

INVERSE TRANSMISSION PROBLEM FOR EQUATIONS OF THE STURM-LIOUVILLE TYPE WITH A DELAY LONGER THAN HALF THE INTERVAL

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Abstract

In this paper, we consider a boundary value problem generated with a second-order Sturm-Liouville type operator with a delay from the interval $\left(\frac{\pi}{2}, \pi\right)$.

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Using Fourier series, we study the inverse spectral problem of the reconstruction of the potential function and delay points of Sturm-Liouville differential operators using their spectral characteristics.

1. Introduction

Inverse spectral problems are problems in which we study an operator via some spectral characteristics of that operator. In this work, we use spectra to reconstruct operator parameters. Inverse problems with discontinuous conditions within intervals play an important role in mathematics, mechanics, radio electronics, geophysics and other fields of science and technology. We consider a boundary value problem generated with a second-order Sturm-Liouville type operator with a delay from the interval $\left(\frac{\pi}{2}, \pi\right)$. Using Fourier series, we study the inverse spectral problem of the reconstruction of the potential function and delay points of Sturm-Liouville differential operators using their spectral characteristics.

Let us now consider the boundary problem.

$$ly := y''(x) + q(x)y(x - \alpha) = \lambda y(x), \quad x \in (0, \pi), \quad (1.1)$$

with boundary conditions

$$y(0) = y^{(j)}(\pi) = 0, \quad \text{for } j = 0, 1, \quad (1.2)$$

and the jump condition

$$y\left(\frac{\pi}{2} + 0\right) = yb\left(\frac{\pi}{2} - 0\right), \quad y'\left(\frac{\pi}{2} + 0\right) = b^{-1}y'\left(\frac{\pi}{2} - 0\right) + c\left(\frac{\pi}{2} - 0\right), \quad (1.3)$$

where is $q(x) \in L^2(0, \pi]$ and $q(x) = 0$ for $x < \alpha$, $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, $b \neq 0$ and c are real number. Coefficients b and c are known and fixed in advance. For simplicity, we will use the label $L_j := L_j(q(x); \alpha; b, c)$ for problems (1.1)-(1.3).

2. Asymptotic Form of the Solution and Eigenvalues

Let it be $\varphi(0, \lambda)$ solution of the Equation (1.1) with initial conditions

$$\varphi(0, \lambda) = 0, \quad \varphi'(0, \lambda) = 1. \quad (2.4)$$

For everyone x and $j = 0, 1$, functions $\varphi^j(x, \lambda)$ are entire functions in λ order $\frac{1}{2}$. Function $\varphi(x, \lambda)$ is the unique solution of the integral equation

$$\varphi(x, \lambda) = \int_0^x \frac{\sin \rho(x-t)}{\rho} q(t) \varphi(t-a, \lambda) dt, \quad (2.5)$$

where is $\rho^2 = \lambda$. Solving (2.5) by the method of successive approximations, we obtain

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \varphi_1(x, \lambda). \quad (2.6)$$

Using the jump conditions (1.3) and the initial conditions (2.4), we get

$$\varphi_0(x, \lambda) = \begin{cases} \frac{\sin \rho x}{\rho}, & x < \frac{\pi}{2}, \\ a^+ \frac{\sin \rho x}{\rho} + a^- \frac{\sin \rho(\pi-x)}{\rho} + \frac{c}{2\rho^2} (\cos \rho(\pi-x) - \cos \rho x), & x > \frac{\pi}{2}, \end{cases} \quad (2.7)$$

$$\varphi'_0(x, \lambda) = \begin{cases} \cos \rho x, & x < \frac{\pi}{2}, \\ a^+ \cos \rho x - a^- \cos \rho(\pi-x) + \frac{c}{2\rho} (\sin \rho(\pi-x) + \sin \rho x), & x > \frac{\pi}{2}, \end{cases} \quad (2.8)$$

where is

$$a^+ = \frac{b + b^{-1}}{2}, \quad a^- = \frac{b - b^{-1}}{2}, \quad (2.9)$$

and

$$\varphi_1(x, \lambda) = \begin{cases} 0, & x < a \\ \int_a^x \frac{\sin \rho(x-t)}{\rho} \rho(t) \varphi_0(t-a, \lambda) dt, & x \geq a \end{cases} \quad (2.10)$$

$$\varphi_1'(x, \lambda) = \begin{cases} 0, & x < a \\ \int_a^x \frac{\cos \rho(x-t)}{\rho} \rho(t) \varphi_0(t-a, \lambda) dt, & x \geq a \end{cases}. \quad (2.11)$$

