**Journal of Mathematical Sciences: Advances and Applications**
Volume 76, 2024, Pages 13-37
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122305

# SPECIAL NOETHERIZATION APPROACH FOR AN INTEGRO-DIFFERENTIAL OPERATOR *A*

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## Abstract

The main purpose of this work is to realize the noetherization construction theory for an integro-differential operator defined by a third-kind integral equation in a specific well-chosen functional space. Several works carried out by our predecessors in connection with the construction of noetherian theory for integro-differential operators defined by integral equations of the third kind when the main part of the operator is defined in the form of the product of a function $a(x)$ by the unknown function have been published. The particularity of our research

2020 Mathematics Subject Classification: 34A25, 45A05, 45B05, 46A32, 46A50, 47A52.
Keywords and phrases: noetherian theory, third kind integral equation, singular linear integro-differential operator, characteristic numbers, index of the operator.
Received July 28, 2024

focuses on the case, where the main part of the operator $A$ this time contains the product of a function $a(x)$ by the derivative of the unknown function itself. To achieve the noetherization of the operator $A$, it was necessary to use a rather special approach involving the notion of derivative in the Taylor sense of the unknown function. The noetherization of operator $A$ is constructed and the solvability conditions of the integral equation have been released. An illustrative example has been realized at the end of the paper.

## 1. Introduction

The research leading to the construction of the noetherian theory for certain types of integro-differential operators defined by integral equations of the third kind has been widely inventoried and published in certain scientific books and articles (see, for example, the references [1, 2, 4, 6, 8, 10, 11, 12, 23]) using several specific methods and approaches depending on the cases studied. In particular, the case of integro-differential operators defined by integral equations of the third kind whose main part is presented as the product of a function $a(x)$ by the unknown function has been widely studied when the function $a(x)$ under the unknown function admits a finite number of zeros, and several illustrations have been presented. The methodology for investigating such case studies is a well-known process, dealing with serious results from the theories of differential equations, functional analysis and integral equations. However, in the presence of certain peculiarities and specificities, expressed as the main difficulties associated with such an investigation, we are sometimes faced in this case with the problem of choosing the indicated approach to achieve the goal. Let us recall that the study of the solvability of integral equations of the third type while not constructing any other theory for the operators defined by such integral equations leads us to carefully choose the necessary approaches which lead us to the expected goal. In several articles and published works, see ([1, 2, 4, 6, 7, 8, 10, 13, 18]) and illustrated in [22, 26, 31], also followings Prossdorf's research. S. G. Samko, A. A. Kilbas, O. I. Marichev, V. S.

Raslambekov, N. S. Gobassov, Roghozin, T. N. Radtchenko, N. N. Karapetiants and Abdourahman, it was presented for example the methods of normalization and hypersingular integrals to achieve the construction of the noetherian theory. Let us underline the importance of a special specific approach, called the method of approximate inverse operators developed for integral equations of the third kind in the functional space of continuous and generalized functions, perfectly illustrated in several research activities of imminent mathematicians and researchers in this area of operator theory. These different methods cited and mentioned above allowed us to properly pose the noetherization problem when we approach the investigation of integral equations of the first type, or non-Fredholm integral equations of the second type defining the different integro-differential operators. Note also that the researchers D. A. Shulaia and E. I. Gugushvili, in some of their works carried out the investigations of the inverse problem of the spectral analysis of the theory of linear multigroup neutron transport in plane geometry in their article Transp. Theory Stat. Phys. 29 (2000), No. 6, 711-722. Namely, Shulaia carried out work in the functional space of Hôlder functions on a non-homogeneous linear integral equation with coefficient cos x and brought out the necessary and sufficient conditions for the solvability of the equation considered under certain hypotheses mentioned on its core. On the other hand, the researcher also succeeds in carrying out the construction of the desired solution analytically, applying the theory of singular integral equations and the theory of Fredholm, which is clearly illustrated with all the details in reference [28].

Similarly, let us also mention that the researcher G. Bal published an interesting article on Inverse problems concerning homogeneous transport equations specifically, the one-dimensional case is published in the work titled Inverse Problems 16 (2000), no 4, 997-1028.

Let’s also remember the work of investigating the solvency of a specific form of an integral equation defined in the following way:

$$A(x)\varphi(x) + \int_a^b K(x,\ y)\varphi(y)dy = f(x),\ x \in\ ]a,\ b[,$$

where $A(x)$ has at least one zero, carried out in 2017 by the mathematician Shulaia. Following his scientific paper titled "Integral equation of the third kind for the case of piecewise monotone coefficients" published in Transactions of A Razmadze Mathematical Institute 171 (2017). 396-410 and devoted to a third kind integral equations with coefficients, which are piecewise strictly monotone functions having simple zeros investigated in Hôlder class, Shulaia using singular integral theory equations had released the necessary and sufficient conditions for the solvability of the considered equation.

