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ON SOME FINITE DIMENSIONAL EXTENSIONS FOR A NOETHER OPERATOR IN A SPECIFIC FUNCTIONAL SPACE

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Abstract

The main purpose of this work is to realize and to establish the extension of a noether operators A defined by a third kind singular integro-differential equation of first and n-th-order. The extended operator of the initial n-th-order operator A is realized in $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$ and it noetherity is investigated. In addition, we investigate the noetherity of the extended initial noether operator noted $\ddot{A}$ in $T_m = C_{-1}^{1}[-1, 1] \oplus \{\sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x)\} \oplus \{\sum_{j=1}^{m} \beta_j F.p \frac{1}{x^j}\}$. The index of such noether operator $\chi(\ddot{A})$ is calculated.

1. Introduction

The construction of noether theory for some integro-differential operators defined by linear third kind integral equations in some specific functional spaces is well known and still interest many scientits around the world. In most scientific research works, the solution of the integral equation of the third kind seems to be in a generalized space of continuous functions. Specific approaches needed when constructing noether theory for integro-differential operators, we can always set to problem of the investigation of the extension of the noether operator. For example, in [1], authors constructed noether theory of an integro-differential operator in generalized

$$D_m = C_{-1}^{1}[-1, 1] \oplus \left\{ \sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x) \right\}. \tag{1}$$

In [2, 3], authors found conditions for solvability of a linear integral equation of the third kind in the class of Hölder functions and in the class of Hölder functions with the coefficients which are piecewise strictly monotone function. A class of functions with Taylor derivative has considered in [4], and noetherity of an integro-differential equation of third kind has constructed.

Following such previous researches cited and others works done by many scientists, related to the realization of various types of extensions of noether operators, we realize in this work a particular type of extension when we add, at this time, simultaneously some functions from the space of delta Dirac distributions and principal-parts values of the following indicated form:

$$g(x) \in \{\sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x)\} \oplus \{\sum_{j=1}^{m} \beta_j F.p(\frac{1}{x^j})\}.$$

Namely, here in this paper we establish the noetherity of the extended operators define in the following:

$$(A\varphi)(x) = x^p \varphi'(x) + \int_{-1}^{1} k(x, t)\varphi(t)dt = f(x);\ x \in [-1, 1],$$

and

$$(A\varphi)(x) = x^p \varphi^{(n)}(x) + \int_{-1}^{1} k(x, t)\varphi(t)dt = f(x);\ x \in [-1, 1],$$

where $f(x) \in C_0^{\{p\}}[-1, 1]$ and $K(x, t) \in C_0^{\{p\}}[-1, 1]XC[-1, 1]$, in spaces

$$T_m = C_{-1}^{1}[-1, 1] \oplus \{\sum_{k=0}^{n} \alpha_k \delta^k(x)\} \oplus \{\sum_{j=1}^{m} \beta_j F.p. \frac{1}{x^j}\},$$

and $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$. The structure of this paper is the following: Section 2 is devoted to some fundamental well known notions and concepts of noether theory, Fredholm third kind integral equation, Taylor derivatives, associated spaces and associated operators. In Section

3, we realize the extension of the operator A when taking the unknown function from the space $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$. Section 4 deals with the realization of the extension of the operator A when taking the unknown function from the space T_m. Lastly after making a small important remark, we conclude our paper in Section 5, followed by some recommendations for the follow-up or future scientific works to undertake, stated in Section 6.

2. Preliminaries

Before presenting in full details our main results, the following definitions and concepts well known from the noether theory of operators are required for the realization of this research. We also recall the notions of Taylor derivatives and linear Fredholm integral equation of the third kind, widely studied in [5].

Definition 2.1. Let X, Y be Banach spaces, $A \in l(X, Y)$ a linear operator. The dimensions $\alpha(A) = \dim\ker A$, $\beta(A) = \dim\operatorname{coker} A$ are called the nullity and the deficiency of the operator A, respectively.

Note that, if at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite, then the difference $\operatorname{Ind} A = \alpha(A) - \beta(A)$ is called the index of the operator A.

Definition 2.2. Let X, Y be Banach spaces, an operator $A \in l(X, Y)$ is normally solvable, if it possesses the following property: The equation $Ax = y(y \in Y)$ has at least one solution $x \in D(A)$ ($D(A)$ is the domain of A) if and only if $< y, f >= 0, \forall f \in (\operatorname{im} A)^{\perp}$. It's prove in [5] that the operator A is normally solvable if and only if, its image space $\operatorname{im} A$ is closed.

Definition 2.3. A closed normally solvable operator A is called a noether operator if its index is finite.

By the way, we briefly review this important notions of Taylor derivatives which is widely used when constructing noether theory of the considered operator A.

Definition 2.4. A continuous function $\varphi(x) \in C[-1, 1]$ admits at the point $x = 0$ Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1, 2, p$ the following limits $\varphi^{\{k\}}(0) = k! \lim_{x \to 0} x^{-k}$ $[\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j]$, and $\varphi^{(0)}(0) = \varphi(0)$. The class of such functions $\varphi(x)$ is denoted $C_0^{\{p\}}[-1, 1]$.

