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NOETHERIAN THEORY FOR A SINGULAR LINEAR DIFFERENTIAL OPERATOR OF HIGHER ORDER *L* IN THE SPACE OF DISTRIBUTIONS

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Abstract

The main objective set in this research is the construction of noetherian theory for a singular linear integro-differential operator L defined by a linear singular differential equation of higher order in a specific functional space well chosen to achieve the goal. It should be emphasized that the case where $n = 1$ has been completely studied in the two situations separately when $p = 1$ and $p \geq 2$. Our previous various published research was related to this topic. The methodology adopted on a case-by-case basis, and depending on the values and sign of the parameter $\gamma \in \mathbb{R}$, leads us to solve the linear differential equation studied with a well-known second specific right-hand side $f(x) \in C_0^{\{p\}}[-1, 1]$, systematically identifying the conditions solvency. This takes us straight to the investigation and construction of the noetherity (neotherian theory) of the operator L. Finally, depending on each case, we evaluate and calculate the deficient numbers and the index of the operator considered in various situations, relative to the parameter $\gamma \in \mathbb{R}$, thus, parallel to the construction of the noetherian theory of the differential operator L, we bring out the solvability conditions of the equation studied in space $C^n[-1, 1]$.

1. Introduction

The construction of the noetherian theory for the integro-differential operators defined by certain forms of integral equations of the third type is widely illustrated by the investigations carried out in certain scientific research works (see, for example, the articles [1-6, 8, 10-12, 23]). However, let us mention that the main difficulty we face in the case of the investigation and study of the solvability of integral equations of the third type during the construction of the noetherian theory for the operators defined by such integral equations lies in the choice of necessary approaches and methods leading to the expected goal. It should be noted that in several works carried out by researchers related to the said subject, see [3-10, 13, 18] and well-illustrated from the general theory in [22, 26], also following Prossdorf Samko, Kilbas, Marichev, Raslambekov, Gabassov and Abdourahman, the application of the normalization method, the method of hypersingular integrals and also the method of approximate inverse operators made it possible to approach the investigations integral equations of the third type in the space of continuous functions and distributions.

These different approaches and methods make it possible to clearly define and pose the problem, when carrying out studies relating to integral equations of the first type, or non-Fredholm integral equations of the second type which define the operators thus considered. At the same time, let us also mention that among others, the mathematicians Shulaia and Gugushvili for their part, carried out studies of the inverse problem of the spectral analysis of the theory of linear multigroup neutron transport in plane geometry in the work of their scientific article titled Transp. Theory Stat. Phys. 29 (6) (2000), 711-722.

Recall that the researcher Shulaia carried out investigations in the space of Holder functions on a non-homogeneous linear integral equation with coefficient cos x and was able to identify the necessary and sufficient conditions for the solvability of the equation considered under certain indicated hypotheses on its core. On the other hand, let us also note that he was able to succeed in constructing the solution to the equation studied analytically, using the application of Fredholm's theory and that of singular integral equations. In relation to said investigations one can refer with more details to [29]. Let us also cite the work of researcher Bal who carried out the investigations in his article relating to inverse problems for homogeneous transport equations, I. The results of the studies undertaken in the one-dimensional case have been published. Inverse Problems 16 (4) (2000), 997-1028. Relative to our previous investigations, we have already carried out the construction of the noetherian theory of an integro-differential operator defined by a singular linear integral equation of the third type of the following form:

$$(A\varphi)(x) = x^p \varphi'(x) + \int_{-1}^{1} K(x, t)\varphi(t)dt = f(x);\ x \in [-1, 1],$$

with the unknown function $\varphi \in C^1_{-1}[-1, 1]$, the second right hand side $f(x) \in C_0^{\{p\}}[-1, 1]$, and the kernel $K(x, t) \in C_0^{\{p\}}[-1, 1] \times C[-1, 1]$.

