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ON ZERO-CENTERED SOLUTIONS OF EULER HIGHER ORDER LINEAR SINGULAR DIFFERENTIAL EQUATION IN THE SPACE K'

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Abstract

The main objective set in this research work is to carry out investigations on the study of the existence of solutions (solutions centered on zero), in the distributional sense, of the non-homogeneous *Euler* linear singular differential equation of order l of the general following form $\sum_{i=0}^{l} a_i x^{k_i} y^{(i)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x)$,

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where l is a natural number not equal to zero; μ, s are two natural numbers, $(a_i)_{0\leq i\leq l}$ real numbers and more $a_l \neq 0, k_i \in \mathbb{Z}_+, k_i \geq i, i = 0, 1, \ldots, l$. $\delta^{(0)}(x) = \delta(x))$ is the Dirac distributional centered on 0 and $\delta^{(s)}(x)$ is the s-th-order derivative of the Dirac delta distribution.

In the scientific work undertaken in some of our published articles, we have completely carried out the investigations on the study of the solvency of the singular linear differential equation of the first order. Confer our references. This allowed us to obtain the conditions of solvency of the equation in question and, we were able to exhaustively identify all the solutions, both distributional and classical, according to the different relationships between the parameters appearing in the latter. Even further in depth, we carried out investigations relating to the same type of equation in $(S_0^\beta)'$ a distributional space larger than K' to realize the existence of other solutions of another nature to the homogeneous equation associated with the non-homogeneous equation. All this leads us to imagine, from then on, the situation which could arise from the same questions of investigation in relation to a general singular equation of order l. We make an exhaustive description of all distributional solutions centered on zero of this equation in the functional space of distributions K'.

1. Introduction

The importance of differential equations in the study of physical phenomena no longer needs to be demonstrated, as long as they describe the mathematical behaviour of said phenomena. However, complete mastery of the phenomenon described lies in solving the differential equation which describes it. Something that is not always easy. We remember that a differential equation in a simple way is an equation that establishes a dependence relationship between the independent variable, the function itself as well as its successive derivatives. We therefore note that differential equations are an essentially important tool for the scientific and mathematical description of natural phenomena since they are the foundation of several classical physical theories. Among others, by way of illustration, we can obviously and quite naturally mention in a non-exhaustive way, among others, Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schrödinger's equation for quantum mechanics and Einstein's equation for the general theory of gravitation.

Let us recall a very important aspect with respect to normal linear homogeneous systems of ODE with infinitely smooth coefficients which have no other solutions of generalized functions than the classical solutions, gradually described and presented with all salient details in the article [14]. However, there is reason to agree and also to make the relevant observation that contrary to this case, and concerning the equations which admit singularities at the level of the coefficients, we note cases of appearance of new forms of solutions in generalized functions then that there may be a complete disappearance of certain traditional solutions for this purpose.

Recall that within the formula $y(x) = \sum_{j=0}^{\gamma_0} C_j \delta^{(j)}(x)$ the number γ_0 is called the order of the distribution. It is important to emphasize according to the theory of distributions that solutions in the form of generalized functions in the distributional sense of ordinary linear differential equations can be derived so many times. In relation to these different fields, several scientific researches are conducted and developed daily with continuous progress, thus opening immense perspectives and avenues, scientific aspects and properties in the theory of linear differential equations and functional differential equations.

Knowing that homogeneous linear ordinary differential equations with infinitely smooth coefficients do not have generalized solutions in the distributional sense; while ordinary linear differential equations with polynomial coefficients such as the Cauchy-Euler equation defined by the equation: $\sum_{m=0}^{l} a_m(x) \frac{d^m y}{dx^m} = g(x)$, where $a_m(x) = c_m x^m$ with $c_m \in \mathbb{R}$, $c_l = 1$, and $g(x) = 0$ can admit a classical solution or a generalized solution in the distributional sense and this can be found there as well as all the necessary details in the references [4-8].

It should be emphasized that several scientific works have recently devoted particular interest to the problems of existence of solutions to differential and functional equations (FDE) in various known spaces of generalized functions. Also, we can note that distributional solutions defined as series of Dirac delta functions and its derivatives of a certain order were used in various fields of applied mathematics such as the theory of partial differential equations, operational calculus and functional analysis without forgetting quantum electrodynamics in the field of physics, following Jhanthanam, Nonlaopon, Orankitjaroen and Abdourahman in their papers, see, respectively [12] and [6, 15, 16, 17].