Note that, a linear operator N^k on the space $C_0^{\{p\}}[-1, 1]$ has been defined by the following:

$$(N^k \varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0) x^j}{j!}}{x^k}, \; k = 1, 2, \dots, p.$$

Lemma 2.1 ([4]). *A function* $\varphi(x)$ *belongs to* $C_0^{\{p\}}[-1, 1]$ *if and only if, the following representation*

$$\varphi(x) = x^p \Phi(x) + \sum_{k=0}^{p-1} \alpha_k x^k, \tag{2}$$

holds with the function $\Phi(x) \in C[-1, 1]$, *and* α_k *being constants.*

Consequently, the linear operator N^p establishes a relation between the spaces $C_0^{\{p\}}[-1, 1]$ and $C[-1, 1]$. The space $C_0^{\{p\}}[-1, 1]$ with the norm.

$$\|\varphi\|_{C_0^{\{p\}}[-1, 1]} = \|N^p \varphi\|_{C[-1, 1]} + \sum_{k=0}^{p-1} |\varphi^{\{k\}}(0)|, \tag{3}$$

becomes a Banach space one.

Let note also that we can define the previous norm by the following way:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\alpha_k| = \|\Phi(x)\|_{C[-1,1]} + \sum_{k=0}^{p-1} |\alpha_k|.$$

Sometimes it is comfortable and suitable to consider as norm in the space $C_0^{\{p\}}[-1, 1]$ the equivalent norm defined as follow:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p} \|N^j\varphi\|_{C[-1,1]}.$$

We can also note a very useful and clearly helpful next inequality:

$$\|\varphi\|_{C[-1,1]} + \sum_{j=0}^{p-1} |\varphi^{\{j\}}(0)| = \|\varphi\|_{C_0^{\{p\}}[-1,1]}.$$

Therefore, it is obvious to see that $\|\varphi\|_{C[-1,1]} \le \|\varphi\|_{C_0^{\{p\}}[-1,1]}$. Finally, note that from the definition $\varphi(x) \in C[-1, 1]$, imply $x^p\varphi(x) \in C_0^{\{p\}}[-1, 1]$. This assertion may be generalized as follows.

Lemma 2.2. *Let* $p \in \mathbb{N}$, $s \in \mathbb{Z}_+$. *If* $\varphi(x) \in C_0^{\{s\}}[-1, 1]$ *then,* $x^p\varphi(x) \in C_0^{\{p+s\}}[-1, 1]$, *and the formula holds*

$$x^p\varphi(x)^{\{j\}}(0) = \begin{cases} 0, & j = 0, 1, \dots, p-1, \\ \dfrac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), & j = p, \dots, p+s. \end{cases} \tag{4}$$

Proof. Note that a stronger assumption on the function $\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1, 1]$ would allow us to easily prove the lemma just applying Leibniz formula.

For $s = 0$ the statement has already been proved above, so $x^p\varphi(x) \in C_0^{\{p\}}[-1, 1]$, and $(x^p\varphi(x))^{\{j\}}(0)$, $j = 0, \dots, p-1$ and $(x^p\varphi(x))^{\{p\}}(0) = p!\,\varphi(0)$. Now let us prove that $(x^p\varphi(x))^{\{j\}}(0) = \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0)$, $j = p+1, \dots, p+s$. Since the derivatives are defined recurrently, and (4) is true for $j = p$, then it is sufficient to verify the passage from j to $j+1$. We have:

$$(x^p\varphi(x))^{\{j+1\}}(0) = (j+1)! \lim_{x\to 0} \frac{x^p\left(\varphi(x) - \sum_{l=p}^{j} \frac{x^l}{(l-p)!}\varphi^{\{l-p\}}(0)\right)}{x^{j+1}} \tag{5}$$

$$= (j+1)! \lim_{x\to 0} \frac{\varphi(x) - \sum_{l=0}^{j-p} \frac{x^l}{(l)!}\varphi^{\{l\}}(0))}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!}\varphi^{\{j+1-p\}}(0). \tag{6}$$

Lemmas 2.1 and 2.2 imply the next important lemma.

Lemma 2.3. *Let* $f(x) \in C_0^{\{p\}}[-1, 1]$, $p \in \mathbb{N}$ *and* $f(0) = \dots = f^{\{r-1\}}(0) = 0$, $1 \le r \le p$. *Then* $\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[-1, 1]$. *We say that the kernel* $k(x, t) \in C_0^{\{P\}}[-1, 1] \times C[-1, 1]$, *if and only if* $k(x, t) \in C[-1, 1] \times C[-1, 1]$ *and admits Taylor derivatives according to the variable x at the point* $(0, t)$ *whatever* $t \in [-1, 1]$.

Instead of talking about adjoint operator when establishing the noetherity of an operator, we can note that also noether property of an operator may depend of the concept of associated operators and associated spaces. Therefore, we start by recalling these two important concepts and we give some associated spaces that we are going to use later within the work.

Definition 2.6. The Banach space $E' \subset E^*$ is called associated space with a Banach space E, if

$$| < f, \varphi > | \leq c\|f\|_{E'}\|\varphi\|_E, \; \forall \varphi \in E, \; f \in E'. \tag{7}$$

Definition 2.7. Let $E_j, (j = 1, 2)$ be Banach spaces and E'_j their associated spaces, operators $A \in l(E_1, E_2)$ and $A' \in l(E'_2, E'_1)$ are associated if and only if:

$$(A'f, \varphi) = (f, A\varphi) \, \forall f \in E'_2 \text{ and } \varphi \in E_1. \tag{8}$$

Lemma 2.4. *Let $E_j, j = 1, 2$ be Banach spaces, E'_j their associated spaces, $A \in l(E_1, E_2)$ and $A' \in l(E'_2, E'_1)$ are associated noether operators we have $\chi(A) = -\chi(A')$ (where χ means the index), and for the solvability of equation $A\phi = f$ it's necessary and sufficient that $(f, \psi) = 0$ for all solutions of the associated homogenous equation $A'\psi = 0$. We finish these reminders with two very important results that define the associated spaces of spaces that we will use later.*