For explanations and other necessary details of such research, one can consult the references [11, 12, 23-25]. In order to approach a generalization of ideas in the construction of the noetherian theory and going in the same direction, we plan to carry out studies which will allow us to construct said theory for an integro-differential operator defined by an integral equation of third type with main part of L shape, and it is for this purpose that we conduct such research as a preliminary step. Various results emanating from numerous investigations devoted to integral equations of the third type can be found with all the necessary details in the references [2-6, 8, 9]. Let us also emphasize that the mathematician Gabbassov undertook special research work on various methods of solving integral equations, which can be consulted in the references [13-16] and also see similar approaches located in [17-20]. In the present work that we carry out in the space of continuous functions, we focus on a differential operator L as the main part of the integral operator A, which is defined by an integral equation of the third type. We focus our attention only on the investigation of the operator L for which we carry out the construction of the noetherian theory as has been done and illustrated in references [12, 23, 32, 33], with the remarkable particularity, that in this case indicated, the order of the differential equation is higher and the study interval considered is closed $[-1, 1]$ rather than that $[0, 1]$ with the interior singular point zero in this situation on the middle of the interval considered. This being said and to know here, we consider for the study the linear differential operator:

$$Ly(x) = x^p y^{(n)}(x) + \gamma y^{(n-1)}(x) = f(x);\ n \geq 2,\ x \in [-1, 1], \tag{1}$$

where $\gamma \in \mathbb{R},\ p \in \mathbb{N}$ with $f(x) \in C_0^{\{p\}}[-1, 1]$ and $y(x) \in C^n[-1, 1]$.

The establishment of the noetherity property of the operator L is carried out systematically through the study of the solvability of the linear differential equation of higher order n, defined by the formula (1). The results obtained are linked to the construction of the continuity of the regularizers, as illustrated in our previous research when we first considered the interval $[0, 1]$, of which the rest of the study was extended until the set of the closed interval $[-1, 1]$. For further details, see the article [12] as well as the references on such approaches studied by Yurko with all the related explanations located in [30-33]. All this allows us to calculate and determine the deficient numbers $\alpha(L)$ and $\beta(L)$ as well as the index $\chi(L)$ of the operator L, depending on the different situations analyzed in relation to the sign and the values of the parameter γ.

The content of this work is organized as follows: first and foremost, we make a detailed presentation in Section 2 of the preliminaries related to the concept and notions of the well-known noetherian theory. Section 3 presents the main important results, formulated through different theorems, obtained from this work and is clearly dedicated to the cases studied, with dependence on the sign and values of γ, in other words (related to the cases (a) $\gamma > 0$; (b) $\gamma < 0, \gamma \neq -1$; and (c) $\gamma = 0$). And to close this work, we finally summarize the content of the investigations in Section 4 entitled conclusion, followed by some recommendations for the continuation of future scientific work to be undertaken.

2. Preliminaries

Before presenting in full details our main results, the following definitions and concepts well known from the noetherian theory of operators and used in our previous researches are required for the realization of this study. We also recall the notions of Taylor derivatives and linear Fredholm integral equation of the third kind, widely studied in many works done by different authors among many of them Bart, Sukavanam, Shulaia, Gabbassov, See [4-7, 10, 14, 27] for more details.

First of all let us move to the following concept.

(A) Noetherian operator

Definition 1. Let X, Y be Banach spaces, $A \in l(X, Y)$ a linear operator. The quotient space $\operatorname{coker} A = Y/\operatorname{im}A$ is called the cokernel of the operator A. The dimensions $\alpha(A) = \dim \ker A$, $\beta(A) = \dim \operatorname{coker} A$ are called the nullity and the deficiency of the operator A, respectively. If at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite, then the difference $IndA = \alpha(A) - \beta(A)$ is called the index of the operator A.

Definition 2. Let X, Y be Banach spaces, $A \in l(X, Y)$ is said to be normally solvable if it possesses the following property: The equation $Ax = y(y \in Y)(y \in Y)$ has at least one solution $x \in D(A)$ ($D(A)$ is the domain of A) if and only if $< y, f >= 0 \ \forall\ f \in (im\ A)^{\perp}$ holds.

We recall that by the definition of the adjunct operator $(im\ A)^{\perp} = \ker A^{*}$ and it's prove in [11] that the operator A is normally solvable if and only if its image space imA is closed.

Definition 3. A closed normally solvable operator A is called a noetherian operator if its index is finite.