Indeed, solving a first order linear differential equation of the following simple form $x^n y(x) = \delta^{(s)}(x)$ via the Fourier transform shows that the distributional solution is not unique and depends on n arbitrary constants. The form of the solution of the latter allows us to have an idea of the fairly complex situations that can appear when we consider studying the solvability of an equation of general form. By applying the principle of superposition of the solutions of an ordinary linear differential equation and taking into account the results obtained in our previous publications, we can formulate theorems to inventory the different solutions for this purpose.

So that here in this work, we investigate the question of the solvability of a linear singular differential equation of l-order with singularity and Dirac delta function (or it derivatives of some order) in the second right hand side in the space of generalized functions K'. Otherwise, we investigate the existence of solutions in the sense of distributions of such equation.

Namely, we focus our consideration to the *Euler* equation of the following kind:

$$\sum_{i=0}^{l} a_i x^{k_i} y^{(i)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x), \tag{1}$$

where l is a natural number not equal to zero; μ, s are two natural numbers, $(a_i)_{0 \leq i \leq l}$ are real numbers and more $a_l \neq 0$, $k_i \in \mathbb{Z}_+$, $k_i \geq i$, $k_i = k_0 + i$; $i = 0, 1, \ldots, l$, $\delta^{(s)}(x) = \delta(x))$ is the Dirac distributional centered on 0 and $\delta^{(s)}(x)$ is the s-th-order derivative of the Dirac delta distribution.

Pursuing the objectives in the same direction relative to Equation (1), we are interested and we focus our attention in this article only on the problem of finding all solutions centered on zero of this equation considered. We are not looking for other types of solutions in this work.

The work carried out in this article is organized as follows: First in Section 2, we recall some well-known basic notions and concepts previously used in our published papers related to the theory of distributions (generalized functions). Then Section 3 itself presents an exhaustive description of the *Euler case* and some results. Section 4 is dedicated to the investigation of the solvency (existence of solutions centered on zero) of the equation considered in the so-called *Euler type* case. We present in Section 5 entitled conclusion the summary of the work carried out in the article.

2. Preliminaries

In this section, we briefly review the notion and necessary concepts of generalized functions (we refer to [2], [3], [9] and [12] for a detailed study) used as previously in our published papers and in these investigations.

Definition 2.1. Let K be the space consisting of all real-valued functions $\varphi(x)$ with continuous derivatives of all orders and compact support. The support of $\varphi(x)$ is the closure of the set of all elements $t \in \mathbb{R}$ such that $\varphi(x) \neq 0$. Then, $\varphi(x)$ is called a *test function*.

Definition 2.2. A distribution T is a continuous linear functional on the space K on the space of the real-valued functions with infinitely-differentiable and bounded support. The space of all such distributions is denoted by K'.

For every $T \in K'$ and $\varphi(x) \in K$, the value T has on $\varphi(x)$ is denoted by $(T, \varphi(x))$. Note that $(T, \varphi(x)) \in \mathbb{R}$. Below, let us give some examples of distributions.

(a) A locally-integrable function $g(x)$ is a distribution generated by the locally-integrable function $g(x)$. Here, we define the following relationship:

$(g(x), \varphi(x)) = \int g(x)\varphi(x)dx$ integration on the support Ω, and $\varphi(x) \in K$.

In this case the distribution is called regular distribution.

(b) The Dirac delta function is a distribution defined by $(\delta(x), \varphi(x)) = \varphi(0)$, and the support of $\delta(x)$ is $\{0\}$.

In this case the distribution is called irregular distribution or singular distribution.

Definition 2.3. The s-th-order derivative of a distribution T, denoted by $T^{(s)}$, is defined by $(T^{(s)}, \varphi(x)) = (-1)^s (T, \varphi(x)^{(s)})$ for all $\varphi(x) \in K$.

