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TOTAL VALUATION RINGS IN $K(\mathbb{Q}, \sigma)$

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Abstract

Let V be a total valuation ring of a skew field K, $\mathbb{Q}$ be the additive group of the rational numbers, and $Aut(K)$ be the group of automorphisms of K. Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism, $K[\mathbb{Q}, \sigma]$ be the skew group ring of $\mathbb{Q}$ over K, and $K(\mathbb{Q}, \sigma)$ be its quotient ring. Let $R_0 = \{\sum a_{r_i} x^{r_i} | a_{r_i} \in K, r_i \in \mathbb{Q}, r_i \geq 0\}$ and $P = \cup_{r>0} x^r R_0$. Consider the natural map φ from R_{0_P} to K and set $\widetilde{V} = \varphi^{-1}(V)$. It is shown that $\widetilde{V}$ is a total valuation ring of $K(\mathbb{Q}, \sigma)$

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and it is characterized by x^r and V. If V is an invariant valuation ring, σ is classified into three types, in order to study the structure of $\Gamma_{\widetilde{V}}$ (the value group of $\widetilde{V}$).

1. Introduction

Let K be a skew field and V be a total valuation ring of K. We assume that $V \neq K$ throughout this paper. Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism and $K[\mathbb{Q}, \sigma]$ be the skew group ring of $\mathbb{Q}$ over K. In [7], it was proved that $K[\mathbb{Q}, \sigma]$ had a quotient skew field $K(\mathbb{Q}, \sigma)$. In [1], the authors initiated an investigation of total valuation rings $K(x, \sigma)$. It was shown that there were at least two total valuation rings in $K(x, \sigma)$. In [2], [5], [8] and [9], extensions of V in $K(x, \sigma)$ had been studied.

Let Q be a simple Artinian ring and σ be an automorphism of Q. Let $Q[x, \sigma]$ be the skew polynomial ring over Q in an indeterminate x. Then $Q[x, \sigma]$ has a quotient ring $Q(x, \sigma)$. In [6], the authors studied the noncommutative valuation rings in $Q(x, \sigma)$. In the case Q is a skew field and V is an invariant valuation ring, in order to study the structure of $\Gamma_{\widetilde{V}}$, σ is classified into five types.

In this paper, we will construct the total valuation ring $\widetilde{V}$ in $K(\mathbb{Q}, \sigma)$. In [10], the authors give a complete classification of graded extensions in $K(\mathbb{Q}, \sigma)$.

In [4], Wadsworth defined the Γ_R of a Dubrovin valuation ring R of a simple Artinian ring Q as follows; let $st(R) = \{q \in U(Q) | qRq^{-1} = R\}$, the stabilizer of R under the action of $U(Q)$ and $\Gamma_R = st(R)/U(R)$. If R is invariant in a skew field Q, then Γ_R coincides with the usual value group of R.

Let V be an invariant valuation ring of a skew field K, we classify σ into three types and give the complete structure of $st(\widetilde{V})$.

2. Preliminaries

In this section, we collect some notations, definitions and known results. Let R be a ring, we denote the Jacobson radical of R by $J(R)$ and the units of R by $U(R)$. Set $\mathbb{Q}^+ = \{r \in \mathbb{Q} | r > 0\}$.

Definition 2.1 ([3]). Let K be a skew field with subring V, for any $a \in K$, if $a \notin V$ implies $a^{-1} \in V$, then V is called a total valuation ring of K.

Definition 2.2 ([3]). Let V be a total valuation ring of a skew field K, we say that V is invariant if $kVk^{-1} = V$ for any non-zero $k \in K$.

Definition 2.3 ([3]). Let V be an invariant valuation ring of a skew field K, $\Gamma_V = U(K)/U(V)$ is called the value group of V.

Definition 2.4 ([4]). Let V be a total valuation ring of a skew field K. $st(V) = \{k \in U(K) | kVk^{-1} = V\}$ is called the stabilizer of V under the action of $U(K)$.

Set $R_0 = \{\sum a_{r_i} x^{r_i} | a_{r_i} \in K, r_i \in \mathbb{Q}, r_i \geq 0\}$. We can easily get the following lemmas.

Lemma 2.5. *For any* $f_1, f_2, \cdots, f_l \in R_0$, *there exists a natural number m, such that* $f_1, f_2, \cdots, f_l \in K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]$.