By the way, we briefly review this important notions of Taylor derivatives which is widely used when constructing noether theory of the considered operator A.

Definition 4. A continuous function $\varphi(x) \in C[-1, 1]$ admits at the point $x = 0$ Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1, 2, \ldots, p$, the following limits:

$$\varphi^{\{k\}}(0) = k! \lim_{x \to 0} x^{-k}\left[\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^{j}\right]. \tag{2}$$

The class of such functions $\varphi(x)$ is denoted $C_0^{\{p\}}[-1, 1]$.

Next let us move to the following part.

Let $C^{m}[-1, 1]$, $m \in \mathbb{Z}_{+}$, noted the Banach space of continuous functions on $[-1, 1]$, having continuous derivatives up to order m, for which the norm is defined as following:

$$\|\varphi(x)\|_{C^m[-1,\,1]} = \sum_{j=0}^{m} \max_{-1\le x\le 1}\left|\varphi^{(j)}(x)\right|. \tag{3}$$

So, that we can consider $\varphi^{\{k\}}(0)$ are defined for all $k = 1, 2, \ldots, p$.

We define $C_0^{\{p\}}[-1, 1]$ as a subspace of continuous functions, having finite Taylor derivatives up to order $p \in \mathbb{Z}_+$; and when $p = 0$, we put $(C_0^{\{p\}}[-1, 1] = C_0^{\{0\}}[-1, 1] = C[-1, 1])$.

Let us also define a linear operator N^k on the space $C_0^{\{p\}}[-1, 1]$ by the formula:

$$(N^k\varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j}{x^k}, \; k = 1, 2, \ldots, p. \tag{4}$$

One can easily verify the property $N^k = N^{k_1} N^{k-k_1}$, $0 \le k_1 \le k$, $k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$.

Definition 5. The operator N^p is called characteristical operator of the space $C_0^{\{p\}}[-1, 1]$.

Remark. The sense of the previous definition can be seen from the verification of the following lemma and also for more details, see [23, 28, 29].

Note also that full description of the special space $C_0^{n,\{p+n-1\}}[-1, 1]$ used in this work has been presented in our previous published scientific paper [12].

Lemma 2.1. *A function* $\varphi(x)$ *belongs to* $C_0^{\{p\}}[-1, 1]$ *if and only if, the following representation*:

$$\varphi(x) = x^p \phi(x) + \sum_{k=0}^{p-1} \alpha_k x^k, \tag{5}$$

holds with the function $\phi(x) \in C[-1, 1]$, *and* α_k *being constants.*

To prove Lemma 2.1 it is enough to observe that (5) implies that the Taylor derivatives of $\varphi(x)$ up to the order p exists, and more $\varphi^{\{k\}}(0) = k!\,\alpha_k$, $k = 0, 1, 2, \dots, p-1$, $\varphi^{\{0\}}(0) = p!\,\phi(0)$ with $\phi(x) = (N^k\varphi)(x)$. Conversely, if $\varphi(x)$ belongs to $C_0^{\{p\}}[-1, 1]$, and we define $\phi(x) = (N^k\varphi)(x)$ with $\alpha_k = \dfrac{\varphi^{\{k\}}(0)}{k!}$, $k = 0, 1, 2, \dots, p-1$, then the representation (5) holds. From Lemma 2.1, it follows that for $\varphi(x) \in C_0^{\{p\}}[-1, 1]$ the inequality

$$\varphi(x) = x^p (N^k\varphi)(x) + \sum_{k=0}^{p-1} \frac{\varphi^{\{k\}}(0)}{k!} x^k, \tag{6}$$

is valid.

Consequently, the linear operator N^p establishes a relation between the spaces $C_0^{\{p\}}[-1, 1]$ and $C[-1, 1]$. The space $C_0^{\{p\}}[-1, 1]$ with the norm

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\varphi^{\{k\}}(0)|, \tag{7}$$

becomes a Banach space one.