Let give an example of derivatives of the singular distribution $T = \delta$ we have:

(a) $(\delta'(x), \varphi(x)) = -(\delta(x), \varphi'(x)) = -\varphi'(0);$

(b) $(\delta^{(s)}, \varphi(x)) = (-1)^s (\delta(x), \varphi(x)^{(s)}) = (-1)^s \varphi(0)^{(s)}.$

Definition 2.4. Let ω(x) be an infinitely-differentiable function. We define the product of $\omega(x)$ with any distribution T in K' by $(\omega(x)T, \varphi(x)) = (T, \omega(x)\varphi(x))$ for all $\varphi(x) \in K$.

We carry out our actual investigative activities on the linear singular differential equation of order l of general form, in the case of the situation known as *Euler-type* equation, by focusing only our attention on the search for the existence of all the solutions centered on zero for the non-homogeneous equation (1).

Note that a simple equation $x^n y(x) = \delta^{(s)}(x)$ in the space K' admitting a distributional solution denoted through $y(x) = \dfrac{\delta^{(s)}(x)}{x^n}$ is such that $y(x) \in K'$ and $(x^n y(x), \varphi(x)) = (\delta^{(s)}(x), \varphi(x)), \varphi(x) \in K$. Therefore, we find that the quotient (the division) of an s-th-order derivative of the Dirac delta function $\delta^{(s)}(x)$ by an n-th-power of x, i.e., x^n is a distribution depending on n constants. It has been established by applying the Fourier transform and its inverse to both sides of the equation, that this quotient is defined by the following formula, see [6].

$$\frac{\delta^{(s)}(x)}{x^n} = \frac{(-1)^n s!}{(s+n)!}\delta^{(s+n)}(x) + \sum_{k=1}^{n} C_k \delta^{(k-1)}(x), \qquad (*)$$

where C_k, $k = 1, \dots, n$ are arbitrary constants.

Otherwise the zero-centered solutions of the considered differential equation are defined by the formula:

$$y_s(x) = \frac{(-1)^n s!}{(s+n)!}\delta^{(s+n)}(x) + \sum\nolimits_{k=1}^{n} C_k \delta^{(k-1)}(x), \qquad (**)$$

where C_k, $k = 1, \ldots, n$ are arbitrary constants.

Therefore, from the previous it is not difficult to obtain for the general equation of the following kind:

$$x^n y(x) = \sum\nolimits_{s=0}^{\mu} \delta^{(s)}(x), \qquad (***)$$

the zero-centered solutions given by the formula:

$$y(x) = \sum\nolimits_{s=0}^{\mu} \frac{(-1)^n s!}{(s+n)!}\delta^{(s+n)}(x) + \sum\nolimits_{k=1}^{n} C_k \delta^{(k-1)}(x), \qquad (****)$$

where C_k, $k = 1, \ldots, n$ are arbitrary constants.

This leads us to suppose that investigating some specific linear singular differential equations in the space of generalized functions K' is interesting and quite challenging as the solutions, in some particular cases, are expressed by rather huge and unexpected formulas.

Now, let move to the main result of our investigations conducted in the following section:

3. The Euler Case

In this part, we investigate as indicated upstairs the Euler case situation when the following equalities $k_i = k_0 + i;\ i = 0, 1, \ldots, l$ are satisfied.

Further, in the next part we will use the following necessary definitions and lemmas:

Definition 3.1. The polynomial denoted

$$P_l(j) = \frac{1}{j!}\sum_{i=0}^{l}(-1)^i a_i\ (j + k_0 + i)!$$

is called characteristic polynomial of the differential equation (1).

Definition 3.2. The set of zeros or roots of the defined polynomial $P_l(j)$ is denoted in the following way:

$$Nul\ P_l(j) = \{j_* : j_* \in \mathbb{Z}_*,\ P_l(j_*) = 0\} = \mathrm{Ker} P_l.$$

Let formulate the following needed lemma which can be found in some books related to generalized functions.

Lemma 3.1. *Let* $\omega(x)$ *be an infinitely-differentiable function. Then*

$$\omega(x)\delta^{(s)}(x) = (-1)^s \omega_{(0)}^{(s)}\delta(x) + (-1)^{s-1} s\omega_0^{(s-1)}\delta'(x)$$

$$+ (-1)^{s-2}\frac{s(s-1)}{2!}\omega_{(0)}^{(s-2)}\delta''(x) + \ldots + \omega(0)\delta^{(s)}(x), \qquad (2)$$

and

$$[\omega(x)H(x)]^{(m)} = \omega_{(x)}^{(m)}H(x) + \omega_{(0)}^{(m-1)}\delta(x) + \omega_{(0)}^{(m-2)}\delta'(x) + \ldots + \omega(0)\delta^{(m-1)}(x). \qquad (3)$$

For the proof of Lemma 3.1 refer to [10].