Thus, in this present work that we carry out, the approach which is used in the construction of the noetherian theory of the integro-differential operator defined by the operator $A$ is based on the application of the notion of derivative in the sense of Taylor at the point 0 due to the existence of the continuous first derivative of the unknown function within the integral equation.

Namely, we investigate for noetherization the following integro-differential operator defined by the following third kind integral equation

$$A\varphi(x) = x^p\varphi'(x) + \int_{-1}^{1} K(x,\ t)\varphi(t)dt = f(x),$$

where the unknown function $\varphi \in C_{-1}^{1}[-1,\ 1]$, $p \in \mathbb{N}$, $f(x) \in C_0^{\{p\}}[-1,\ 1]$, and $K(x,\ t)$ satisfying the condition $K(x,\ t) \in C_0^{\{p\}}[-1,\ 1]XC[-1,\ 1]$.

We organize this work as follows: first of all at the beginning, we present in Section 2 the important preliminaries related to the concept and the notions of well-known noetherian theory. Section 3 is properly devoted to the main results of the investigation presenting the whole description of the analysis of the considered problem followed by an illustrative example. Lastly, after an important remark, we summarize the content of the work in Section 4 titled Conclusion.

## 2. Preliminaries

Before presenting in detail our main results, the following definitions and concepts well-known from the noetherian theory of operators are required. For details in full see also [6, 10, 12, 21, 22].

By the way, we briefly also review these important notions of Taylor derivatives and concepts of associated operators and associated spaces which are widely used when constructing noetherian theory of the considered operator $A$.

**(A) Noetherian operator**

**Definition 1.** Let $X, Y$ be Banach spaces, $A \in l(X, Y)$ a linear operator. The quotient space $coker\ A = Y/imA$ is called the cokernel of the operator $A$. The dimensions $\alpha(A) = \dim \ker A$, $\beta(A) = \dim coker\, A$ are called the nullity and the deficiency of the operator $A$, respectively. If at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite, then the difference $IndA = \alpha(A) - \beta(A)$ is called the index of the operator $A$.

**Definition 2.** Let $X, Y$ be Banach spaces, $A \in l(X, Y)$ is said to be normally solvable if it possesses the following property: The equation $Ax = y(y \in Y)$ has at least one solution $x \in D(A)(D(A)$ is the domain of $A$) if and only if $< y, f >= 0 \forall f \in (imA)^{\perp}$ holds.

We recall that by the definition of the adjunct operator $(im\ A)^{\perp} = \ker A^*$ and it's prove in [11] that the operator $A$ is normally solvable if and only if its image space $imA$ is closed.

**Definition 3.** A closed normally solvable operator $A$ is called a noetherian operator if its index is finite.

By the way, we briefly review this important notions of Taylor derivatives which is widely used when constructing noetherian theory of the considered operator $A$.

**Definition 4.** A Continuous function $\varphi(x) \in C[-1, 1]$ admits at the point $x = 0$. Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1, 2, \ldots, p$, the following limits:

$$\varphi^{\{k\}}(0) = k! \lim_{x \to 0} x^{-k}\left[\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j\right]. \tag{2}$$

The class of such functions $\varphi(x)$ is denoted $C_0^{\{p\}}[-1, 1]$.

Next, let us move to the following part.

Let $C^m[-1, 1]$, $m \in \mathbb{Z}_+$, note the Banach space of continuous functions on $[-1, 1]$, having continuous derivatives up to order $m$, for which the norm is defined as follows:

$$\|\varphi(x)\|_{C^m[-1, 1]} = \sum_{j=0}^{m} \max_{-1 \le X \le 1} \left|\varphi^{(j)}(x)\right|. \tag{3}$$

So, we can consider $\varphi^{\{k\}}(0)$ are defined for all $k = 1, 2, \ldots, p$.

We define $C_0^{\{p\}}[-1, 1]$ as a subspace of continuous functions, having finite Taylor derivatives up to order $p \in \mathbb{Z}_+$; and when $p = 0$, we put $(C_0^{\{p\}}[-1, 1] = C_0^{\{0\}}[-1, 1] = C[-1, 1]))$.

Let us also define a linear operator $N^k$ on the space $C_0^{\{p\}}[-1, 1]$ by the formula:

$$(N^k \varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j}{x^k}, \ k = 1, 2, \ldots, p. \tag{4}$$

One can easily verify the property $N^k = N^{k_1} N^{k-k_1}$, $0 \le k_1 \le k$, $k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$.

**Definition 5.** The operator $N^p$ is called characteristical operator of the space $C_0^{\{p\}}[-1, 1]$.

**Remark.** The sense of the previous definition can be seen from the verification of the following lemma and also for more details, see [23, 28, 29].