Using formulas (2.7), (2.8), (2.10) and (2.11), we calculate

$$\varphi_1(x, \lambda) = \frac{1}{\rho^2} \int_a^x q(t) \sin \rho(x-t) \sin \rho(t-a) dt, \quad (2.12)$$

and

$$\varphi_1'(x, \lambda) = \frac{1}{\rho} \int_a^x q(t) \cos \rho(x-t) \sin \rho(t-a) dt. \quad (2.13)$$

Let us introduce the notation $\Delta_1(\lambda) := \varphi_{(j)}(\pi, \lambda)$, $j = 0, 1$. The functions $\Delta_j(\lambda)$ are entire functions in λ of order $\frac{1}{2}$ and the zeros of $\Delta_j(\lambda)$ coincide with the eigenvalues λ_n and μ_n of $L_0(q(x); a; b, c)$ and $L_1(q(x); a; b, c)$, respectively. The functions $\Delta_j(\lambda)$ are called *characteristic functions* of $L_j(q(x); a; b, c)$. From Equations (2.6)-(2.8) and (2.12)-(2.13), we obtain the following asymptotic formula for $|\rho| \rightarrow \infty$;

$$\begin{aligned} \Delta_0(\lambda) &:= \varphi(\pi, \lambda) \\ &= a^+ \frac{\sin \rho \pi}{\rho} + \frac{c}{2\rho^2} (1 - \cos \rho \pi) - \frac{1}{2\rho^2} \cos \rho(\pi - a) \int_a^\pi q(t) dt \\ &\quad + O\left(\frac{\exp(|\tau|(\pi - a))}{\rho^3}\right), \end{aligned} \quad (2.14)$$

and

$$\Delta_0(\lambda) := \varphi'(\pi, \lambda)$$

$$\begin{aligned} &= a^+ \cos \pi \rho - a^- + \frac{c}{2\rho} \sin \rho \pi + \frac{1}{2\rho} \sin \rho(\pi - a) \int_a^\pi q(t) dt \\ &\quad + 0 \left(\frac{\exp(|\tau|(\pi - a))}{\rho^2} \right). \end{aligned} \quad (2.15)$$

Using Equation (2.14) and (2.15), we obtain the asymptotic formula for the eigenvalues

$$\lambda_n = \rho_{n0}^2 \text{ and } \mu_n = \rho_{n1}^2 \text{ when } n \rightarrow \infty,$$

$$\rho_{n0} = n + \frac{1}{2\pi n a^+} \left(\cos n a \int_a^\pi q(t) dt + c(1 - (-1)^n) \right) + o\left(\frac{1}{\pi}\right), \quad (2.16)$$

$$\rho_{n1} = \rho_{n1}^0 + O\left(\frac{1}{\pi}\right), \quad (2.17)$$

where ρ_n^0 is root of $\Delta_1^0(\lambda) = a^+ \cos \pi \rho - a^-$. So, we get that

$$\rho_{n1}^0 = n - \frac{1}{2} + \frac{1}{\pi} \arccos \frac{a^-}{a^+}.$$

Lemma 1. *Spectrum specifications $\{\lambda_{nj}\}$, $n \geq 1$, $j = 0, 1$, $\Delta_j(\lambda)$ uniquely determine the characteristic function $\Delta_j(\lambda)$ by the formulas:*

$$\Delta_0(\lambda) = a^+ \pi \prod_{n=1}^\infty \left(\frac{\lambda_n - \lambda}{n^2} \right), \quad \Delta_1(\lambda) = a^+ \prod_{n=1}^\infty \left(\frac{\mu_n - \lambda}{\rho_{n1}^0 2} \right). \quad (2.18)$$

Proof. By Hadamard's factorization theorem, $\Delta_0(\lambda)$ is uniquely determined up to the product of a constant and its zeros:

$$\Delta_0(\lambda) = a^+ C \prod_{n=1}^\infty \left(1 - \frac{\lambda}{\lambda_n} \right). \quad (2.19)$$

(The case when $\Delta_0(0) = 0$ requires minimal modifications).