As useful formula that is deduced from (2), we can easily establish an important result for any monomial $\omega(x) = x^k$ in the following lemma:

Lemma 3.2. *Let* $k, s \in \mathbb{N} \cup \{0\}$. *Then it is holding place*

$$x^k\delta^{(s)}(x) = \begin{cases} 0, & s < k; \\ \dfrac{(-1)^k s!}{(s-k)!}\delta^{(s-k)}(x), & s \geq k. \end{cases}$$

The proof of this lemma can be found in some special mathematical books related to the theory of distributions, see also [9].

Now let move to the following interesting situation.

Below, we study the non-degenerated case and look for zero-centered solutions.

We consider the most interesting case when it is fulfilled the condition $\prod_{i=0}^{l} a_i \neq 0$ and call this case *non-degenerated case*.

The following theorem gives the necessary conditions of the existence of zero-centered solutions of the Equation (1) in the space K'.

4. Main results

In this section, we will state our main results and give their proofs.

Let formulate the following theorem:

Theorem 4.1. *Let* $\prod_{i=0}^{l} a_i \neq 0$; $k_i \in \mathbb{Z}_+$, $i = 0, \ldots, l$; k_0, $s \in \mathbb{N} \cup \{0\}$. *For the existence of zero-centered solutions of the Equation* (1) *in the space* K', *it is necessary and sufficient that*

$$\sum_{i=1}^{l} (k_0 - k_1 + i)^2 + \left(\sum_{i=0}^{l} (-1)^i a_i (k_0 + s + i)!\right)^2 \neq 0. \tag{4}$$

Proof. We simply prove the necessary part of the theorem while the sufficient part will be deduced directly at the same time as the construction of the solutions.

It is clear that we should find the particular solution $y(x)$ of the non-homogeneous equation (1) in the form of functional centered at zero, i.e.,

$$y(x) = \sum_{j=0}^{\gamma_0} C_j \delta^{(j)}(x), \tag{5}$$

where γ_0 is sufficient a large number. We suppose contrary, when

$$k_i = k_0 + i;\ i = 1, \ldots, l; \text{ and } \sum_{i=0}^{l} (-1)^i a_i (k_0 + s + i)! = 0. \tag{6}$$

Simple calculations when taking into account Lemma 2.2 give us after setting (5) into the Equation (1) the following result:

$$\sum_{i=0}^{l} a_i x^{k_i} \sum_{j=0}^{\gamma_0} C_j \delta^{(j+1)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x) \sum_{j=0}^{\gamma_0} C_j x^{k_i} \delta^{(j+i)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x), \tag{7}$$

or that is the same as:

$$\sum_{i=0}^{l} a_i \sum_{j=k_i-i}^{\gamma_0} C_j (-1)^{k_i} \frac{(j+i)!}{(j+i-k_i)!} \delta^{(j+i-k_i)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x), \tag{8}$$

when changing $j + i - k_i$ into j we obtain

$$\sum_{i=0}^{l} a_i \sum_{j=k_i-i}^{\gamma_0+i-k_i} (-1)^{k_i} C_{j+i-k_i} \frac{(j+k_i)!}{j!} \delta^{(j)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x), \tag{9}$$

Until now, the investigation has been conducted for the arbitrary k_i and the equality (9) is obtained without any supplementary suppositions. We next take into account $k_i = k_0 + i$ following the supposition (6) and we rewrite the system (9) in the form as:

$$\sum_{i=0}^{l} a_i \sum_{j=0}^{\gamma_0-k_0} (-1)^{k_i} C_{j+k_0} \frac{(j+k_0+i)!}{j!} \delta^{(j)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x). \tag{10}$$

Or when changing the order of the summation we obtain

$$\sum_{j=0}^{\gamma_0-k_0} \delta^{(j)}(x) \Big(\sum_{i=0}^{l} (-1)^{k_i} a_i (j+k_0+i)!\Big) \frac{C_{j+k_0}}{j!} = \sum_{s=0}^{\mu} \delta^{(s)}(x), \tag{11}$$