**Lemma 2.1.** *A function* $\varphi(x)$ *belongs to* $C_0^{\{p\}}[-1, 1]$ *if and only if, the following representation*:

$$\varphi(x) = x^p \phi(x) + \sum_{k=0}^{p-1} \alpha_k x^k, \tag{5}$$

*holds with the function* $\phi(x) \in C[-1, 1]$, *and* $\alpha_k$ *being constants.*

To prove Lemma 2.1, it is enough to observe that (5) implies that the Taylor derivatives of $\varphi(x)$ up to the order $p$ exist, and more $\varphi^{\{k\}}(0) = k!\,\alpha_k$, $k = 0, 1, 2, \ldots, p-1$, $\varphi^{\{0\}}(0) = p!\,\phi(0)$ with $\phi(x) = (N^k \varphi)(x)$. Conversely, if $\varphi(x)$ belongs to $C_0^{\{p\}}[-1, 1]$, and we define $\phi(x) = (N^k \varphi)(x)$ with $\alpha_k = \dfrac{\varphi^{\{k\}}(0)}{k!}$, $k = 0, 1, 2, \ldots, p-1$, then the representation (5) holds. From Lemma 2.1, it follows that for $\varphi(x) \in C_0^{\{p\}}[-1, 1]$ the equality

$$\varphi(x) = x^p (N^k \varphi)(x) + \sum_{k=0}^{p-1} \frac{\varphi^{\{k\}}(0)}{k!} x^k, \tag{6}$$

is valid.

Consequently, the linear operator $N^p$ establishes a relation between the spaces $C_0^{\{p\}}[-1, 1]$ and $C[-1, 1]$. The space $C_0^{\{p\}}[-1, 1]$ with the norm

$$\|\varphi\|_{C_0^{\{p\}}[-1,\,1]} = \|N^p\varphi\|_{C[-1,\,1]} + \sum_{k=0}^{p-1}|\varphi^{\{k\}}(0)|, \tag{7}$$

becomes a Banach space one.

Let note also that we can define the previous norm by the following way:

$$\|\varphi\|_{C_0^{\{p\}}[-1,\,1]} = \|N^p\varphi\|_{C[-1,\,1]} + \sum_{k=0}^{p-1}|\alpha_k| = \|\phi(x)\|_{C[-1,\,1]} + \sum_{k=0}^{p-1}|\alpha_k|.$$

Sometimes it is comfortable and suitable to consider as norm in the space $C_0^{\{p\}}[-1,\,1]$ the equivalent norm defined as follow:

$$\|\varphi\|^1_{C_0^{\{p\}}[-1,\,1]} = \sum_{j=0}^{p}\|N^j\varphi\|_{C[-1,\,1]}. \tag{$\therefore$}$$

We can also note a very useful and clearly helpful next inequality:

$$\|\varphi\|_{C[-1,\,1]} \leq \|N^p\varphi\|_{C[-1,\,1]} + \sum_{j=0}^{p-1}|\varphi^{\{j\}}(0)| = \|\varphi\|_{C_0^{\{p\}}[-1,\,1]}. \tag{8}$$

Therefore, it is obvious to see that $\|\varphi\|_{C[-1,\,1]} \leq \|\varphi\|_{C_0^{\{p\}}[-1,\,1]}$.

Finally, note that from the Lemma 2.1 follows the following fact, if $\varphi(x) \in C[-1,\,1]$, then $x^p\varphi(x) \in C_0^{\{p\}}[-1,\,1]$. This assertion may be generalized as follows.

**Lemma 2.2.** *Let* $p \in \mathbb{N}$, $s \in \mathbb{Z}_+$. *If* $\varphi(x) \in C_0^{\{s\}}[-1,\,1]$ *then,* $x^p\varphi(x) \in C_0^{\{p+s\}}[-1,\,1]$, *and the formula holds*

$$(x^p\varphi(x))^{\{j\}}(0) = \begin{cases} 0, \; j = 0, 1, \dots, p-1, \\ \dfrac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), \; j = p, \dots, p+s. \end{cases} \tag{9}$$

**Proof.** Note that a stronger assumption on the function $\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1, 1]$ would allow us to easily prove the lemma just by applying the Leibniz formula.

For $s = 0$ the statement has already been proved above, so $x^p \varphi(x) \in C_0^{\{p\}}[-1, 1]$, and $(x^p \varphi(x))^{\{j\}}(0) = 0$, $j = 0, \dots, p-1$ and $(x^p \varphi(x))^{\{p\}}(0) = p!\,\varphi(0)$. Now, let us prove that $(x^p \varphi(x))^{\{j\}}(0) = \frac{j!}{(j-p)!} \varphi^{\{j-p\}}(0)$, $j = p+1, \dots, p+s$. Since the derivatives are defined recurrently, and (8) is true for $j = p$, then it is sufficient to verify the passage from $j$ to $j+1$. We have:

$$(x^p \varphi(x))^{\{j+1\}}(0) = (j+1)! \lim_{x \to 0} \frac{x^p \varphi(x) - \sum_{l=p}^{j} \frac{x^l}{(l-p)!} \varphi^{\{l-p\}}(0)}{x^{j+1}} \tag{10}$$

$$= (j+1)! \lim_{x \to 0} \frac{\varphi(x) - \sum_{l=0}^{j-p} \frac{x^l \varphi^{\{l\}}(0)}{l!}}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!} \varphi^{\{j+1-p\}}(0). \tag{11}$$

Lemmas 2.1 and 2.2 imply the next following important lemma.