Lemma 2.5. *Space $C^1_{x_0}[-1, 1]$ is associated to space $C[-1, 1]$, where $C^1_{x_0}[-1, 1]$ means the space of functions $\varphi \in C[-1, 1]$ satisfying $\varphi(x_0) = 0$.*

Proof. Let $f \in C^1_{x_0}[-1, 1]$ and $\varphi \in C[-1, 1]$. Then we have: $| < f, \varphi > |$ $= |\int_{-1}^{1} f(x)\varphi(x)dx| \leq 2 \max_{-1\leq x\leq 1}|f(x)|. \max_{-1\leq x\leq 1}|\varphi(x)|.$

□

Definition 2.8. Through P^1 we denote the space of distributions ψ on the space of test functions $C^{\{p\}}_{-1}[-1, 1]$ such that: $\psi(x) = \frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)$, where $z(x) \in C^{\{p\}}_{-1}[-1, 1] \cap C^1_{-1}[-1, 1]$, β_k

are arbitrary constants, $\delta^{\{k\}}(x)$ is the k-th Taylor derivative of the Dirac-delta function defined by

$$(\delta^{\{k\}}(x),\ \varphi(x)) = (-1)^{k}(0). \tag{9}$$

In the space P^1 let us introduce the norm by the following way:

$$\|\psi\|_{P^1} = \|z\|_{C_{-1}^{\{p\}}[-1,\,1]} + \|z\|_{C^1[-1,\,1]} + \sum_{k=0}^{p-1}|\beta_k|, \tag{10}$$

it is obvious that P^1 is a Banach space with this norm.

Lemma 2.6 ([1, 4])**.** *The space* P^1 *is a Banach space associated to the space* $C_{-1}^{\{p\}}[-1,\,1]$.

Definition 2.9. A linear Fredholm integral equation of the third kind is an equation of the form

$$A_n\varphi = f, \tag{11}$$

where f is a given function of the variable $x \in [a,\,b]$; φ is the unknown function of $x \in [a,\,b]$, and A_n is the operator defined by

$$A_n\varphi = \prod_{k=1}^{n}(x - x_k)\varphi(x) - \int_a^b k(x,\,t)\varphi(t)dt,\ x_k \in\ ]a,\,b[, \tag{12}$$

$k(x,\,t)$ is a given function of variables $(x,\,t) \in [a,\,b]\times[a,\,b]$.

Full details on the notion of linear Fredholm integral equation of the third can be found in [6-8].

3. Extension of the *n*-th Operator *A* in Space $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$

Let us consider the following third kind integral equation:

$$(A\varphi)(x) = x^p \varphi^{(n)}(x) + \int_{-1}^{1} k(x, t)\varphi(t)dt = f(x), \tag{13}$$

$x \in [-1, 1]$, $f \in C_0^{\{p\}}[-1, 1]$, $\varphi \in C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$, where $p, n \in \mathbb{N}$, and $k(x, t)$, $f(x)$ are continuous functions $\varphi \in C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\} \Leftrightarrow \exists h \in C_{-1}^{n}[-1, 1] : \varphi(x) = h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)$, with $h(-1) = h'(-1) = \dots = h^{(n-1)}(-1) = 0$.

Therefore, (13) can be write as follow:

$$(A\varphi)(x) = x^p \Big(h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\Big)^{(n)} + \int_{-1}^{1} k(x, t)\Big(h(t) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(t)\Big)dt = f(x). \tag{14}$$

Now consider the principal part of operator (14) define by

$$(L\varphi)(x) = x^p \Big(h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\Big)^{(n)} = f(x), \tag{15}$$

$x \in [-1, 1]$, $f \in C_0^{\{p\}}[-1, 1]$.

It homogeneous equation is define by:

$$x^p \Big(h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\Big)^{(n)} = 0. \tag{16}$$

Let us solve above equation in space $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$.

$$x^p (h^{(n)}(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k+n\}}(x)) = 0,$$

we deduce that, $h^{(n)}(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k+n\}}(x) = 0$. By doing n successives integrations and take to the account that $h(-1) = h'(-1) = \ldots = h^{(n)}(-1) = 0$, we have

$$h(x) = -\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x).$$

Then we have

$$\varphi(x) = h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x) = -\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x) = 0.$$

Now, we continue with the study of non-homogeneous equation define as follow:

$$(L\varphi)(x) = x^p (h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x))^{(n)} = f(x), \tag{17}$$

with $f \in C_0^{\{p\}}[-1, 1]$. f can be write as follow:

$$f(x) = \sum_{k=0}^{p-1} c_k x^k + x^p g(x),$$

with $g \in C[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$, see [9]. By replacing the value of f in (17), we have

$$x^p (h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x))^{(n)} = \sum_{k=0}^{p-1} c_k x^k + x^p g(x),$$

i.e.,

$$(h(x)+\sum_{k=0}^{l}\alpha_k\delta^{\{k\}}(x))^{(n)} = \frac{1}{x^p}\sum_{k=0}^{p-1}c_kx^k + g(x),\ x \neq 0, \tag{18}$$

where $c_k = \frac{f^{\{k\}}(0)}{k!}$.