Let us turn to the previous equation in the case when considering having at the right-hand side only the s-th derivative of the Dirac distributional function, i.e.,

$$\sum_{j=0}^{\gamma_0-k_0} \delta^{(j)}(x) \Big(\sum_{i=0}^{l} (-1)^{k_i} a_i (j+k_0+i)!\Big) \frac{C_{j+k_0}}{j!} = \delta^{(s)}(x), \tag{12}$$

From our previous achieved investigations whose results were known, it is easy to see that by equalizing the coefficients under Dirac delta functions and it derivatives, we obtain a linear non homogeneous algebraic system for the determination of the unknown coefficients C_j of the following form:

$$(-1)^{k_0} \frac{C_{j+k_0}}{j!} \sum_{i=0}^{l} (-1)^i a_i (j + k_0 + i)! = \begin{cases} 1, & j = s, \\ 0, & j \neq s. \end{cases} \tag{13}$$

It is not difficult to see that such equation when $j = s$ has the following form:

$$\frac{(-1)^{k_i}}{s!} C_{s+k_0} \sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i)! = 1, \tag{14}$$

which contradicts the assumption made of the proved theorem. The theorem is proved.

Further, we consider the simplest and easy case as announced at the beginning when it is fulfilled the next conditions:

$$k_i = k_0 + i;\ i = 0, 1, \dots, l, \text{ and } \sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i)! \neq 0. \tag{15}$$

Now by applying the principle of superposition of the solutions of a differential equation we reach to the following results. So, it takes place the following theorem:

Theorem 4.2. *Let* $\prod_{i=0}^{l} a_i \neq 0;\ k_i \in \mathbb{N},\ i = 1, \dots, l;\ k_0, s \in \mathbb{N} \cup \{0\}$ *and be realized the conditions* (15) *then, the general zero-centered solution of the Equation* (1) *is given by the following form*:

$$y(x) = \sum_{s=0}^{\mu} y_s(x),$$

where

$$y_s(x) = \frac{(-1)^{k_i} s!}{\sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i)!} \delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j \delta^{(j)}(x), \tag{16}$$

i.e.,

$$y(x) = \sum_{s=0}^{\mu} \frac{(-1)^{k_i} s!}{\sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i)!} \delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j \delta^{(j)}(x), \tag{17}$$

in the case, when

$$\sum_{i=0}^{l} (-1)^i a_i (j + k_0 + i)! \neq 0, \qquad \forall j \in \mathbb{Z}_+,$$

where C_j, $j = 0, \ldots, k_0 - 1$, *are arbitrary constants.*

And if there exists at least one $j_*^m \in \mathbb{Z}_+ / \{s\}$, $m \leq l$, *such that*

$$j_*^m \in Nul\ P_l(j), \tag{18}$$

then the zero-centered solution is found and defined by the following form:

$$y(x) = \sum_{s=0}^{\mu} \frac{(-1)^{k_i} s!}{\sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i)!} \delta^{(s+k_0)}(x)$$

$$+ \sum_{j=0}^{k_0-1} C_j \delta^{(j)}(x) + \sum_{j_*^m \in Nul\ P_l(j)} C_{j_*^m + k_0} \delta^{(j_*^m + k_0)}(x), \tag{19}$$

C_j, $j = 0, \ldots, k_0 - 1$, *are arbitrary constants.*

Proof. As we notice upstairs, in this case we can use the system (13) from which with the condition (15) we may find directly the following relationship:

$$C_{s+k_0} = \frac{(-1)^{k_0} s!}{\sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i)!}. \tag{20}$$

Concerning the other remaining coefficients C_{j+k_0}, $j \in \mathbb{Z}_+ / \{s\}$, so they are equal to zero, when $\sum_{i=0}^{l} (-1)^i a_i (j + k_0 + i)! \neq 0$ for all $j \in \mathbb{Z}_+$ and not more l from them will remain free, if there exist $j_*^m \in \mathbb{Z}_+ / \{s\}$, such that

$$\sum_{i=0}^{l} (-1)^i a_i \, (j_*^m + k_0 + i)! = 0.$$

All what has been said lead us to (15)-(18) with the consideration that $\delta^{(j)}(x)$, $j = 0, 1, \ldots, k_0 - 1$ are solutions of the homogeneous equation.