Let note also that we can define the previous norm by the following way:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\alpha_k| = \|\phi(x)\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\alpha_k|.$$

Sometimes it is comfortable and suitable to consider as norm in the space $C_0^{\{p\}}[-1, 1]$ the equivalent norm defined as follows:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p} \|N^j\varphi\|_{C[-1,1]}.$$

We can also note a very useful and clearly helpful next inequality:

$$\|\varphi\|_{C[-1,\,1]} \le \|N^{p}\varphi\|_{C[-1,\,1]} + \sum_{j=0}^{p-1} |\varphi^{\{j\}}(0)| = \|\varphi\|_{C_0^{\{p\}}[-1,\,1]}.$$

Therefore, it is obvious to see that $\|\varphi\|_{C[-1,\,1]} \le \|\varphi\|_{C_0^{\{p\}}[-1,\,1]}$.

Finally, note that from the Lemma 2.1 it follows the following fact, if $\varphi(x) \in C[-1, 1]$, then $x^{p}\varphi(x) \in C_0^{\{p\}}[-1, 1]$. This assertion may be generalized as follows.

Lemma 2.2. *Let* $p \in \mathbb{N}$, $s \in \mathbb{Z}_{+}$. *If* $\varphi(x) \in C_0^{\{s\}}[-1, 1]$, *then* $x^{p}\varphi(x) \in C_0^{\{p+s\}}[-1, 1]$, *and the formula holds*

$$(x^{p}\varphi(x))^{\{j\}}(0) = \begin{cases} 0, \; j = 0, 1, \ldots, p-1, \\ \dfrac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), \; j = p, \ldots, p+s. \end{cases} \tag{8}$$

Proof. Note that a stronger assumption on the function $\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1, 1]$ would allow us to easily prove the lemma just by applying Leibniz formula.

For $s = 0$, the statement has already been proved above, so $x^{p}\varphi(x) \in C_0^{\{p\}}[-1, 1]$ and $(x^{p}\varphi(x))^{\{j\}}(0) = 0,\ j = 0, \ldots, p-1$ and $(x^{p}\varphi(x))^{\{p\}}(0) = p!\,\varphi(0)$. Now, let us prove that $(x^{p}\varphi(x))^{\{j\}}(0) = \dfrac{j!}{(j-p)!}$ $\varphi^{\{j-p\}}(0),\ j = p+1, \ldots, p+s.$ Since the derivatives are defined recurrently, and (8) is true for $j = p$, then it is sufficient to verify the passage from j to $j+1$. We have,

$$(x^{p}\varphi(x))^{\{j+1\}}(0) = (j+1)!\lim_{x\to 0}\frac{x^{p}\varphi(x) - \sum_{l=p}^{j}\frac{x^{l}}{(l-p)!}\varphi^{\{l-p\}}(0)}{x^{j+1}} \tag{9}$$

$$= (j+1)! \lim_{x \to 0} \frac{\varphi(x) - \sum_{l=0}^{j-p} \frac{x^l \varphi^{\{l\}}(0)}{l!}}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!} \varphi^{\{j+1-p\}}(0). \quad (10)$$

Lemmas 2.1 and 2.2 imply the next important lemma.

Lemma 2.3. *Let* $f(x) \in C_0^{\{p\}}[-1, 1]$, $p \in \mathbb{N}$ *and* $f(0) = \ldots = f^{\{r-1\}}(0) = 0$, $1 \leq r \leq p$. *Then* $\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[-1, 1]$.

(B) Associated operator and associated space

Definition 6. The Banach space $E' \subset E^*$ is called associated space with the space E, if $|(f, \varphi)| \leq c\|f\|_{E'}\|\varphi\|_E$ for every $\varphi \in E$, $f \in E'$.

We note that the initial space E can be considered associated with the space E'. Moreover, the norm $\|f\|_{E'}$ is not obliged to be equivalent to the norm $\|f\|_{E^*}$.

Let be noted $\mathcal{L}(E_1, E_2)$ the Banach algebra of all linear bounded operators from E_1 into E_2.

Definition 7. Let E_j, $j = 1, 2$ be two Banach spaces and E'_j be their associated spaces. The operators $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ are called associated, if $(A'f, \varphi) = (f, A\varphi)$ for all $f \in E'_2$ and $\varphi \in E_1$.