Therefore (17) is solvable in $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l}\alpha_k\delta^{\{k\}}(x)\}$ if and only if $c_k = 0$, $k = 0, \dots, p-1$. In addition, $\exists u \in C[-1, 1]$ such that $g(x) = u(x) + \sum_{k=0}^{l}\beta_k\delta^{\{k\}}(x)$ As a result,

$$(h(x)+\sum_{k=0}^{l}\alpha_k\delta^{\{k\}}(x))^{(n)} = u(x)+\sum_{k=0}^{l}\beta_k\delta^{\{k\}}(x),$$

i.e., $$h^{(n)}(x)+\sum_{k=0}^{l}\alpha_k\delta^{\{k+n\}}(x) = u(x)+\sum_{k=0}^{l}\beta_k\delta^{\{k\}}(x),$$

i.e., $$h^{(n)}(x) = u(x)+\sum_{k=0}^{l}\beta_k\delta^{\{k\}}(x)-\sum_{k=0}^{l}\alpha_k\delta^{\{k+n\}}(x).$$

By doing n successives integrations, one has

$$h(x) = \frac{1}{(n-1)!}\int_{-1}^{x}(t+1)^{n-1}u(t)dt$$

$$+\sum_{k=0}^{l}\beta_k\int_{-1}^{x}dt\int_{-1}^{t}dv\int_{-1}^{v}dw\dots\int_{-1}^{s}\delta^{\{k\}}(\theta)d\theta-\sum_{k=0}^{l}\alpha_k\delta^{\{k\}}(x),$$

$\varphi(x) = h(x)+\sum_{k=0}^{l}\alpha_k\delta^{\{k\}}(x),$

where

$$\varphi(x) = \frac{1}{(n-1)!}\int_{-1}^{x}(1+t)^{n-1}u(t)dt+\sum_{k=0}^{l}\beta_k\int_{-1}^{x}dt\int_{-1}^{t}dv\int_{-1}^{v}dw\dots\int_{-1}^{s}\delta^{\{k\}}(\theta)d\theta,$$

under the following P-conditions: $(\delta^{\{k\}}(x), f(x)) = 0$, $k = 0, \dots, p-1$.

Now, let show that the above solvability conditions is exactly orthogonality conditions of homogeneous equation.

Let show that, adjoint L' of operator (15) is define as follows $(L'\psi)(x) = (-1)^n (x^p \psi)^{(n)}$ from P^1 (see [9]) to $C^n_{-1}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$. In fact, let verify that $\forall \psi \in P^1$ and $\varphi \in C^n_{-1}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$ we have: $(L\varphi, \psi) = (\varphi, L'\psi)$.

$$
\begin{aligned}
(L\varphi, \psi) &= (x^p \varphi^{(n)}(x), \psi) \\
&= \Big(x^p \big(h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\big)^{(n)}, \frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)\Big) \\
&= \Big(x^p \big(h^{(n)}(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k+n\}}(x)\big), \frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)\Big) \\
&= (h^{(n)}, z(x)) + \Big(x^p h^{(n)}(x), \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)\Big) \\
&\quad + \Big(x^p \sum_{k=0}^{l} \alpha_k \delta^{\{k+n\}}(x), \frac{z(x)}{x^p}\Big) \\
&\quad + \Big(x^p \sum_{k=0}^{l} \alpha_k \delta^{\{k+n\}}(x), \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)\Big) \\
&= (h^{(n)}(x), z(x)) + \Big(x^p h^{(n)}(x), \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)\Big) \\
&\quad + \Big(\sum_{k=0}^{l} \alpha_k \delta^{\{k+n\}}(x), z(x)\Big) + \sum_{k=0}^{l} \alpha_k \sum_{k=0}^{p-1} \beta_k (x^p \delta^{\{k+n\}}(x), \delta^{\{k\}}(x))
\end{aligned}
$$

$$= ((h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x))^{(n)},\ z(x)) + (h^{(n)}(x),\ \sum_{k=0}^{p-1} \beta_k x^p \delta^{\{k\}}(x))$$

$$+ \sum_{k=0}^{l} \alpha_k \sum_{k=0}^{p-1} \beta_k (x^p \delta^{\{k+n\}}(x),\ \delta^{\{k\}}(x)),$$

or $p > k,\ \forall k \in \{0, 1, \ldots, p-1\}$ then, $x^p \delta^{\{k\}}(x) = 0.$

Therefore

$$(L\varphi,\ \psi) = ((h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x))^{(n)},\ z(x)). \tag{19}$$

In the other hand,

$$(\varphi,\ L'\psi) = (\varphi,\ (-1)^n (x^p \psi)^{(n)})$$

$$= (h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x),\ (-1)^n (x^p (\frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)))^{(n)}$$

$$= (h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x),\ (-1)^n (z(x) + x^p \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x))^{(n)})$$

$$= (-1)^n (h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x),\ (z(x) + \sum_{k=0}^{p-1} \beta_k x^p \delta^{\{k\}}(x))^{(n)}),$$

or $x^p \delta^{\{k\}}(x) = 0 \forall k = 0, 1, \ldots, p-1.$

Therefore

$$(\varphi,\ L'\psi) = (-1)^n (h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x),\ z^{(n)}(x)),$$

$$(\varphi,\ L'\psi) = ((h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x))^{(n)},\ z(x)), \tag{20}$$

taking into the account (19) and (20) we deduce that $(L\varphi, \psi) = (\varphi, L'\psi)$, therefore $L'\psi = (-1)^n (x^p \psi)^{(n)}$. Let study the following homogeneous equation: $(L'\psi)(x) = (-1)^n (x^p \psi)^{(n)} = 0$ in space P^1.