Lemma 2.6. R_0 *is an Ore ring and* $K(\mathbb{Q}, \sigma)$ *is its quotient skew field.*

3. Construction of $\widetilde{V}$

Let K be a skew field and $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism. Let $R_0 = \{\sum a_r x^r | a_r \in K, r \in \mathbb{Q}, r \geq 0\}$. For any $a \in K$, $r \in \mathbb{Q}^+$, $x^r a = \sigma(r)(a)x^r$, by Lemma 2.6, R_0 has a quotient ring $K(\mathbb{Q}, \sigma)$.

Let $P = \bigcup_{r>0} x^r R_0$. Then P is a maximal ideal of R_0. We can easily prove that P is localizable. Let $T = R_{0_P} = \{fg^{-1} | f \in R_0, g \in R_0 \backslash P\}$ be the localization of R_0 at P. Then T is a total valuation ring of $K(\mathbb{Q}, \sigma)$ with $J(T) = \bigcup_{r>0} x^r T$.

For any $\alpha = fg^{-1} \in T$, where $f = a_0 + a_{r_1} x^{r_1} + \cdots + a_{r_n} x^{r_n}$, $g = c_0 + c_{s_1} x^{s_1} + \cdots + c_{s_m} x^{s_m}$ with $c_0 \neq 0$. We denote the map

$$\varphi : T \to K$$

by $\varphi(\alpha) = a_0 c_0^{-1}$. We start with the following lemmas:

Lemma 3.1. *With the notations above,* φ *is a ring epimorphism with* $\ker \varphi = \bigcup_{r>0} x^r T$.

Proof. By using the Ore condition for $R_0 \backslash P$, it is easy to see that φ is well defined and a ring homomorphism. It is also clear that φ is an epimorphism and $\ker \varphi = \bigcup_{r>0} x^r T$ by the definition of φ.

Let V be a total valuation ring of K. Set $\widetilde{V} = \varphi^{-1}(V) = V + J(T)$, the complete inverse image of V by φ.

Lemma 3.2. $\widetilde{V}$ *is a total valuation ring of* $K(\mathbb{Q}, \sigma)$.

Proof. For any non-zero $\alpha \in K(\mathbb{Q}, \sigma)$, let $\alpha = fg^{-1}$, $f, g \in R_0$. We can write $f = f_1 x^r$ and $g = g_1 x^s$ with $f_1, g_1 \in R_0 \setminus P$ for some $r, s \geq 0$. So $fg^{-1} = f_1 x^{r-s} g_1^{-1}$. If $r - s > 0$, then $\alpha \in J(T) \subseteq \widetilde{V}$. If $r - s < 0$, then $\alpha^{-1} \in J(T) \subseteq \widetilde{V}$. Let $r - s = 0$ and $\varphi(\alpha) = a$. If $a \in V$, then $\alpha \in \widetilde{V}$. If $a \notin V$, then $a^{-1} \in V$, $\varphi(\alpha^{-1}) = a^{-1} \in V$, so $\alpha^{-1} \in \widetilde{V}$. Hence $\widetilde{V}$ is a total valuation ring of $K(\mathbb{Q}, \sigma)$.

Set $V^* = V \setminus \{0\}$. Then we have:

Lemma 3.3. *V^* is an Ore set of $\widetilde{V}$ and* $T = \widetilde{V}_{V^*} = \{\alpha c^{-1} \mid \alpha \in \widetilde{V}, c \in V^*\}$.

Proof. For any $\alpha \in T$, there exists $c \in V^*$ with $\varphi(\alpha)c \in V$. Then $\varphi(\alpha c) = \varphi(\alpha) c \in V$, $\alpha c \in \widetilde{V}$. Set $\beta = \alpha c$, $\alpha = \beta c^{-1}$. For any $\gamma \in \widetilde{V}$, $d \in V^*$, $d^{-1}\gamma \in T$. Hence there exist $c \in V^*$ and $\beta \in \widetilde{V}$ with $d^{-1}\gamma = \beta c^{-1}$, i.e., $\gamma c = d\beta$. Therefore, V^* is a right Ore set of $\widetilde{V}$. Similarly, we can prove that V^* is a left Ore set of $\widetilde{V}$. Now it is clear that $T = \widetilde{V}_{V^*}$

Now we consider the Jacobson radical of $\widetilde{V}$ and its residue skew field.