It seems that we can formalize the noetherity in terms of associated operator and associated space. See [22, 26].

Lemma 2.4. *Let* E_j, $j = 1, 2$ *two Banach spaces and* E'_j *their associated spaces and, let* $A \in \mathcal{L}(E_1, E_2)$ *and* $A' \in \mathcal{L}(E'_2, E'_1)$ *be associated noetherian operators and more,*

$$\alpha(A) = -\alpha(A').$$

Then, for the solvability of the equation $A\varphi = f$ *it is necessary and sufficient that* $(f, \psi) = 0$ *for all solutions of the homogeneous associated equation* $A'\psi = 0$.

Next, let us move to the presentation of the general important results of the work in the following section.

3. Main Results

In this section, we undertake properly the noetherization theory for the investigation of the operator L.

Namely, here we consider as a model to be investigated, the higher order linear differential equation defined by the formula (1).

It is clear that the results on such equation could be receive from those obtained in the situation $n = 1$, making the change of function $z(x) = y^{(n-1)}(x)$ transforms the Equation (1) into the following equation:

$$x^p z'(x) + \gamma z(x) = f(x), \tag{11}$$

which, we already investigated with full details in our previous published research, see paper [32].

As previously, we consider $f(x) \in C_0^{\{p\}}[-1, 1]$. Note that the results of the study of such an equation that we have already carried out for the simple case when $n = 1$ are mentioned as well as all the detail in our previous published research, see the articles [32, 33]. Therefore, let us underline that the proofs of all the theorems formulated below are obtained in the same way as those carried out in relation to the different simple cases, meticulously analyzed and exposed in our previous scientific articles in the situations studied when $n = 1$ and published in the references [12, 32, 33]. Consequently, it would be judicious and useful to refer to it to similarly obtain the proofs of said theorems in the new cases mentioned in the present work.

We begin the analysis of the Equation (1) with the case $p = 1$.

3.1. The Euler case situation

Let the operator L defined by the formula (1) and $p = 1$.

(a) The case $\gamma > 0$

From our previous results obtained from paper [32] when $p = 1$ in the space $C^1[-1, 1]$, we obtained the form of the solution by the following way:

$$y^{(n-1)}(x) = z(x) = \begin{cases} \int_0^x \left(\frac{t}{x}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, & x > 0, \\ \frac{f(0)}{\gamma}, & x = 0, \\ -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}, & x > 0, \end{cases} \tag{12}$$

with respect to the specific way of the formula (12) the solution of the equation $y'(x) = g(x)$ on the segment $[-1, 1]$ we will take by the formula

$$y(x) = \begin{cases} \int_0^x g(t)dt, & x \geq 0, \\ -\int_x^0 g(t)dt, & x \leq 0. \end{cases} \tag{13}$$

In the case of the equation of higher order than one in (13) then we obtain multiple integration, to which (separately by $x > 0$ and by $x < 0$) the Dirichlet formula is applicable. All that after some small computations leads us to the following:

$$y(x) = P_{n-2}(x) + \frac{f(0)}{\gamma^{(n-1)!}} x^{n-1} + \begin{cases} \int_0^x k(x, s)(Nf)(s)ds, & x \geq 0, \\ \int_x^0 k(x, s)(Nf)(s)ds, & x < 0, \end{cases} \tag{14}$$

where the kernel $k(x, s)$ in (14) is defined by the following formula:

$$k(x, s) = \begin{cases} \int_s^x \frac{(x-t)^{n-2}}{(n-2)!} \left(\frac{s}{t}\right)^\gamma dt, & x > 0, s > 0, \\ \frac{(-1)^n}{(n-2)!} \int_x^s \left(\frac{|s|}{|t|}\right)^\gamma (t-x)^{n-2} dt, & x < 0, s < 0. \end{cases} \tag{15}$$

Therefore, it holds the following theorem.

Theorem 1. *Let* $\gamma > 0,$ *and* $f(x) \in C_0^{\{1\}}[-1, 1].$ *The Equation* (11) *is solvable in the space* $C^n[-1, 1]$ *and has the solution of the form* (14). *The corresponding operator* $L : C^n[-1, 1] \to C_0^{\{1\}}[-1, 1]$ *is a noetherian operator with the characteristic numbers of the form* $(n-1, 0)$ *and the index* $\chi(L) = n - 1.$

Proof. Let us carry out the proof for this case indicated by noting that the other demonstrations relating to the following theorems are obtained in an analogous way.