$$(L'\psi)(x) = 0 \Leftrightarrow (x^p \psi)^{(n)} = 0$$

$$\Leftrightarrow (x^p (\frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)))^{(n)} = 0$$

$$\Leftrightarrow (z(x) + x^p \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}})^{(n)} = 0$$

$$\Leftrightarrow (z(x))^{(n)} + (\sum_{k=0}^{p-1} \beta_k x^p \delta^{\{k\}}(x))^{(n)} = 0$$

$$\Leftrightarrow ((z(x))^n, \varphi) + (\sum_{k=0}^{p-1} \beta_k x^p \delta^{\{k\}}(x)^{(n)}, \varphi) = 0$$

$$\Leftrightarrow ((z(x))^n, \varphi) + (\sum_{k=0}^{p-1} (-1)^n \beta_k x^p \delta^{\{k\}}(x), \varphi^{(n)}) = 0$$

$$\Leftrightarrow ((z(x))^{(n)}, \varphi) = 0, \text{ if } (\delta^{\{k\}}, f) = 0\ k = 0, \dots, p-1$$

$$\Leftrightarrow (z(x))^{(n)}, (x) = 0.$$

Therefore solvability conditions $(\delta^{\{k\}}, f) = 0$, $k = 0, \dots, p-1$ of non-homogeneous equation (17) is the same of orthogonal conditions of right part $f(x)$ of (17) with all solution of homogenous equation $L'\psi = 0$.

In fact Let $\psi \in P^1$ solution of $L'\psi = 0$, one has

$$\psi(x) = \frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x),$$

with $z(-1) = z'(-1) = \dots = z^{(n-1)}(-1) = 0$ and

$$(\psi, f) = \Big(\frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x), f\Big)$$

$$= \Big(\frac{z(x)}{x^p}, f\Big) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f)$$

$$= \Big(\frac{z(x)}{x^p}, x^p \varphi^{(n)}(x)\Big) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f)$$

$$= (z(x), \varphi^{(n)}(x)) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f)$$

$$= (-1)^n (z^{(n)}(x), \varphi) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f)$$

$$= 0 \text{ because } (z^{(n)}, \varphi) = 0 \text{ and } (\delta^{\{k\}}(x), f) = 0, k = 0, \dots, p-1.$$

The following lemma, give noetherity of principal part L and its adjoint L' of Equation (13).

Lemma 3.1. *Let* $L : C_{-1}^n \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\} \to C_0^{\{p\}}[-1, 1]$ *defined by*

$$(L\varphi)(x) = x^p \Big(h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\Big)^{(n)} = f(x),$$

with $h \in C^n$ *such that* $h(-1) = h'(-1) = \dots = h^{(n-1)}(-1) = 0$. *L is noether with d-characteristics* $(0, p)$. *In addition, adjoint operator of L is define by the following*: $L' : P^1 \to C_{-1}^n[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$ *such that* $L'\psi = (-1)^n (x^p \psi)^{(n)} = h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)$. *$L'$ is noether with indexe* $\chi(L') = p$.

Proof. Let us show that L is noether. We start by showing that, L is normally solvable, we show above that $L\varphi = f$ is solvable in $C^n[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$ under conditions $(\delta^{\{k\}}, f) = 0$, let $\psi \in \ker L'$, then $L'\psi = 0$, i.e.,

$$\begin{aligned}
L'\psi = 0 &\Leftrightarrow (-1)^n (x^p \psi)^{(n)} = 0 \\
&\Leftrightarrow (x^p \psi)^{(n)} = 0 \\
&\Leftrightarrow ((x^p \psi)^{(n)}, \varphi) = 0 \\
&\Leftrightarrow (-1)^n (x^p \psi, \varphi^{(n)}) = 0 \\
&\Leftrightarrow (-1)^n (z(x), \varphi^{(n)}(x)) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f) = 0 \\
&\Leftrightarrow (-1)^n (-1)^n (z^{(n)}(x), \varphi) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f) = 0 \\
&\Leftrightarrow (z^{(n)}, \varphi) = 0 \\
&\Leftrightarrow z^{(n)} = 0.
\end{aligned}$$

$$\begin{aligned}
\psi(f) &= (\psi, f) \\
&= \left(\frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x), f \right) \\
&= \left(\frac{z(x)}{x^p}, f \right) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f) \\
&= \left(\frac{z(x)}{x^p}, x^p \varphi^{(n)}(x) \right) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x), f)
\end{aligned}$$

$$= (z(x),\ \varphi^{(n)}(x)) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x),\ f)$$

$$= (-1)^n (z^{(n)}(x),\ \varphi) + \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x),\ f)$$

$$= \sum_{k=0}^{p-1} \beta_k (\delta^{\{k\}}(x),\ f) \text{ because } (z^{(n)},\ \varphi) = 0 \forall \varphi,$$

$$= 0 \text{ because } (\delta^{\{k\}}(x),\ f) = 0,\ \forall k = 0, \ldots, p-1.$$

Therefore L is normally solvable.

Let us show that $\alpha(L)$ and $\beta(L)$ are finite.

Let $\varphi \in \ker L$, with $\varphi(x) = h(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)$, $h(x) \in C_{-1}^{n}$ then

$$L\varphi = 0,$$

$$x^p \varphi^{(n)}(x) = 0,$$

$$x^p h^{(n)}(x) + \sum_{k=0}^{l} \alpha_k x^p \delta^{\{k+n\}}(x) = 0,$$

$$h(x) = -\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x) \text{ car } h(-1) = h'(-1)$$

$$= \ldots = h^{(n-1)}(-1) = 0,$$

where $\varphi(x) = -\sum_{k=0}^{l} \alpha_k x^p \delta^{\{k\}}(x) + \sum_{k=0}^{l} \alpha_k x^p \delta^{\{k\}}(x) = 0$, then dimker $L = 0$.

