Finally, if there exists \( R \geq t^* \), such that

\[
U(x, R) \subseteq \mathcal{D},
\]

and

\[
R \leq \frac{2}{M_0} - t^*,
\]

then, the solution \( x^* \) is unique in \( U(x, R) \).

**Example 3.4.** Let \( \mathcal{X} = \mathcal{Y} = \mathbb{R} \), \( \mathcal{D} = [2, 4] \), \( x_0 = 2 \), and define function \( F \) on \( \mathcal{D} \) by:

\[
F(x) = x^3 - 2x - 5.
\]

Using (2.1)-(2.7), we obtain

\[
\eta = .1, \quad M = 1.8, \quad N = .6, \quad M_0 = 1.5, \quad L = 0, \quad K = 1.8183312,
\]

and

\[
h = .18183312 < .46568.
\]

Hence, the conclusions of Theorem 2.1 hold for equation \( F(x) = 0 \).

Concerning the hypotheses of Theorem 3.3, we have:

\[
\delta_0 = .18, \quad \frac{1}{M_0} \left( \frac{2 - \delta_0}{2 + \delta_0} \right) = .556574924,
\]

\[
\delta_1 = .313552873, \quad \frac{1}{M_0} \left( \frac{1 - \delta_1}{\delta_1} \right) = 1.459503189,
\]

\[
a = 2.064, \quad b = 4.008, \quad c = 4.4533, \quad d = 4.948148148,
\]

\[
\delta_2 = .664620037, \quad \frac{1}{M_0} \left( \frac{1 - \delta_2}{\delta_2} \right) = .336412731,
\]

\[
\frac{2M_0}{M + 4M_0} = .384615384, \quad \eta_0 = \eta_1 = .336412731,
\]
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\[ \overline{K} = 1.486269555, \quad h_A = 1.48626956 < .5, \]

\[ t_1 = .10738, \quad t_1 - s_0 = .00738, \]

set \( \delta = 2 \delta_1 \), then, we get:

\[ M_0 t_1 = .16107 < 1, \]

and

\[ s_1 - t_1 = .000304457 < \frac{\delta}{2} (s_0 - t_0) = .031355287. \]

Hence, conditions of Theorem 3.3 are also satisfied.

Let us compare the bounds, between Theorems 2.1, and 3.3:

\[ v_1 = .107438492, \quad v_1 - s_0 = .007438492 > t_1 - s_0, \]

and

\[ w_1 - v_1 = .003798007 > s_1 - t_1, \]

which shows that the bounds of Theorem 3.3 are finer than the ones of Theorem 2.1.

Concerning the hypotheses of Proposition 3.2, we have for \( q = 4 \):

\[ \sqrt[3]{b} = 1.597913947, \quad p = 2.252937342, \quad p_0 = .400347676, \]

\[ g_3(\eta) = .378228768 < p_0, \]

and

\[ M_0 t_1 = .16107 < p_0. \]

Hence, the conclusions of Proposition 3.2 also hold for equation \( F(x) = 0 \).

Note that in practice, one will test hypotheses of both theorems, and use the combination producing the best results.

**Remark 3.5.** (a) Condition (3.66) can be replaced by a stronger, but easier to check:
for $\delta \in I$ (see, (3.21) and (3.13)).

The best possible choice for $\delta$ seems to be $\delta = \delta_3$. Let

$$\delta_3 = \max\{2 \delta_1, 2 \delta_2, \delta_0\}.$$ 

In this case, (3.80) is written as

$$\eta \leq \frac{(2 - \delta_3)p_0}{2}. \tag{3.81}$$

(b) The ratio of convergence "q\eta" given in Proposition 3.2 can be smaller than "$\frac{3}{\sqrt{5}} \theta$" given in Theorem 2.1 for $q$ close to $\frac{3}{\sqrt{b}}$, and $M, N, L$ not all zero, and $\eta > 0$.

Set $\alpha = \frac{3}{\sqrt{b}} \eta$, and $\beta = \frac{3}{\sqrt{5}} \theta$. Note that $b < K$, and $40 K^3 > b$. By comparing $\alpha$ and $\beta$, we have for

$$h_0 = \frac{1}{2K} \left[ 1 - \left( \frac{40 K^3}{b} \right)^{\frac{1}{6}} - 1 \right]^2.$$ 

**Case 1.** If

$$2.666 M^3 + .444 N M - 6.740740 L \leq 0,$$

or

$$2.666 M^3 + .444 N M - 6.740740 L > 0,$$

and

$$\eta > h_0,$$

then, we have:

$$\alpha < \beta.$$
Case 2. If
\[ 2.666 M^3 + .444 N M - 6.740740 L > 0, \]
and
\[ \eta < h_0, \]
then, we have:
\[ \alpha > \beta. \]

Case 3. If
\[ 0 < \eta = h_0, \]
then, we have:
\[ \alpha = \beta. \]

Note that the \( p \)-Jarratt-type method (\( p \in [0, 1] \)), given in [7], uses (2.1)-(2.5), but the sufficient convergence conditions are different from the ones given in this study, and guarantees only third order of convergence (not fourth obtained here), in the case of Jarratt method (for \( p = \frac{2}{3} \)).

4. Conclusion

We provided a semilocal convergence analysis for Jarratt-type method in order to approximate a locally unique solution of a nonlinear equation in a Banach space, involving a thrice differentiable operator.

Using our new idea of recurrent functions, we provided a semilocal convergence analysis with the following advantages over the work in [1]: larger convergence domain, and weaker sufficient convergence conditions. Note that these advantages are obtained under the same computational cost as in [1].

A numerical example, and some favorable comparisons with the previous works are also provided.
References