Similarly as in the simple case when $n = 1$ completely investigated in the paper [32]. In other words, first give the exact operational interpretation by introducing the inverse operator L^{-1}, where L is defined by (1). So that let $L^{-1} : C_0^{\{1\}}[-1, 1] \to C^n[-1, 1]$ and L^{-1} defined by the obtained following formula:

$$(L^{-1}f)(x) = P_{n-2}(x) + \frac{f(0)}{\gamma(n-1)!}x^{n-1} + \begin{cases} \int_0^x k(x, s)(Nf)(s)ds, & x \geq 0, \\ \int_x^0 k(x, s)(Nf)(s)ds, & x < 0, \end{cases}$$

Of course, it is simply sufficient to take the expression of the solution defined by (14) as $(L^{-1}f)(x)$ analogous to that of the case of $n = 1$ expressed by (17) defining the operator L^{-1} similar to our formula (14), with the kernel designated by the term (15). It is then necessary to obtain by the same scheme, the result on the fact that the two operators L and L^{-1} are bounded, by relying on Lemma 3.1 demonstrated in [32]. From this, it follows that both operators $L : C^n[-1, 1] \to C_0^{\{1\}}[-1, 1]$ and $L^{-1} : C_0^{\{1\}}[-1, 1] \to C^n[-1, 1]$ are bounded and invertible. Next we can see easily the relations $LL^{-1}f = f,\ f \in C_0^{\{1\}}[-1, 1]$ and $(L^{-1}Ly)(x) = y(x)$ take place as proven in the formulas (27) and (28).

(b) The case $\gamma < 0$, $\gamma \neq -1$

Under investigation of the Equation (11) similarly to the previous case investigated when $n = 1$ in the paper [32], we obtain the following relationship:

$$y^{(n-1)}(x) = z(x) = c_1|x|^{-\gamma} + c_2|x|^{-\gamma} sign\, x + \frac{f(0)}{\gamma}(1 - |x|^{-\gamma})$$

$$+ \begin{cases} -\int_x^1 \left(\frac{t}{x}\right)^{\gamma} (Nf)(t)dt, & x \geq 0, \\ \int_{-1}^x \left(\frac{|t|}{|x|}\right)^{\gamma} (Nf)(t)dt, & x < 0, \end{cases} \tag{16}$$

and some computations similarly as those realized when we investigated the case $\gamma > 0$ give us the description of the solution in the following way:

$$y(x) = P_{n-2}(x) + \frac{f(0)}{\gamma(n-1)!} x^{n-1}$$

$$+ \begin{cases} \int_0^1 k(x, s)(Nf)(s)ds + \frac{-f(0)}{\gamma(n-2)!} \int_0^x (x-t)^{n-2}(-t)^{-\gamma} dt \\ \int_{-1}^0 k(x, s)(Nf)(s)ds + \frac{(-1)^n f(0)}{\gamma(n-2)!} \int_x^0 (x-t)^{n-2}|t|^{-\gamma} dt \end{cases}$$

$$+ \begin{cases} \frac{c_1}{(n-2)!} \int_0^x (x-t)^{n-2} t^{-\gamma} dt, & x \geq 0, \\ \frac{(-1)^n c_1}{(n-2)!} \int_x^0 (t-x)^{n-2}|t|^{-\gamma} dt, & x < 0, \end{cases} \tag{17}$$

where the kernel $k(x, s)$ has the following form:

$$k(x, s) = \begin{cases} \frac{-1}{(n-2)!} \int_0^{\min(x, s)} \left(\frac{s}{t}\right)^{\gamma} (x-t)^{n-2} dt, & x > 0, s > 0, \\ \frac{(-1)^{n-1}}{(n-2)!} \int_{\max(x, s)}^0 \left(\frac{|s|}{|t|}\right)^{\gamma} (x-t)^{n-2} dt, & x < 0, s < 0. \end{cases} \tag{18}$$