**Lemma 2.3.** *Let* $f(x) \in C_0^{\{p\}}[-1, 1]$, $p \in \mathbb{N}$ *and* $f(0) = \dots = f^{\{r-1\}}(0) = 0$, $1 \le r \le p$. *Then it holds the assertion*:

$$\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[-1, 1]. \tag{12}$$

*Let us state the following important lemma relative to the characteristical operator of the space* $C_0^{\{p\}}[-1, 1]$.

**Lemma 2.4.** *The operator* $N^p : C_0^{\{p\}}[-1, 1] \to C[-1, 1]$ *has the following properties*:

(1) $N^p$ *is bounded, and* $\|N^p\varphi\|_{C[-1,\,1]} \le \|\varphi\|_{C_0^{\{p\}}[-1,\,1]}$;

(2) $N^p$ *is right invertible*;

(3) $\alpha(N^p) = p$, *where* $\alpha(N^p)$ *is the dimension of the null subspace for* $N^p$.

**Proof.** Statement (1) follows from the definition of the norm in (7). The invertibility is justified by the fact that the equation $N^p\varphi = f$ with an arbitrary $f(x) \in C[-1, 1]$ has a solution $\varphi(x) = x^p f(x) \in C_0^{\{p\}}[-1, 1]$, which follows from Lemma 2.1 and the equality $N^p(x^p f(x)) \equiv f(x)$. By (2) and noticing that $N^p x^k = 0$ for all $k = 0, 1, \dots, p-1$, we arrive at to the point (3).

The lemma is proved.

Note that from the proof of this lemma, it follows that the equation $N^p f = g$ is always solvable in the space $C_0^{\{p\}}[-1, 1]$ for every $g(x) \in C[-1, 1]$, and its general solution has the form

$$f(x) = \sum_{k=0}^{p-1} c_k x^k + x^p g(x), \tag{13}$$

where $c_k$ are arbitrary constants. Note that for the Taylor derivatives, many formulas are valid similar to those for ordinary derivatives; in particular, Leibniz formula, l'Hospitale rule and others see [11, 13]. Nevertheless, for example, the formula

$$|\varphi^{(k)}(x)|^{\{r\}}(0) = \varphi^{\{k+r\}}(0) \tag{14}$$

is not always valid.

**Definition 6.** We designate through $P^1 = P^{1,\{p\}}_{1,0}[-1, 1]$ the space of distributions on the space of test functions $C_0^{\{p\}}[-1, 1]$ such that: $\psi(x) = \frac{z(x)}{x^P} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)$, where $z(x) \in C_0^{\{p\}}[-1, 1] \cap C_{-1}^1[-1, 1]$, $\beta_k$ - arbitrary constants.

We underline that $\delta^{\{k\}}(x)$ is the $k$-th derivative of the Dirac delta function defined by the following way $(\delta^{\{k\}}(x), \varphi(x)) = (-1)^k \varphi^{\{k\}}(0)$.

Let us introduce a norm in the space $P^1$ in the following way:

$$\|\psi\|_{P^1} = \|z\|_{C_0^{\{p\}}[-1, 1]} + \|z\|_{C^1[-1, 1]} + \sum_{k=0}^{p-1} |\beta_k|.$$

**(B) Associated operator and associated space**

**Definition 7.** The Banach space $E' \subset E^*$ is called associated space with the space E, if $|(f, \varphi)| \le c\|f\|_{E'}\|\varphi\|_E$ for every $\varphi \in E$, $f \in E'$.

We note that the initial space E can be considered associated with the space E$'$. Moreover, the norm $\|f\|_{E'}$ is not obliged to be equivalent to the norm $\|f\|_{E^*}$.

Let be noted $\mathcal{L}(E_1, E_2)$ the Banach algebra of all linear bounded operators from $E_1$ into $E_2$.

**Definition 8.** Let $E_j$, $j = 1, 2$ two Banach spaces and $E'_j$, $j = 1, 2$ their associated spaces. The operators $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ are called associated, if $(A'f, \varphi) = (f, A\varphi)$ for all $f \in E'_2$ and $\varphi \in E_1$.

It seems that we can formalize the noetherity in terms of associated operator and associated space. See [22, 26].