Lemma 3.4. $J(\widetilde{V}) = J(V) + J(T)$ *and* $\widetilde{V}/J(\widetilde{V}) \cong V/J(V)$.

Proof. Let I be a maximal right ideal of $\widetilde{V}$. Then $I \supseteq J(T)$. Furthermore, $\widetilde{V}/J(T) \cong V$. Hence $J(\widetilde{V}) = J(V) + J(T)$ and $\widetilde{V}/J(\widetilde{V}) \cong V/J(V)$.

Since $x^r a = \sigma(r)(a)x^r$ for any $a \in K$, $r \in \mathbb{Q}$, $\sigma(r)$ naturally extends to an automorphism of $K(\mathbb{Q}, \sigma)$ for any $r \in \mathbb{Q}$, which is the conjugation by x^r. We denote it by the same symbol $\sigma(r)$.

Lemma 3.5. *Let* $r \in \mathbb{Q}^+$, V *be a total valuation ring of K. Then the following are equivalent*:

(1) $\sigma(r)(\widetilde{V}) = \widetilde{V}$;

(2) $x^r \in st(\widetilde{V})$;

(3) $\sigma(r)(V) = V$.

Proof.

(1) $\Leftrightarrow$ (2): This is clear from the definition of $\sigma(r)$.

(1) $\Rightarrow$ (3): $\sigma(r)(V) = \sigma(r)(\widetilde{V} \cap K) = \sigma(r)(\widetilde{V}) \cap \sigma(r)(K) = \widetilde{V} \cap K = V$.

(3) $\Rightarrow$ (1): Obviously, $\sigma(r)(J(T)) = J(T)$, $\sigma(r)(\widetilde{V}) = \sigma(r)(V + J(T)) = \sigma(r)(V) + \sigma(r)(J(T)) = V + J(T) = \widetilde{V}$.

Lemma 3.6. *Let V be a total valuation ring of K. Then*

(1) $st(V) = st(\widetilde{V}) \cap K$;

(2) *Let* $\alpha = fg^{-1}$ *be any non-zero element in* $K(\mathbb{Q}, \sigma)$ *with* $f = a_0 + a_{r_1}x^{r_1} + \cdots + a_{r_n}x^{r_n} \in R_0$ *and* $g = c_0 + c_{s_1}x^{s_1} + \cdots + c_{s_m}x^{s_m} \in R_0$, $a_0 \neq 0$, $c_0 \neq 0$. *Then* $\alpha\widetilde{V} = a_0c_0^{-1}\widetilde{V}$ *and* $\widetilde{V}a_0c_0^{-1} = \widetilde{V}\alpha$. *In particular,* $\alpha \in st(\widetilde{V})$ *if and only if* $a_0c_0^{-1} \in st(V)$.

Proof. (1) We note that $\widetilde{V} = V + J(T)$ and $kJ(T) = J(T)$ for any $0 \neq k \in K$. Then $st(V) \subseteq st(\widetilde{V}) \cap K$. Let $k \in st(\widetilde{V}) \cap K$. Then

$k(V + J(T)) = (V + J(T))k$. Hence $kV = Vk$, $k \in st(V)$. Therefore $st(V) = st(\widetilde{V}) \cap K$.

(2) Since $\varphi(\alpha) = a_0 c_0^{-1}$, $\alpha - a_0 c_0^{-1} \in J(T)$ and so $c_0 a_0^{-1} a - 1 \in J(T) \subseteq J(\widetilde{V})$ by Lemma 3.4. Therefore $c_0 a_0^{-1} \alpha \in U(\widetilde{V})$ and thus $\alpha \widetilde{V} = a_0 c_0^{-1} \widetilde{V}$. Similarly, we can get $\widetilde{V}\alpha = \widetilde{V} a_0 c_0^{-1}$.

Next we consider the case where V is an invariant valuation ring.

Proposition 3.7. *Let V be an invariant valuation ring of K. Then $\widetilde{V}$ is an invariant valuation ring of $K(\mathbb{Q}, \sigma)$ if and only if $\sigma(r)(V) = V$ for any $r \in \mathbb{Q}^+$.*

Proof. Let $P = \cup_{r>0} x^r R_0$ and $S = R_0 \setminus P$. For any $0 \neq \alpha \in K(\mathbb{Q}, \sigma)$, $\alpha = x^r f g^{-1}$ for some $r \in \mathbb{Q}$ and $f, g \in S$. Hence the statement follows from Lemmas 3.2, 3.5, and 3.6.