The theorem is proved.

We want to remark immediately if the conditions $k_i = k_0 + i$; $i = 0, 1, \ldots, l$ are violated, then in the general situation the problem will be complicated. So that for the simplicity, it is recommendable to continue the already started investigations in these cases by considering the following differential equation:

$$ax^m y''(x) + bx^n y'(x) + cx^r y(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x), \qquad (21)$$

where $m = n + 1 \neq r + 2$.

By the way we recalled that we have completely realized the investigation of the simplest case of the *first order* linear singular differential equation of the following form:

$$ax^p y'(x) + bx^q y(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x), \qquad (22)$$

with $q > p - 1$ or $q < p - 1$ in the same space K', whose scientific results had been published in [16].

The case of the second order differential equation evocated in the remark done, will be one of the scientific work that we can plan to investigate in a brief future to help us looking for generalizing the general similar cases when considering equation

$$\sum_{i=0}^{l} a_i x^{k_i} y^{(i)}(x) = \sum_{s=0}^{\mu} \delta^{(s)}(x),$$

with conditions $k_i = k_{i-1} + 1$; $i = 1, \ldots, l - 1$ but $k_l \neq k_{l-1} + 1$. We may call (in the two situations appearing) this equation *left Euler case equation* or *right Euler case equation,* depending of the cases $k_l > k_{l-1} + 1$ or $k_l < k_{l-1} + 1$.

Before concluding, let says that the zero-centered solution $y(x) \in K'$ is solution in the sense of distributions of the Equation (1) if and only if it is realized the following relationship.

$$\forall \varphi(x) \in K, \left(\sum_{i=0}^{l} a_i x^{k_i} y^{(i)}(x),\ \varphi(x) \right) = \left(\sum_{s=0}^{\mu} \delta^{(s)}(x),\ \varphi(x) \right) = \sum_{s=0}^{\mu} (-1)^s \varphi_{(0)}^{(s)}. \tag{23}$$

In other words, and with regard to the situations arising in connection with the study of the solvency of the so-called *Euler-type* singular linear differential equation that we have carried out in our previous published papers, it is necessary to verify that the following two equalities take place and are realized exactly.

$$\left(\sum_{i=0}^{l} a_i x^{k_i} \left[\begin{array}{c} \sum_{s=0}^{\mu} \dfrac{(-1)^{k_i} s!}{\sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i) i} \delta^{(s+k_0)}(x) \\ + \sum_{j=0}^{k_0 - 1} C_j \delta^{(j)}(x) \end{array} \right]^{(i)},\ \varphi(x) \right)$$

$$= \left(\sum_{s=0}^{\mu} \delta^{(s)}(x),\ \varphi(x) \right) = \sum_{s=0}^{\mu} (-1)^s \varphi_{(0)}^{(s)}, \tag{24}$$

in the case, when $\sum_{i=0}^{l} (-1)^i a_i\ (j + k_0 + i)! \neq 0,\ \forall j \in \mathbb{Z}_+$. And in the other case, we should obtain:

$$\left(\sum_{i=0}^{l} a_i x^{k_i} \left[\begin{array}{c} \sum_{s=0}^{\mu} \dfrac{(-1)^{k_i} s!}{\sum_{i=0}^{l} (-1)^i a_i (s + k_0 + i) i} \delta^{(s+k_0)}(x) \\ + \sum_{j=0}^{k_0 - 1} C_j \delta^{(j)}(x) + \sum_{j_*^m \in Nul\ P_l(j)} C_{j_*^m + k_0} \delta^{(j_*^m + k_0)}(x) \end{array} \right]^{(i)},\ \varphi(x) \right)$$

$$= \left(\sum_{s=0}^{\mu} \delta^{(s)}(x),\ \varphi(x) \right) = \sum_{s=0}^{\mu} (-1)^s \varphi_{(0)}^{(s)}, \tag{25}$$

if there exists at least one $j_*^m \in \mathbb{Z}_+ / \{s\},\ m \leq l,$ such that $j_*^m \in Nul\ P_l(j)$.