**Lemma 2.5.** *Let* $E_j$, $j = 1, 2$ *two Banach spaces and* $E'_j$, $j = 1, 2$ *their associated spaces and, let* $A \in \mathcal{L}(E_1, E_2)$ *with* $A' \in \mathcal{L}(E'_2, E'_1)$ *be associated noetherian operators and more,*

$$\alpha(A) = -\alpha(A').$$

Then, for the solvability of the equation $A\varphi = f$ it is necessary and sufficient that $(f, \psi) = 0$ for all solutions of the homogeneous associated equation $A'\psi = 0$.

Let us state without proof the following important theorem:

**Theorem 2.1.** *The space* $P^1 = P_{1,0}^{1,\{p\}}[-1, 1]$ *is a banach space associated with the space* $C_0^{\{p\}}[-1, 1]$.

**Definition 9.** We say that the kernel $K(x, t) \in C_0^{\{p\}}[-1, 1]XC[-1, 1]$, if $K(x, t) \in C[-1, 1]XC[-1, 1]$ and it holds the representation

$$K(x, t) = \sum_{k=0}^{p-1} c_k(t)x^k + x^p \widetilde{K}_1(x, t), \tag{$*$}$$

where $\widetilde{K}_1(x, t)$ is continuous by the arguments $x, t \in [-1, 1]$ and $c_k(t) \in C[-1, 1]$.

In other words, $K(x, t)$ has Taylor derivatives by the argument $x$ at the point $(0, t)$ under any $t \in [-1, 1]$.

Next, let us move to the presentation of the general important results of the work in the following section.

## 3. Main Results

In this section, we undertake properly the noetherization investigation of the integro-differential operator $A$ defined by the formula (1).

Namely, here we consider as a model to be investigated, the integro-differential operator defined by the formula (1).

**(A) The main part of the equation**

First of all let consider the main part designated $L$ of the operator $A$ with the multiplication operator with a function of the following form:

$$L\varphi(x) = x^p \varphi'(x) = f(x);\ x \in [-1, 1], \tag{15}$$

where we suppose $p \in \mathbb{N},\ f(x) \in C_0^{\{p\}}[-1, 1]$. It is obvious that the homogeneous equation $x^p y'(x) = 0$ in the space $C^1[-1, 1]$ admits only trivial solution $\varphi(x) = 0$. Under consideration of the nonhomogeneous equation with respect to $f(x) \in C_0^{\{p\}}[-1, 1]$ represented in the form $f(x) = \sum_{k=0}^{p-1} c_k x^k + x^p g(x)$, where $c_k = \dfrac{f^{\{k\}}(0)}{k!}$ and $g(x) = (N^p f)(x) \in C[-1, 1]$, we have $\varphi'(x) = g(x) + \dfrac{1}{x^p} \sum_{k=0}^{p-1} c_k x^k$. From that for the solvability in the space of continuous functions, it is necessary and sufficient that $c_k = 0,\ k = 0, \dots, p-1,$ or the same as $\left(\delta^{\{k\}}(x),\ f(x)\right) = 0.$ In this case, with respect to the condition $\varphi(-1) = 0,$ the equation (15) has only a unique solution $\varphi(x) = \int_{-1}^{x} t^{-p} f(t)dt.$

Therefore the operator $L$ has the characteristic numbers $d(0,\ p)$, i.e., $\alpha(L) = 0,\ \beta(L) = p,$ and the index $\chi(L) = -p.$

Our following objective is to verify that the obtained solvability conditions are the same as the orthogonality conditions of the solutions of the homogeneous associated equation. For this matter, consider the operator $L'$ of the following form:

$$(L'\psi)(x) = -(x^p \psi)' = h(x), \tag{16}$$

which acts from $P^1$ into $C[-1, 1]$. It is not difficult to verify with respect to the condition $\varphi(-1) = z(1) = 0$, that for every $\psi(x) \in P^1$ and $\varphi(x) \in C^1_{-1}[-1, 1]$, it holds the relationship $(L\varphi, \psi) = (\varphi, L'\psi)$ as $L$ and $L'$ are associated operators.

The associated homogeneous equation $(L'\psi)(x) = -(x^p\psi)' = 0$ in the space $P^1$ with respect to the representation from definition 6 for the $\psi(x)$, has the solution of the form $\psi(x) = \sum_{k=0}^{p-1}\beta_k\delta^{\{k\}}(x)$, where $\beta_k$ - are arbitrary constants. We note that the $\delta^{\{k\}}(x)$ are linearly independent distributional functions.

From the previous analysis, let us now formulate the following lemma.