The valuation ring $\widetilde{V}$ can be characterized as a total valuation ring R with one of the equivalent properties.

Theorem 3.8. *Let V be a total valuation ring of K and R be a total valuation ring of $K(\mathbb{Q}, \sigma)$ with $R \cap K = V$. Then the following are equivalent*:

(1) $R = \widetilde{V}$;

(2) $x^r a \in J(R)$ *for any* $a \in K$ *and* $r \in \mathbb{Q}^+$;

(3) $1 + x^{r_1} a_{r_1} + \cdots + x^{r_n} a_{r_n} \in U(R)$ *for any* $r_i \in \mathbb{Q}^+$, $a_{r_i} \in K$.

Proof.

(1) $\Rightarrow$ (2): For any $a \in K$, $r \in \mathbb{Q}^+$, $x^r a \in J(T) \subseteq J(\widetilde{V})$.

(2) $\Rightarrow$ (3): $x^{r_i}a_{r_i} \in J(R)$, then $x^{r_1}a_{r_1} + \cdots + x^{r_n}a_{r_n} \in J(R)$. Hence $1 + x^{r_1}a_{r_1} + \cdots + x^{r_n}a_{r_n} \in U(R)$.

(3) $\Rightarrow$ (1): For any $a \in K$, $r \in \mathbb{Q}^+$, $x^r a = (1 + x^r a) - 1 \in R$. We shall prove $\bigcup_{r>0} x^r T \subseteq R$. Let $x^s fg^{-1} \in \bigcup_{r>0} x^r T$, where $f = a_0 + x^{r_1}a_{r_1} + \cdots + x^{r_n}a_{r_n} \in R_0$, $g = c_0 + x^{t_1}c_{t_1} + \cdots + x^{t_m}c_{t_m}$ with $c_0 \neq 0$, $s \in \mathbb{Q}^+$. So there is $g_1 \in R_0$ such that $g = g_1 c_0$ and $g_1 \in U(R)$ by the assumption. Furthermore, there is a non-zero element $d \in V$ such that the constant coefficient of dfc_0^{-1} belongs to V, so $dfc_0^{-1} \in R$. Hence, it follows that $x^s fg^{-1} = (x^s d^{-1})(dfc_0^{-1})g_1^{-1} \in R$. Therefore, we proved $\bigcup_{r>0} x^r T \subseteq R$ and so $\widetilde{V} \subseteq R$. Hence T and R are both $(\widetilde{V}, \widetilde{V})$-bimodules, which implies that either $T \supsetneq R$ or $R \supseteq T$. The latter case shows that $V = R \cap K \supseteq T \cap K = K$, a contradiction. Thus we have $T \supsetneq R$. Assume that $R \supsetneq \widetilde{V}$. Then $K \supseteq \varphi(R) \supsetneq \varphi(\widetilde{V}) = V$, since $\widetilde{V} = \varphi^{-1}(V)$. Let $k \in K \setminus V$ with $\varphi(\alpha) = k$ for some $\alpha \in R$. Then $0 = \varphi(\alpha) - k = \varphi(\alpha - k)$ implies $\alpha - k \in \ker \varphi = \bigcup_{r>0} x^r T \subseteq R$. Thus $k \in R \cap K = V$, a contradiction. Hence $R = \widetilde{V}$.

Corollary 3.9. *There are no total valuation rings R of $K(\mathbb{Q}, \sigma)$ with $R \cap K = V$ and either $R \supsetneq \widetilde{V}$ or $\widetilde{V} \supsetneq R$.*

Proof. First assume that $R \cap K = V$. If $\widetilde{V} \supseteq R$, then $J(R) \supseteq J(\widetilde{V}) \supseteq J(T)$. So $R \supseteq V + J(T) = \widetilde{V}$ and hence $R = \widetilde{V}$. If $R \supseteq \widetilde{V}$, then as in the proof of Theorem 3.8, we have $T \supsetneq R$, then $J(R) \supseteq J(T) = \bigcup_{r>0} x^r T \supseteq x^r K$ for any $r \in \mathbb{Q}^+$. Hence $R = \widetilde{V}$ by Theorem 3.8.