Let's consider the function $\tilde{\Delta}_0(\lambda) := a^+ \frac{\sin \rho}{\rho}$.

Using Hadamard factorization we get an infinite product

$$\frac{\sin \rho}{\rho} = \pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n^2} \right),$$

relatively

$$\tilde{\Delta}_0(\lambda) := a^+ \pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2} \right).$$

Then,

$$\frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = \frac{C}{\pi} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - n^2}{n^2 - \lambda} \right).$$

Introducing (2.14) and (2.19), we calculate

$$\lim_{\lambda \rightarrow -\infty} \frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = 1, \quad \lim_{\lambda \rightarrow -\infty} \left(1 + \frac{\lambda_n - n^2}{n^2 - \lambda} \right) = 1,$$

because of that

$$C = \pi \prod_{n=1}^{\infty} \frac{\lambda_n}{n^2}.$$

Substituting this into (2.19) we arrive at (2.18).

3. Reconstruction of potential

In this section, we will first show that the delay point is uniquely determined by the spectrum.

Lemma 2. *If it is $\{\lambda_{nj}\}_{n=1}^{\infty}$ spectrum of $L_0(q(x); a; b, c)$, then the delay point is uniquely determined.*

Proof. There are infinitely many numbers $k \in N$ such that $\sin(ka) \neq 0$.

From (2.16), it follows

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{\lambda_{k+2} - (k+2)^2 - \lambda_{k-2} + (k-2)^2}{\lambda_{k+1} - (k+1)^2 - \lambda_{k-1} + (k-1)^2} \\ &= \lim_{k \rightarrow \infty} \frac{\cos a(k+2) - \cos a(k-2)}{\cos a(k+1) - \cos a(k-1)} = \lim_{k \rightarrow \infty} \frac{\sin ka \sin 2a}{\sin ka \sin a} = 2 \cos a. \end{aligned} \quad (3.20)$$

Finally,

$$a = \arccos \left(\frac{1}{2} \lim_{k \rightarrow \infty} \frac{\lambda_{k+2} - (k+2)^2 - \lambda_{k-2} + (k-2)^2}{\lambda_{k+1} - (k+1)^2 - \lambda_{k-1} + (k-1)^2} \right). \quad (3.21)$$

Lemma 3. *If it is $\{\lambda_{nj}\}_{n=1}^{\infty}$ spectrum of $L_0(q(x); a; b, c)$, then it is $\int_a^{\pi} q(t)dt$ uniquely determined.*

Proof. There are infinitely many numbers $k \in N$ such that $\cos(ka) \neq 0$.

From (2.16), we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{\pi a^+ \left(\lambda_k - \left(k^2 + \frac{c}{\pi a^+} (1 - (-1)^k) \right) \right)}{\cos ka} \\ &= \lim_{k \rightarrow \infty} \frac{\pi a^+ \left(\frac{\cos ka}{\pi a^+} \int_a^{\pi} q(t)dt + o(1) \right)}{\cos ka} = \int_a^{\pi} q(t)dt. \end{aligned} \quad (3.22)$$

Using $q(x) = 0$ for $x \in [0, a)$, we get that $\int_0^{\pi} q(t)dt = \int_a^{\pi} q(t)dt$.

Using Lemma 3, we get the first Fourier coefficient of the potential q on $[0, \pi]$. Now we will get the other Fourier coefficients of the potential $q(x)$ and we will prove that these coefficients are uniquely determined by the spectra L_0 and L_1 .

Let's mark it with $a_n = \int_0^\pi q(t) \cos 2nt \, dt$ and $b_n = \int_0^\pi q(t) \sin 2nt \, dt$

Fourier potential coefficients q . We come to the main result.