**Lemma 3.1.** *The operator* $L : C^1_{-1}[-1, 1] \to C_0^{\{p\}}[-1, 1]$ *is noetherian and with the characteristic numbers* $(0, p)$. *The nonhomogeneous equation* $L\varphi = f$ *is solvable under the accomplishment of p conditions of orthogonality* $\left(\delta^{\{k\}}(x), f(x)\right) = 0$, $k = 0, \ldots, p-1$ *to all the solutions of the associated homogeneous equation* $L'\psi = 0$ *in the associated space* $P^1$. *Under accomplishment of these conditions, the nonhomogeneous equation has a unique solution* $\varphi(x) = \int_{-1}^{x} t^{-p}f(t)dt$ *in the space* $C^1_{-1}[-1, 1]$.

The associated nonhomogeneous equation defined by the operator $L' : P^1 \to C[-1, 1]$ with respect to (16) under every $h(x) \in C[-1, 1]$ has in the associated space $P^1$ the solution of the following form:

$$\psi(x) = \frac{1}{x^p}\int_x^1 h(t)dt + \sum_{k=0}^{p-1}\beta_k\delta^{\{k\}}(x), \tag{17}$$

with arbitrary constants $\beta_k$, $k = 0, \ldots, p-1$.

Let us globalize all of that in the form of a lemma.

**Lemma 3.2.** *The operator* $L' : P^1 \to C[-1, 1]$ *is noetherian and with the characteristic numbers* $(p, 0)$. *The nonhomogeneous equation* $L'\psi = h$ *has a solution defined by the formula* (17).

Next, let us investigate the compactness of the integral operator.

**(B) Compactness criterion of the integral operator**

Let us consider the following integral operator

$$(K\varphi)(x) = \int_{-1}^{1} K(x, t)\varphi(t)dt, \ x \in [-1, 1], \tag{18}$$

with the kernel $K(x, t)$ satisfying the condition

$$K(x, t) \in C_0^{\{p\}}[-1, 1]XC[-1, 1]. \tag{19}$$

**Theorem 3.1.** *Let be accomplished the condition* (19). *Then the operator* $K$ *of the form* (18) *completely continuous operator from* $C_{-1}^1[-1, 1]$ *into* $C_0^{\{p\}}[-1, 1]$.

**Proof.** The proof follows from the schema of the proof of the same fact from [13, 14]. Let $\varphi(t) \in C_{-1}^1[-1, 1]$.

Let's reassure ourselves at the beginning that $(K\varphi)(t) \in C_0^{\{p\}}[-1, 1]$. Using the representation $K(x, t)$ in the form $(*)$, we have:

$$(K\varphi)(x) = \sum_{k=0}^{p-1} \tilde{c}_k x^k + x^p \Omega(x), \tag{20}$$

where it is denoted

$$\tilde{c}_k = \int_{-1}^{1} c_k(t)\varphi(t)dt, \ \Omega(x) = \int_{-1}^{1} \tilde{K}_1(x, t)\varphi(t)dt.$$

Considering that $\Omega(x) \in C[-1, 1]$, we obtain $(K\varphi)(x) \in C_0^{\{p\}}[-1, 1]$. From (20), it is easy to deduce the following approximation:

$$\|K\varphi\|_{C_0^{\{p\}}[-1, 1]} \leq c\|\varphi\|_{C[-1, 1]}, \tag{21}$$

where $c = 2(M + \sum_{k=0}^{p-1} M_k)$, $M_k = \max_t |c_k(t)|$, and $M = \max_{x,t} |\widetilde{K}_1(x, t)|$.

Lastly, let us prove that the operator $K$ translates any bounded set $\mathcal{M}_0 = \{\varphi\} \subset C_{-1}^1[-1, 1]$ in a relatively compact set into $C_0^{\{p\}}[-1, 1]$.

For this matter, we will use the following compactness criterion in the space $C_0^{\{p\}}[-1, 1]$ refer to [13, 14].

**Lemma 3.3.** *The set* $\mathcal{M} \subset C_0^{\{p\}}[-1, 1]$ *is relatively compact into the space* $C_0^{\{p\}}[-1, 1]$ *if and only if, when*:

(a) *The set* $\mathcal{M}$ *is bounded*;

(b) *The family of continuous functions* $N^p(\mathcal{M})$ *is equicontinuously on the segment* $[-1, 1]$.

The proof of this lemma from [13] consists of the repetition with some not great changes of the proof of the Arzela criterion.

Here we conduct another form of the proof which will allow us next to expand it also to our further research.

**Proof.** The necessary conditions (a) of the theorem follows directly from the boundedness relatively compact set. Concerning the condition (b), then the boundedness operator $N^p : C_0^{\{p\}}[-1, 1] \to C[-1, 1]$ translates the relatively compact set $\mathcal{M}$ into $N^p(\mathcal{M})$ in $C_0^{\{p\}}[-1, 1]$ relatively compact in $C[-1, 1]$ and then (b) follows from the Arzela criterion.