5. Conclusion

In this paper, we have completely investigated the question of the existence of zero-centered solutions and the solvability of an *l-order* linear singular differential equation in the *Euler case situation* in the space of generalized functions K' with a second right hand side in the form of Dirac delta function or it derivatives of some order.

We looked for the zero-centered solutions by replacing the general form of the solution $y(x) = \sum_{j=0}^{\gamma_0} C_j \delta(x)^{(j)}$ in the differential equation (1) that leads us to obtain and analyze a linear non-homogeneous algebraic system for the determination of the unknown coefficients C_j. Case by case depending of the situations arising we reached and made an exhaustive description of all distributional solutions centered on zero of this equation in the functional space of distributions K'.

References

[1] A. Kananthai, Distribution solutions of the third order Euler equation, Southeast Asian Bulletin of Mathematics 23(4) (1999), 627-631.

[2] I. M. Gelfand and G. E. Shylov, Space of Test and Generalized Functions, V.1.M.: Phyzmatgiz, 1958, 307 Pages. Russian Edition.

[3] L. Schwartz, Théorie des distributions, Tomes I et II, Hermann, Paris, 1957.

[4] A. Liangprom and K. Nonlaopon, On the generalized solutions of a certain fourth order Euler equations, Journal of Nonlinear Sciences and Applications 10(8) (2017), 4077-4084.

DOI: http://dx.doi.org/10.22436/jnsa.010.08.04

[5] L. Fuchs, Zur theorie der linearen differentialgleichungen mit veränderlichen coefficienten, Journal für die Reine und Angewandte Mathematik 66 (1866), 121-160.

[6] Abdourahman, On a linear differential equation in the spaces of generalized functions, Rostov on Don. Preprint at VINITI 02.08.2000. No. 2035 (2000), 27 pages.

[7] L. Fuchs, Zur theorie der linearen differentialgleichungen mit veränderlichen coefficienten, Journal für die Reine und Angewandte Mathematik 66 (1866), 121-160.

[8] A. Kananthai, The distribution solutions of ordinary differential equation with polynomial coefficients, Southeast Asian Bulletin of Mathematics 25(1) (2001), 129-134.

DOI: https://doi.org/10.1007/s10012-001-0129-7

[9] I. M. Gel'fand and G. E. Shilov, Generalized Functions, Vol. II, Spaces of Fundamental and Generalized Functions, Academic Press, New York, 1968.

[10] R. P. Kanwal, Generalized functions: Theory and Technique, 3rd Edition, Springer: New York, NY, USA, 2004.

[11] Abdourahman, On a linear differential equation with singular coefficients, Rostov-On-Don,1999, Collection of papers, Integro-differential operators and their Applications, Rostov-Na-Donu: Izdat.DGTU,1999.N°.5.PP.4-7.

Available at: http://chat.ru/volume1999

[12] S. Jhanthanam, K. Nonlaopon and S. Orankitjaroen, Generalized Solutions of the Third-Order Cauchy-Euler Equation in the Space of Right-Sided Distributions via Laplace Transform, Mathematics 7(4), (2019); Article 376.

DOI: https://doi.org/10.3390/math7040376

[13] Abdourahman, On a linear singular differential equation with singular coefficients, Collection of papers, Integro-differential operators and their applications, Rostov-na-Donu: Izdat. DGTU, No.5 (2001), 4-10.

[14] Joseph Wiener, Generalized-Function Solutions of Differential and Functional Differential Equations, Journal of Mathematical Analysis and Applications 88 (1982), 170-182. Edinburg, Texas 78539.

[15] Abdourahman, Generalized-function solutions of a differential equation of L-order in the space K', International Journal of Theoretical and Applied Mathematics 8(2) (2022), 40-44.

DOI: https://doi.org/10.11648/j.ijtam.20220802.12

[16] Abdourahman Haman Adji and Shankishvili Lamara Dmitrievna, On a first order linear singular differential equation in the space K', Journal of AppliedMath 1(2) (2023), 88.

DOI: https://doi.org/10.59400/jam.v1i2.88

[17] Abdourahman, On generalized-function solutions of a first order linear singular differential equation in the space K' via Fourier transform, Journal of Mathematical Sciences: Advances and Applications 70 (2022), 27-51.

DOI: https://doi.org/10.18642/jmsaa_7100122244

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