Let $\psi \in \ker L'$, then $L'\psi = 0$, i.e., $(-1)^n (x^p \psi)^{(n)} = 0$, we deduce by the above that $z^{(n)}(x) = 0$, then $z(x) = 0$ because $z(-1) = z'(-1) = \ldots = z^{(n-1)}$ $(-1) = 0$ then

$$\psi(x) = \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x).$$

Hence $\ker \psi =< \delta^{\{0\}}, \delta^{\{1\}} \ldots \delta^{\{p-1\}} >$, i.e., $\dim \ker = p$.

Therefore L is noether with d-characteristics $(0, p)$, then L' is also noether with index

$$\begin{aligned}\chi(L') &= -\chi(L)\\ &= -(0-p)\\ &= p.\end{aligned}$$

□

Adjoint A', of operator A is define below:

Lemma 3.2. *Operator* A' *define by*

$$A'\psi = (-1)^n (x^p \psi)^{(n)} + \int_{-1}^{1} k(t, x)\psi(t)dt$$

is adjoint of operator A.

Proof. Let $\psi \in P^1$, then

$$\psi(x) = \frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x),$$

$$\begin{aligned}(\psi, A\varphi) &= (\psi, (L+K)\varphi)\\ &= (\psi, L\varphi + K\varphi)\end{aligned}$$

$$= (L'\psi, \varphi) + (\psi, K\varphi)$$

$$= ((-1)^n (x^p \psi)^{(n)}, \varphi) + (\psi, K\varphi)$$

$$= ((-1)^n (x^p \psi)^{(n)}, \varphi) + (K'\psi, \varphi)$$

with $K'\psi = \int_{-1}^{1} k(x, t)\psi(t)dt,$

where

$$(\psi, A\varphi) = ((-1)^n (x^p \psi)^{(n)}, \varphi) + (\int_{-1}^{1} k(x, t)\psi(t)dt, \varphi)$$

$$= ((-1)^n (x^p \psi)^{(n)} + \int_{-1}^{1} k(x, t)\psi(t)dt, \varphi),$$

$$= (A'\psi, \varphi),$$

where

$$A'\psi = (-1)^n (x^p \psi)^{(n)} + \int_{-1}^{1} k(x, t)\psi(t)dt.$$

□

Noether property of extension of operator A, define by Equation (13), in the $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$, is given in the following result.

Theorem 3.1. *Operator* $A : C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\} \to C_0^{\{p\}}[-1, 1]$,

define by

$$(A\varphi)(x) = x^p \varphi^{(n)}(x) + \int_{-1}^{1} k(x, t)\varphi(t)dt = f(x),$$

is noether with index $\chi(A) = -p$. *In addition, its adjoint* A' *is noether with index* $\chi(A) = p$.

Proof. One has $A = L + K$ with $L : C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$ $\to C_0^{\{p\}}[-1, 1]$ such that $L\varphi = x^p \varphi^{(n)}(x)$ and $K\varphi = \int_{-1}^{1} k(x, t)\varphi(t)dt.$

By Lemma 3.1, L is noether with d-characteristics $(0, p)$ and $\chi(L) = -p$. Therefore $A = L + K$ is also noether and $\chi(A) = \chi(L) = -p$ because K is compact.

Since A' is the adjoint of A, A' is also noether and $\chi(A') = -\chi(A) = p$.

□

4. Extension of Operator A in Space

$$T_m = C_{-1}^{1}[-1, 1] \oplus \{\sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x)\} \oplus \{\sum_{j=1}^{m} \beta_j F.p(\tfrac{1}{x^j})\}$$

Let give definitions of our fundamental spaces.

Definition 4.1. We denote through $T_m = C_{-1}^{1}[-1, 1] \oplus \{\sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x)\}$ $\oplus \{\sum_{j=1}^{m} \beta_j F.p.(\frac{1}{x^j})\}$ the space of all functions of the form:

$$\varphi(x) = \varphi_0(x) + \sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x) + \sum_{j=1}^{m} \beta_j F.P.(\frac{1}{x^j}), \tag{21}$$

where $\varphi_0(x) \in C[-1, 1]$ and $\varphi_0(-1) = 0$ with the natural norm

$$\|\varphi\| = \|\varphi_0\|_{C[-1, 1]} + \sum_{k=1}^{n} |\alpha_k| + \sum_{j=1}^{m} |\beta_j|. \tag{22}$$

In the following, we present results related to singular integral for functions from the space $C_0^{\{p\}}[-1, 1]$.

If the function $g(x)$ has feature (singularity) in $x = 0$, then we say that $\int_{-1}^{1} g(x) dx$ exists in the sense of Hadamard if it is true the following representation:

$$\int_{-1}^{-\epsilon} g(x) dx + \int_{\epsilon}^{1} g(x) dx = a + \sum_{k=1}^{l} a_k \epsilon^{-k} + a_{l+1} \ln \frac{1}{\epsilon} + O(\epsilon), \ \epsilon \to 0. \quad (23)$$

In this case, we put $F.P. \int_{-1}^{1} g(x) dx = a$, i.e., it remains the finite parts. Note that under the definition of the convergence by Hadamard often we take $a_{l+1} = 0$, but we do not exclude that possibility as this can allow us to consider the convergence (V.p.) in the sense of Cauchy principal part as particular case of convergence in the sense of Hadamard.

Now let $\varphi(x) \in C_0^{\{p\}}[-1, 1]$, $p \in \mathbb{N}$ and consider $\int_{-1}^{1} \frac{\varphi(x)}{x^p} dx$, $p \in \mathbb{N}$.

The following result deal with compactness of an operator on $C_0^{\{p\}}[-1, 1]$.