Theorem 1. *If $\{\lambda_n\}_{n \geq 1}$ and $\{\mu_n\}_{n \geq 1}$ are the eigenvalues of the boundary problems L_0 and L_1 , respectively, then the Fourier coefficients a_n and b_n of the potential q are uniquely determined for each $n \in N$.*

Proof. Using existing knowledge, we get that the characteristic functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ have the following form:

$$\Delta_0(\lambda) = a^+ \frac{\sin \rho \pi}{\rho} + \frac{c}{2\rho^2} (1 - \cos \rho \pi) + \frac{1}{\rho^2} \int_a^\pi q(t) \sin \rho(\pi - t) \sin \rho(t - a) dt,$$

and

$$\Delta_1(\lambda) = a^+ \cos \rho \pi - a^- + \frac{c}{2\rho^2} \sin \rho \pi + \frac{1}{\rho^2} \int_a^\pi q(t) \cos \rho(\pi - t) \sin \rho(t - a) dt.$$

We transform the product of trigonometric functions into a sum and get

$$\begin{aligned} \Delta_0(\lambda) = & a^+ \frac{\sin \rho \pi}{\rho} + \frac{c}{2\rho^2} (1 - \cos \rho \pi) - \frac{\cos \rho(\pi - a)}{2\rho^2} \int_a^\pi q(t) dt \\ & + \frac{\cos \rho(\pi + a)}{2\rho^2} \int_a^\pi q(t) \cos(2\rho t) dt + \frac{\sin \rho(\pi + a)}{2\rho^2} \int_a^\pi q(t) \sin(2\rho t) dt, \end{aligned}$$

and

$$\Delta_1(\lambda) = a^+ \cos \rho \pi - a^- + \frac{c}{2\rho} \sin \rho \pi \frac{\sin \rho(\pi + a)}{2\rho} \int_a^\pi q(t) dt$$

$$+ \frac{\sin \rho(\pi + \alpha)}{2\rho} \int_a^\pi q(t) \cos(2\rho t) dt + \frac{\cos \rho(\pi + \alpha)}{2\rho} \int_a^\pi q(t) \sin(2\rho t) dt.$$

Now putting $\rho = n$, for $n \in N$, and using

$$a_n = \int_0^\pi q(t) \cos(2nt) dt = \int_a^\pi q(t) \cos(2nt) dt$$

and

$$b_n = \int_0^\pi q(t) \sin(2nt) dt = \int_a^\pi q(t) \sin(2nt) dt,$$

we get

$$a_n = 2n(nA_n \cos n(\pi + \alpha) - B_n \sin n(\pi + \alpha)),$$

$$b_n = 2n(nA_n \sin n(\pi + \alpha) + B_n \cos n(\pi + \alpha)),$$

where is

$$A_n = \Delta_0(n) = a^+ \frac{\sin n\pi}{n} - \frac{C}{2n^2} (1 - \cos n\pi) + \frac{\cos n(\pi - \alpha)}{2n^2} \int_a^\pi q(t) dt,$$

$$B_n = \Delta_1(n) - a^+ \cos n\pi + a^- - \frac{c}{2n} \sin n\pi - \frac{\sin n(\pi - \alpha)}{2n} \int_a^\pi q(t) dt.$$

Using Lemmas 2 and 3, we can calculate A_n and B_n . We determined the coefficients of the Fourier series and calculated the potential $q(x)$.

4. Conclusion

In this paper, we started from the assumption that the given spectra are $\{\lambda_n\}_{n=1}^\infty$ and $\{\mu_n\}_{n=1}^\infty$ of the spaces L_0 and L_1 . We used Equations (3.21) and (3.22) to determine the delay point a and $a_0 = \int_0^\pi q(t) dt$. We applied Lemma 1 to calculate $\Delta_0(n)$ and $\Delta_1(n)$. We calculated the

values of the coefficients a_n and b_n of the Fourier series by applying Theorem 1. This opens up space for further research into the transmission problem with a delay of less than half the interval

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