**Sufficiency.** Using the condition (a) and the equivalent norm in the space $C_0^{\{p\}}[-1, 1]$ given by the formula $(\therefore)$, we see that $|f(t)| \leq r,\ t \in [-1, 1],\ |f^{\{j\}}(0)| \leq r,\ j = 1, \ldots, p,$ for all $f \in \mathcal{M}$. With respect to the obvious equality (7) and taking into consideration the equicontinuity of $(N^p f)(x),\ f \in \mathcal{M}$, uniformly continuity of power functions $x^j,\ j = 0, \ldots, p,$ and uniformly boundedness $|f^{\{j\}}(0)| \leq r$, $j = 0, 1, \ldots, p$, follows the equidistant continuity $f(x) \in \mathcal{M}$. Therefore by the Arzela criterion, $\mathcal{M}$ relatively compact in $C[-1, 1]$, and then with respect to the uniformly boundedness $|f^{\{j\}}(0)|,\ j = 1, \ldots, p$ and Bolzano lemma follows the relatively compactness in the space $C_0^{\{p\}}[-1, 1]$ which was what needed to be proven.

Continuing the proof of the theorem, let us apply this criterion. As $\mathcal{M}_0$ is a bounded set, then by virtue of (21) it follows that and $\mathcal{M} = K(\mathcal{M}_0) \subset C_0^{\{p\}}[-1, 1]$ also a bounded set. It remains to convince ourselves that the family of the functions $h = N^p K\varphi \in N^p(\mathcal{M}) = N^p$ $K(\mathcal{M}_0)$, where it is designed $h(x) = \int_{-1}^{1} \widetilde{K}_1(x, t)\varphi(t)dt$, equidistant continuous.

Approximating the difference $|h(x_1) - h(x_2)|$, we obtain

$$|h(x_1) - h(x_2)| = 2\left(\max_{x_1, x_2, t}\left|\widetilde{K}_1(x_1, t) - \widetilde{K}_1(x_2, t)\right|\right)\|\varphi\|_{C[-1, 1]}.$$

It remains to consider that the function $\widetilde{K}_1(x, t)$ is uniformly continuous by the argument $x, t \in [-1, 1]$, and this ends the proof of the Theorem 3.1.

Now, let us consider the associated operator defined by the following formula:

$$(K'\psi)(x) = \int_{-1}^{1} K(t,\ x)\psi(t)dt,\ x \in [-1,\ 1]. \tag{22}$$

The following theorem gives the condition of its compactness from $P^1$ into $C[-1, 1]$.

**Theorem 3.2.** *Let the kernel* $K(t, x)$ *satisfying the condition* (19). *Then the operator* $K'$ *is compact from* $P^1$ *into* $C[-1, 1]$.

**Proof.** For the function $\psi(x)$ from $P^1$ we have the analytical representation: $\psi(x) = \dfrac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)$, where $z(x) \in C_0^{\{p\}}[-1, 1] \cap C_{-1}^1[-1, 1]$ with respect to the previous, we obtain the following result:

$$(K'\psi)(x) = \int_{-1}^{1} K(t,\ x)\frac{z(t)}{t^p}dt + \sum_{j=0}^{p-1}(-1)^j \beta_j K_1^{\{j\}}(0,\ x),$$

where $K_1^{\{j\}}(0, x)$ designates the Taylor derivative of the kernel $K(t, x)$ by the variable $t$ at the point $(0, x)$ under $x \in [-1, 1]$.

By the supposition made in the theorem, the kernel $K(t, x)$ is representable in the following form $K(t, x) = \sum_{k=0}^{p-1} c_k(x)t^k + t^p \widetilde{K}_1(t, x)$, where $c_k(x)$ and $\widetilde{K}_1(t, x)$ are continuous functions. Using the previous, we find:

$$(K'\psi)(x) = \sum_{k=0}^{p-1} c_k(x) \int_{-1}^{1} \frac{z(t)}{t^{p-k}}dt + \int_{-1}^{1} z(t)\widetilde{K}_1(t,\ x)dt$$

$$+ \sum_{j=0}^{p-1}(-1)^j \beta_j K_1^{\{j\}}(0,\ x).$$

There, the first and the third terms are finite-dimensional operators, and the second term - completely continuous from $C[-1, 1]$ into $C[-1, 1]$. The theorem is proved.

**(C) The main operator *A***

Consider the integral operator defined by the formula (1) as operator acting from $C^1_{-1}[-1, 1]$ into $C_0^{\{p\}}[-1, 1]$ with the supposition that the kernel $K(t, x)$ satisfy the condition (19).

**Theorem 3.3.** *The operator* $A : C^1_{-1}[-1, 1] \to C_0^{\{p\}}[-1, 1]$ *defined by the equality* (1) *is noetherian with the index* $\chi(A) = -p$.

**Proof.** The affirmation of Theorem 3.3 easily can be deduced with respect to Lemma 3.1, to Theorem 3.1, and a well-known fact on the conservation of the index under perturbation of a noetherian operator by a completely continuous operator. For this, it is sufficient to remark that $A = L + K$.