Lemma 4.1 ([4])**.** *The set* $\mathcal{M} \subset C_0^{\{p\}}[-1, 1]$ *is relatively compact in space* $C_0^{\{p\}}[-1, 1]$ *if and only, if*

(1) *the set* $\mathcal{M}$ *is bounded,*

(2) *the family of continuous functions* $N^p(\mathcal{M})$ *is equicontinuous on* $[-1, 1]$.

Let consider the following integro-differential equation:

$$(A\varphi)(x) = x^p \varphi'(x) + \int_{-1}^{1} k(x, t)\varphi(t) dt = f(x), \quad (24)$$

$\varphi \in T_m$ mean that,

$$\varphi(x) = \varphi_0(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x) + \sum_{j=0}^{m} \beta_j F.P. \frac{1}{x_j}. \quad (25)$$

(24) become

$$(A\varphi)(x) = x^p(\varphi_0(x)' + \sum_{k=0}^{l} \alpha_k \delta^{\{k+1\}}(x)) + \int_{-1}^{1} k(x, t)(\varphi_0(x)$$

$$+ \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(t))dt - \sum_{j=0}^{m} \beta_j F.P \int_{-1}^{1} \frac{k(x, t)}{x^j} dt = g(x). \quad (26)$$

Let put

$$K_j = F.P \int_{-1}^{1} \frac{k(x, t)}{x^j}. \quad (27)$$

Lemma 4.2. *The operator* $K_j : C[-1, 1] \to C_0^{\{p\}}[-1, 1]$ *is compact.*

Proof. First, let show that K_j is continuous.

Since $k(x, t) \in C_0^{\{p\}}[-1, 1] \times C[-1, 1]$, $k(x, t)$ is represented as follows:

$$k(x, t) = \sum_{k=0}^{p-1} C_k(t) x^k + x^p \widehat{k_1(x, t)}.$$

where $\widehat{k_1(x, t)}$ is continuous, $c_k(t) \in C[-1, 1]$.

Let $h \in C[-1, 1]$ one has

$$(K_j h)(x) = F.P \int_{-1}^{1} \frac{k(x, t)}{x^j} h(t) dt$$

$$= \int_{-1}^{1} \frac{\sum_{k=0}^{p-1} C_k(t) x^k + x^p \widehat{k_1(x, t)}}{x^j} h(t) dt$$

$$= \int_{-1}^{1} \sum_{k=0}^{p-1} c_k(t) x^{k-j} h(t) dt + \int_{-1}^{1} x^{p-j} \widehat{k_1(x, t)} h(t) dt.$$

Let put $\widehat{c_k} = \int_{-1}^{1} c_k(t)\,dt$, $\Omega(x) = \int_{-1}^{1} \widehat{k_1(x, t)}\, h(t) dt \in C[-1, 1]$, then we have

$$\|K_j h\|_{C_0^{\{p\}}[-1, 1]} = \|\Omega(x)\|_{c[-1, 1]} + \sum_{k=0}^{p-1} \left| \int_{-1}^{1} c_k(t) h(t) \right|$$

$$= \max_{-1 \le x \le 1} |\Omega(x)| + \sum_{k=0}^{p-1} \left| \int_{-1}^{1} c_k(t) h(t) dt \right|$$

$$\le \int_{-1}^{1} \max_{-1 \le t \le 1} |\widehat{k_1(x, t)}| \max_{-1 \le t \le t} |h(t)| dt$$

$$+ \sum_{k=0}^{p-1} \int_{-1}^{1} \max_{-1 \le t \le 1} |c_k(t)| \max_{-1 \le t \le 1} |h(t)| dt$$

$$\le 2 \max_{-1 \le t \le 1} |\widehat{k_1(x, t)}| \, \|h\|_{C[-1, 1]}$$

$$+ \sum_{k=0}^{p-1} 2 \max_{-1 \le t \le 1} |c_k(t)| \, \|h\|_{C[-1, 1]}.$$

Thus,

$$\|K_j h\|_{C_0^{\{p\}}[-1, 1]} \le 2\left(M + \sum_{k=0}^{p-1} M_k\right) \|\varphi\|_{C[-1, 1]}, \tag{28}$$

where $M = \max_{-1 \le t \le 1} |\widehat{k_1(x, t)}|$ and $M_k = \max_{-1 \le t \le 1} |c_k(t)|$.

Therefore, K_j is a continuous operator.

Finally, let show that K_j transform each bounded set $\mathcal{M}_0 = \{\varphi\} \subset C_{-1}^{1}[-1, 1]$, into a relatively compact set, in $C_0^{\{p\}}[-1, 1]$.

Let $\mathcal{M}_0 = \{h \in C^1_{-1}[-1, 1]\}$ a bounded set, then $\exists r > 0$ such that $\|h(t)\|_{C^1_{-1}[-1, 1]} \leq r, \forall h \in \mathcal{M}_0$. Let put $\mathcal{M} = K_j\mathcal{M}_0 \subset C_0^{\{p\}}[-1, 1]\}$, $\mathcal{M}$ is a bounded set by virtue of (28). The remain to show that, the family of continuous functions:

$$N^p(\mathcal{M}) = N^p\big(K_j(\mathcal{M}_0)\big)$$

$$= \{N^p K\varphi, \varphi \in \mathcal{M}_0\}$$

$$= \{h / h(x) = \int_{-1}^{1} \frac{\widehat{k(x, t)}}{x^j} \varphi(t) dt, \varphi \in \mathcal{M}_0\},$$

is equicontinuous. Let $\epsilon > 0$, we are looking for $\delta > 0 / |h(x_1) - h(x_2)| < \epsilon$, $\forall |x_1 - x_2| < \delta$, $h \in N^p(\mathcal{M})$. One has:

$$|h(x_1) - h(x_2)| = |\int_{-1}^{1} [\frac{\widehat{k(x_1, t)}}{x_1^j} - \frac{\widehat{k(x_2, t)}}{x_2^j}]|\varphi(t) dt,$$

$$= \int_{-1}^{1} |\frac{\widehat{k(x_1, t)}}{x_1^j} - \frac{\widehat{k(x_2, t)}}{x_2^j}||\varphi(t)| dt$$

$$\leq \int_{-1}^{1} |\frac{\widehat{k(x_1, t)}}{x_1^j} - \frac{\widehat{k(x_2, t)}}{x_1^j} + \frac{\widehat{k(x_2, t)}}{x_1^j} - \frac{\widehat{k(x_2, t)}}{x_2^j}||\varphi(t)| dt$$

$$\leq \int_{-1}^{1} |\frac{\widehat{k(x_1, t)} - \widehat{k(x_2, t)}}{x_1^j}||\varphi(t)| dt + \int_{-1}^{1} |(\frac{1}{x_1^j} - \frac{1}{x_2^j})\widehat{k(x_2, t)}|\varphi(t)| dt$$

$$\leq \int_{-1}^{1} (\frac{1}{x^j}) \max_{-1 \leq t \leq 1} |k(x_1, t) - k(x_2, t)|| \max_{-1 \leq t \leq 1} \varphi(t)| dt$$

$$+ \int_{-1}^{1} |\frac{1}{x_1^j} - \frac{1}{x_2^j}|| \max_{-1 \leq t \leq 1} (k(x_2, t))|| \max_{-1 \leq t \leq 1} (\varphi(t))| dt$$

$$\leq (c + 2 \max_{-1 \leq t \leq 1} |\hat{k}(x_1, t) - \hat{k}(x_2, t)|) \|\varphi\|_{C^1_{-1}[-1, 1]},$$

with $c = \int_{-1}^{1} (\frac{1}{x_1^j} - \frac{1}{x_2^j}) |\max_{-1 \leq t \leq 1} (k(x_2, t))|$, since $k(x, t)$ is uniformly continuous,

$$\forall x, t \in [-1, 1], \exists \delta' > 0 / |\hat{k}(x_1, t) - \hat{k}(x_2, t)| < \frac{\epsilon - cr}{2r}, \tag{29}$$

taking into account (29)

$$|h(x_1) - h(x_2)| \leq (c + \frac{\epsilon - cr}{cr}) \times r$$

$$|h(x_1) - h(x_2)| \leq \frac{\epsilon}{r} \times r$$

$$|h(x_1) - h(x_2)| = \epsilon.$$

Thus K_j is a compact operator. □

The following result show that the extension of noether operator define by (24), in T_m is noether.

Theorem 4.1. *Let denote by* $\ddot{A}$ *the extension of operator A defined by* (24) *in* T_m. *Operator* $\ddot{A}$ *is noether with index* $\chi(\ddot{A}) = -1$.

Proof. Let put

$$\ddot{L}\varphi = x^p (\varphi_0'(x) + \sum_{k=0}^{l} \alpha_k \delta^{\{k+1\}(x)}) + \int_{-1}^{1} k(x, t)(\varphi_0(t) + \sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(t)) dt,$$

i.e.,

$$\ddot{L}\varphi = x^p \varphi' + \int_{-1}^{1} k(x, t) \varphi(t) dt, \ \varphi \in D_l,$$

by virtue of work [9, 10] $\ddot{A}$ is noether, with index $\chi(\ddot{A}) = -1$.

Thank to Lemma 4.2., we deduce that,

$$\sum_{j=0}^{m} \beta_j \int_{-1}^{1} \frac{k(x, t)}{x^j} dt$$

is compact.

By virtue of the conservation of the noether property of a noether operator when it is perturbed by a compact operator, we deduce that

$$\ddot{A}$$

is noether, with $\chi(\ddot{A}) = -1$.

5. Conclusion

Summarizing our work, we state that we have completely realized the extension of a n-th-order noether operator A in $C_{-1}^{n}[-1, 1] \oplus \{\sum_{k=0}^{l} \alpha_k \delta^{\{k\}}(x)\}$ by using direct computation. In addition, we have realized the extension of a noether operator A defined by the extended operator $\ddot{A}$ in the space T_m. We applied well-known noether theory for integro-differential operators defined by a third kind integral equation and, we used compactness and our previous results to show noetherity of extension $\ddot{A}$. Consequently, we formalized within Theorem 4.1 the global results of the investigation related to the question of noetherity nature of the extended operator, which as proved is noetherian operator. The principle of the conservation of noetherity nature of a noether operator after extension by some finite dimensional space of added functions to the initial space is established firmly as proved in theory. We can also note that, the index of the initial operator after extension remains the same, i.e., $\chi(A) = \chi(\ddot{A})$ no matter the deficient numbers may be others than the previous before extension, i.e., $(\alpha(A), \beta(A)) \neq (\alpha(\ddot{A}), \beta(\ddot{A}))$.

6. Recommendations

The achieved researches in this work completed by those already obtained by many scientific researchers related to the question of the noetherity nature of an extended operator of an initial noether operator in some various functional generalized spaces may lead us to project, and to construct illustrative examples to inter connect theory and practice. This will be the next work to be done in a brief future. We underline once more again that, the main difficulty appearing when realizing such extension is still and always connected with the derivative of the unknown function within the third kind singular integral equation through which is defined the initial integro-differential operator to be extended onto the new generalized functional space.

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