Therefore, it holds the following theorem:

Theorem 2. *Let* $\gamma < -1$, *and* $f(x) \in C_0^{\{1\}}[-1, 1]$. *The Equation* (11) *is solvable in the space* $C^n[-1, 1]$ *and has the solution of the form* (17) *where the kernel* $k(x, s)$ *is defined by the formula* (18) *with* c_1 *and* c_2*-arbitrary constants. The corresponding operator* $L : C^n[-1, 1] \to C_0^{\{1\}}[-1, 1]$ *is a noetherian operator with the characteristic numbers of the form* $(n+1, 0)$ *and the index* $\chi(L) = n + 1$.

(c) The case $\gamma = 0$

This case is trivial and we can easily state the following theorem:

Theorem 3. *Let* $\gamma = 0$ *and* $f(x) \in C_0^{\{p\}}[-1, 1]$. *For the solvability of the Equation* (11) *in the space* $C^n[-1, 1]$, *it is necessary and sufficient the accomplishment of the conditions* $f(0) = f^{\{1\}}(0) = \cdots = f^{\{p-1\}}(0) = 0$. *Under realization of these conditions, the solution of the Equation* (11) *is given by the formula of the following form*:

$$y(x) = P_{n-2}(x) + \frac{1}{(n-2)!}\int_{-1}^{x}(x-t)^{n-2}(N^p f)(t)dt, \tag{19}$$

where $P_{n-2}(x)$ *is an* $(n-1)$*-th-order polynomial with arbitrary coefficients. The corresponding operator* $L : C^n[-1, 1] \to C_0^{\{p\}}[-1, 1]$ *is a noetherian operator with the characteristic numbers of the form* $(n-1, p)$ *and the index* $\chi(L) = n - 1 - p$.

3.2. Non-Euler case (case $p \geq 2$)

Let the operator $L : C_0^{n, \{p+n-1\}}[-1, 1] \to C_0^{\{p\}}[-1, 1]$ given by the left hand side of the equality (11) defined onto the functions from the space $C_0^{n, \{p+n-1\}}[-1, 1]$.

(a) The case $\gamma > 0$

In this case we have the following result easily deduced from the similar case investigated when $n = 1$, refer to the paper [33].

$$y^{(n-1)}(x) = z(x) = ce_{-}^{\frac{\gamma}{p-1}x^{1-p}} + \begin{cases} e^{\frac{\gamma}{p-1}x^{1-p}} \int_0^x e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, & x > 0, \\ \frac{f(0)}{\gamma}, & x = 0, \\ e^{\frac{\gamma}{p-1}x^{1-p}} \int_{-1}^x e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, & x < 0, \end{cases} \tag{20}$$

where c is an arbitrary constant. Taking into account the formula (13) and applying the Dirichlet formula, we obtain the following result:

$$y(x) = P_{n-2}(x) + \frac{(-1)^{n-1}c}{(n-2)!}\left[\int_x^0 (t-x)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt\right]_{-}$$

$$+ \begin{cases} \int_0^x k(x, s) f(s)ds, & x > 0, \\ \int_{-1}^0 k(x, s) f(s)ds, & x < 0, \end{cases} \tag{21}$$

where the kernel $k(x, s)$ is defined by:

$$k(x, s) = \begin{cases} \frac{1}{(n-2)!}\int_s^x \frac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(x-t)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, & x > 0, s > 0, \\ \frac{(-1)^{n-1}}{(n-2)!}\int_{\max(s, x)}^0 \frac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(t-x)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, & x < 0, s < 0, \end{cases} \tag{22}$$

and $P_{n-2}(x)$ a polynomial with arbitrary coefficients.

So that we see it holds the following theorem:

Theorem 4. *Let* $\gamma > 0$, $f(x) \in C_0^{\{p\}}[-1, 1]$, $p \geq 2$ *and* $n \geq 2$. *The Equation* (11) *is solvable in the space* $C_0^{n, \{p+n-1\}}[-1, 1]$ *with a right hand side* $f(x)$ *and the solution is given by the formula* (21) *where the kernel* $k(x, s)$ *has the form* (22). *The corresponding operator* $L : C_0^{n, \{p+n-1\}}[-1, 1] \to C_0^{\{p\}}[-1, 1]$ *is noetherian with the characteristic numbers* $(n, 0)$ *and the index* $\chi(L) = n$.