Let V be an invariant valuation ring of K and $k \in U(K)$. We write $\overline{k}$ for the image of k in $\Gamma_V = U(K)/U(V)$. For any $\alpha \in st(\widetilde{V}) = \{\alpha \in U$

$(K(\mathbb{Q}, \sigma))|\alpha\widetilde{V}\alpha^{-1} = \widetilde{V}\}$. Let $\overline{\alpha}$ denote the image of α in $\Gamma_{\widetilde{V}} = st(\widetilde{V})/U(\widetilde{V})$. Let $P = \cup_{r>0} x^r R_0$, $S = R \setminus P$. Also, let $A = \{fg^{-1} | f \in S, \quad g \in S\}$. Note that $U(K) \subseteq A$ and $U(K(\mathbb{Q}, \sigma)) = \cup_{r \in \mathbb{Q}} x^r A$. Let $M = \{r \in \mathbb{Q} | \sigma(r)(V) = V\}$. Obviously, M is an additive subgroup of $\mathbb{Q}$. Using these notations, we have:

Theorem 3.10. *Let V be an invariant valuation ring of a skew field K, $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism.*

(1) *If $M = \mathbb{Q}$, then $st(\widetilde{V}) = U(K(\mathbb{Q}, \sigma))$ and $\Gamma_{\widetilde{V}} = \cup_{r \in \mathbb{Q}} \{\overline{k}\overline{x^r} | k \in U(K)\}$ with $\overline{x^r}\overline{k} = \overline{\sigma(r)(k)x^r}$.*

(2) *If $M = \{0\}$, then $st(\widetilde{V}) = A$ and $\Gamma_{\widetilde{V}} = \Gamma_V$.*

(3) *If $M \neq \{0\}$ and $M \neq \mathbb{Q}$, then $st(\widetilde{V}) = \cup_{r \in M} x^r A$ and $\Gamma_{\widetilde{V}} = \cup_{r \in M} \{\overline{k}\overline{x^r} | k \in U(K)\}$.*

Proof. We note that $A \subseteq st(\widetilde{V})$ by Lemma 3.6, since V is an invariant.

(1) Since $M = \mathbb{Q}$, $\widetilde{V}$ is invariant by Proposition 3.7. Hence $st(\widetilde{V}) = U(K(\mathbb{Q}, \sigma))$ and so $\Gamma_{\widetilde{V}} = \cup_{r \in \mathbb{Q}} \{\overline{k}\overline{x^r} | k \in U(K)\}$ by Lemma 3.6.

(2) Suppose that $x^r \widetilde{V} x^{-r} = \widetilde{V}$ for some $r \neq 0$. Then $\sigma(r)(V) = \sigma(r)(\widetilde{V} \cap K) = \sigma(r)(\widetilde{V}) \cap \sigma(r)(K) = \widetilde{V} \cap K = V$, a contradiction. So $x^r \notin st(\widetilde{V})$ for any $r \neq 0$. Let $\alpha = x^r fg^{-1} \in U(K(\mathbb{Q}, \sigma)) \setminus A$ with $f \in S$, $g \in S$. Suppose that $\alpha\widetilde{V} = \widetilde{V}\alpha$, then $x^r \widetilde{V} = \widetilde{V} x^r$ by Lemma 3.6, a contradiction. Hence $st(\widetilde{V}) = A$ and so $\Gamma_{\widetilde{V}} = \Gamma_V$ by Lemma 3.6.

Similarly, we can prove (3) as in (2).

Let $r \neq 0$, for any $k \in U(K)$, $\bar{k} = \overline{\sigma(r)(k)}$ if and only if $kV = \sigma(r)(k)V$. So the following corollary can be obtained by Theorem 3.10, which shows the conditions for $\Gamma_{\widetilde{V}}$ to be abelian.

Corollary 3.11. *Let V be an invariant valuation ring of a skew field K, $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism.*

(1) *Suppose that $M = \mathbb{Q}$. Then $\Gamma_{\widetilde{V}}$ is abelian if and only if Γ_V is abelian and $kV = \sigma(r)(k)V$ for every $k \in K$ and $r \in \mathbb{Q}$.*

(2) *Let $M \neq \{0\}$ and $M \neq \mathbb{Q}$. Then $\Gamma_{\widetilde{V}}$ is abelian if and only if Γ_V is abelian and $kV = \sigma(r)(k)V$ for any $k \in K$ and $r \in M$.*

We end this paper with one example.