Besides the operator $A$ defined by the formula (1), now consider the associated operator $A'$ given in the following way:

$$(A'\psi)(x) = -(x^p\psi)' + \int_{-1}^{1} K(t, x)\psi(t)dt, \; x \in [-1, 1]. \tag{23}$$

Supposing previously that the kernel $K(t, x)$ satisfying the condition $(*)$. Similarly to the Theorem 3.3, we can prove the following theorem: Namely, we state.

**Theorem 3.4.** *The operator* $A' : P^1 \to C[-1, 1]$ *is noetherian with the index* $\chi(A') = p$.

Let us convince ourselves that the operators $A$ and $A'$ are associated operators.

**Lemma 3.4.** *The operators* $A : C^1_{-1}[-1, 1] \to C^{\{p\}}_0[-1, 1]$ *and* $A' : P^1 \to C[-1, 1]$ *verify the relationship* $(\varphi \in C^1_{-1}[-1, 1],\ \psi \in P^1)$:

$$\int_{-1}^{1} \psi(t)(A\varphi)(t)dt = \int_{-1}^{1} \psi(t)(A'\varphi)(t)dt. \tag{24}$$

**Proof.** For the proof of (24), it is sufficient from the definition of the space $P^1$ to represent $\psi(t)$ in the form $\psi(x) = x^{-p}z(x) + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x)$, where $z(x) \in C^{\{p\}}_0[-1, 1] \cap C^1_1[-1, 1]$ and $z(1) = 0$. And next with respect to the Lemma 2.1, compare the left and the right terms of the relationship (24).

As the operators $A$ and $A'$ are noetherian associated operators and $\chi(A) = -\chi(A')$, then based on Lemma 2.5 we obtain the main general assertion.

**Theorem 3.5.** *The equation* $A\varphi = f$, *where the operator A is defined by the formula* (1) *and the function* $f(x) \in C^{\{p\}}_0[-1, 1]$ *is solvable in the space* $C^1_{-1}[-1, 1]$ *if and only if*:

$$\int_{-1}^{1} f(t)\psi_k(t)dt = 0,\ k = 1, 2, \ldots, \alpha(A'),$$

*where* $\{\psi_k\}-$ *is the basis of the space of the solutions of the associated homogeneous equation* $A'\psi = 0$ *in the space* $P^1$.

**(D) Illustrative example**

Let us give an illustrative example by considering the following integro-differential operator defined by the next:

$$A\varphi(x) = x\varphi'(x) + \int_{-1}^{1} t\varphi(t)dt = f(x), \tag{25}$$

where $f(x) \in C_0^{\{1\}}[-1, 1]$ and we look for the solution $\varphi(x)$ from the space $C_{-1}^1[-1, 1]$. It is not difficult to verify that this equation has a unique solution $\varphi(x) = \int_{-1}^{x} (Nf)(t)dt,$ only and only if it is accomplished the following condition:

$$f(0) = \int_{-1}^{1} tdt \int_{-1}^{t} (Nf)(s)ds.$$

The last after interchanging the order of integration and simple calculation can be written in the following way:

$$\int_{-1}^{1} [(Nf)(s) - sf(s)]ds - 2f(0) = 0, \tag{26}$$

or in the more simple way

$$\left(f, \frac{1}{s} - s - 2\delta(s)\right) = 0.$$

It is not difficult to prove that the function of the form $\psi(x) = \frac{1}{x} - x - 2\delta(x)$ is the unique linear independent solution of the associated homogeneous equation $A'\psi = 0$ in the associated space $P^1$. Let us recall that in this case it is indicated to look for $\psi(t)$ in the form of $\psi(t) = \frac{z(t)}{t} + \beta\delta(t),$ where $z(t)) \in C_0^{\{1\}}[-1, 1] \cap C_1^1[-1, 1]$. Therefore, the condition (26) on the function $f(x)$ is the condition of the orthogonality to the unique nontrivial solution of the associated homogeneous equation $A'\psi = 0$.

## 4. Conclusion

This achieved scientific work presents in full detail the completed investigation of the establishment and the construction of noetherian theory for the integro-differential operator $A$, defined by a third kind integral equation depending of the parameter $p \in \mathbb{N}$, in the functional space of continuous functions $C^1_{-1}[-1, 1]$.

We firstly found the solvability condition of the equation (16) and this step lead us to determine the deficient numbers of the considered operator $L$ denoted $(\alpha, \beta)$ and, from that therefore is deduced the index $\varkappa(L)$, which is finite in all cases, making clearly the operator $L$ to be noetherian. At the same time we also studied the associated operator $L'$ in the functional spaces indicated, also showing that it is also noetherian. On the basis of the well-known fact on the conservation of the index under perturbation of a noetherian operator by a compact operator, we reach the needed result. At the end of this work, we illustrate the investigated noetherian property by a clear and concise example.

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