(b) The case $\gamma < 0$

From the previous obtained solution of the Equation (1) from paper [33], we have

$$y^{(n-1)}(x) = z(x) = ce_{+}^{\frac{\gamma}{p-1}x^{1-p}} + \begin{cases} -e^{\frac{\gamma}{p-1}x^{1-p}} \int_x^1 e^{-\frac{\gamma}{p-1}t^{1-p}} f(t) \frac{dt}{t^p}, & x > 0, \\ \frac{f(0)}{\gamma}, & x = 0, \\ -e^{\frac{\gamma}{p-1}x^{1-p}} \int_x^0 e^{-\frac{\gamma}{p-1}t^{1-p}} f(t) \frac{dt}{t^p}, & x < 0, \end{cases} \tag{23}$$

where c is an arbitrary constant. From that and using the same method as upstairs we find

$$y(x) = P_{n-2}(x) + \left[\frac{c}{(n-2)!} \int_0^x (x-t)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt \right]_{+}$$

$$+ \begin{cases} \int_0^1 k(x, s) f(s) ds, & x > 0, \\ \int_x^0 k(x, s) f(s) ds, & x < 0, \end{cases} \tag{24}$$

where the kernel $k(x, s)$ in the formula (24) has the following form:

$$k(x, s) = \begin{cases} -\dfrac{1}{(n-2)!}\displaystyle\int_{\min(x, s)}^{0} \dfrac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(x-t)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, & x > 0, s > 0, \\ \dfrac{(-1)^n}{(n-2)!}\displaystyle\int_{x}^{s} \dfrac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(t-x)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, & x < 0, s < 0, \end{cases} \tag{25}$$

and $P_{n-2}(x)$ a polynomial with arbitrary coefficients.

Now let us formulate the last theorem as following.

Theorem 5. *Let* $\gamma < 0$, $f(x) \in C_0^{\{p\}}[-1, 1]$, $p \geq 2$ *and* $n \geq 2$. *The Equation* (11) *is solvable in the space* $C_0^{n, \{p+n-1\}}[-1, 1]$ *with any right hand side* $f(x)$ *and the solution is given by the formula* (24) *where the kernel* $k(x, s)$ *has the form* (25). *The corresponding operator* $L : C_0^{n, \{p+n-1\}}[-1, 1] \to C_0^{\{p\}}[-1, 1]$ *is noetherian with the characteristic numbers* $(n, 0)$ *and the index* $\chi(L) = n$.

To close our research carried out in this work, let's go straight to the next part entitled conclusion.

4. Conclusion

This scientific work carried out presents in detail the different stages of complete investigation of the realization of the construction of the noetherian theory for the differential operator L, defined by a linear singular differential equation of higher order n in the space of continuous functions $C^n[-1, 1]$. We have identified the solvency conditions of Equation (1) taking into account the nature of the parameter $\gamma \in \mathbb{R}$ in various analyzed situations. All this leads us to gradually determine the characteristic numbers (α, β) and, consequently subsequently the index $\chi(L)$ of the operator L, which is a finite number in all the cases studied, making the operator considered a noetherian operator.

It is clear to mention quite naturally that in this investigation carried out, we were able to conduct the most important and necessary study to carry out the construction of noetherity of an integro-differential operator A, defined by a singular integral equation of the third type having for the main part, the studied operator L considered. We know from the general theory that, under perturbation of a noetherian operator by a compact operator and, in the case to be investigated in a brief future, we will reach and maintain the noetherity nature of the initial operator L. This will be the next future work to undertake when we do consider first of all, the operator A as a sum of two operators L and K, where L is the operator defined by $Ly(x) = x^p y^{(n)}(x) + \gamma y^{(n-1)}(x) = f(x)$ and K is a compact operator defined as follows $K\varphi = \int_{-1}^{1} k(x, t)y(t)dt.$

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