Example. Let $K = F(y_r | r \in \mathbb{Q})$ be the rational function field over a field F in indeterminates $y_r (r \in \mathbb{Q})$. Group homomorphism $\sigma : \mathbb{Q} \to Aut(K)$ is determined by the following: for any $r \in \mathbb{Q}$, $\sigma(r)(a) = a$ for all $a \in F$, $\sigma(r)(y_s) = y_{r+s}$ for any y_s.

(1) Let $G_1 = \mathbb{Z}^{(\mathbb{Q})}$, which is a totally ordered abelian group by lexicographical ordering. We define a valuation v_1 of K as follows: $v_1(a) = 0$ for any non-zero $a \in F$ and for any non-zero homogeneous element $\alpha = y_{r_1}^{s_1} y_{r_2}^{s_2} \cdots y_{r_n}^{s_n} (r_1 < r_2 < \cdots < r_n)$, $v_1(\alpha) = (s_k)_{k \in \mathbb{Z}}$ with the r_j component of $v_1(\alpha)$ is $s_j (1 \leq j \leq n)$ and the other components of it are all zeroes. Let $\beta = \beta_1 + \beta_2 + \cdots + \beta_m$ be any element in $F[y_r | r \in \mathbb{Q}]$, where β_i are non-zero homogeneous elements, we define $v_1(\beta) = \min\{v_1(\beta_1) | 1 \leq i \leq m\}$. As usual, we can extend the map v_1 to $K \setminus \{0\}$, which is a valuation of K. Let V_1 be the valuation ring of K determined by v_1. Since $\sigma(t)$ is just shifting and for any $\alpha\beta^{-1} \in K$,

$V_1\alpha\beta^{-1} = V_1 y_{r_1}^{s_1} y_{r_2}^{s_2} \cdots y_{r_n}^{s_n}$ for some $r_1, r_2, \cdots, r_n \in \mathbb{Q}, s_1, s_2, \cdots, s_n \in \mathbb{Z}$. Hence $\sigma(t)(V_1) = V_1$ for any $t \in \mathbb{Q}$. Then $M = \mathbb{Q}$. $\widetilde{V_1}$ is invariant by Theorem 3.10. $\sigma(1)(y_0)V_1 = y_1V_1 \neq y_0V_1$, so $\Gamma_{\widetilde{V_1}}$ is not abelian.

(2) Let $G = \mathbb{Z}$. A valuation v_2 of K is determined by the following: $v_2(a) = 0$ for any non-zero $a \in F$, $v_2(y_r) = 1$ for any $r \in \mathbb{Q}$. Let V_2 be the valuation ring of K determined by v_2. Then, it is easily seen that $M = \mathbb{Q}$ and $\sigma(r)(k)V_2 = kV_2$ for any $k \in K$. Hence $\widetilde{V_2}$ is invariant and $\Gamma_{\widetilde{V_2}}$ is abelian by Theorem 3.10 and Corollary 3.11.

(3) Let $G = \mathbb{Z}$. A valuation v_3 of K is determined by the following: $v_3(a) = 0$ for any non-zero $a \in F$, $v_3(y_r) = 0$ for any $r \neq 0$, $v_3(y_0) = 1$. Let V_3 be the valuation ring of K determined by v_3. For any $r \neq 0$, $\sigma(r)(y_0) = y_r$, $\sigma(r)(y_{-r}) = y_0$. Hence $M = \{0\}$, $st(\widetilde{V_3}) = A$ and $\Gamma_{\widetilde{V_3}} = \Gamma_V$.

(4) Let $G = \mathbb{Z}$. A valuation v_4 of K is determined by the following: $v_4(a) = 0$ for any non-zero $a \in F$, $v_4(y_r) = 0$ for any $r \notin \mathbb{Z}$, $v_4(y_n) = 1$ for any $n \in \mathbb{Z}$. Let V_4 be the valuation ring of K determined by v_4. For any $r \notin \mathbb{Z}$, $\sigma(r)(y_0) = y_r$, which implies that $r \notin M$. For any $n \in \mathbb{Z}$, $\sigma(n)(y_s) = y_{n+s}$, which implies that $\sigma(n)(V_4) = V_4$. Hence $M = \mathbb{Z}$. For any $n \in \mathbb{Z}$, $\sigma(n)(k)V_4 = kV_4$. Therefore, $\Gamma_{\widetilde{V_4}}$ is abelian.

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