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EXTENSIONS OF TOTAL VALUATION RINGS IN $K(\mathbb{Q}, \sigma)$

GUANGMING XIE, ZIFEN HONG and CHUNXIA LIU

School of Mathematics and Statistics
Guangxi Normal University
Guilin, 541006
P. R. China
e-mail: gmxie@mailbox.gxnu.edu.cn
1005124478@qq.com
2508606349@qq.com

Abstract

Let V be a total valuation ring of a skew field K, $\mathbb{Q}$ be the addictive group of the rational numbers, and $Aut(K)$ be the group of automorphisms of K. Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism, $K[\mathbb{Q}, \sigma]$ be the skew group ring of $\mathbb{Q}$ over K and $K(\mathbb{Q}, \sigma)$ be its quotient ring. We consider extensions of V in $K(\mathbb{Q}, \sigma)$. Set $\hat{R} = \{a_0 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} \,|\, r_i \in \mathbb{Q}^+, a_0 \in V, a_{r_i} \in K\}$ and

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$\hat{S} = \{1 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} \mid r_i \in \mathbb{Q}^+, a_{r_i} \in K\}$. It is shown that $\hat{S}$ is an Ore system in $\hat{R}$ and $R_{(1)} = \hat{S}^{-1}\hat{R}$ is an extension of V in $K(\mathbb{Q}, \sigma)$. Similarly, we can get $R_{(-1)}$, an extension of V in $K(\mathbb{Q}, \sigma)$. Let σ be compatible with V. Set $R = \{\sum a_r x^r \mid r \geq 0, a_r \in V$ for any $r\}$ and $S = \{\sum a_r x^r \mid r \geq 0, a_r \in V$ and at least one $a_r \in U(V)\}$. It is shown that S is an Ore system in R and $S^{-1}R$ is an extension of V in $K(\mathbb{Q}, \sigma)$.

1. Introduction

Let K be a skew field and V be a total valuation ring of K. We assume that $V \neq K$ throughout this paper. Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism and $K[\mathbb{Q}, \sigma]$ be the skew group ring of $\mathbb{Q}$ over K. In [6], it was proved that $K[\mathbb{Q}, \sigma]$ had a quotient skew field $K(\mathbb{Q}, \sigma)$. Let σ be a monomorphism of K. In [2], the authors considered the extensions of V in $K(x; \sigma, \delta)$. In [1], extensions of V in $K(x, \sigma)$ have been studied. Also, total valuation rings in Ore extensions or in skew polynomial rings have been studied in [3]. Let σ be an automorphism of K. The structure of graded extensions of V was studied in [5], [7], and [8]. $\mathbb{Q}$ is the simplest divisible group. It seems interesting to study the extensions of V in $K(\mathbb{Q}, \sigma)$. In [9], the authors studied the graded extensions of V in $K[\mathbb{Q}, \sigma]$. The aim of this paper is to study the extensions of V in $K(\mathbb{Q}, \sigma)$.

2. Preliminaries

In this section, we collect some notations, definitions and known results.

Definition 2.1 ([2]). Let V be a subring of a skew field K. If for any non-zero $k \in K$, either $k \in V$ or $k^{-1} \in V$, then V is called a total valuation ring of K.

Definition 2.2 ([1]). Let V be a total valuation ring of a skew field K. Let F be a skew field containing K and R be a total valuation ring of F. If $R \cap K = V$, then R is called an extension of V in F.

Definition 2.3 ([4]). Let R be a ring with no divisor and S be a multiplicatively closed subset of R. If for any $a \in R, c \in S$, there exist $b \in R, d \in S$ such that $da = bc$, then S is called a left Ore system. Similarly, we can define a right Ore system. If S is both left Ore system and right Ore system, then S is called an Ore system.

Theorem 2.4 ([4]). *S is a left (right) Ore system if and only if the left (right) quotient ring $S^{-1}R(RS^{-1})$ exists.*

Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism. Then $K[\mathbb{Q}, \sigma] = \{\sum a_{r_i}x^{r_i} | a_{r_i} \in K, r_i \in \mathbb{Q}\}$ with $x^{r_i}a = \sigma(r_i)(a)x^{r_i}$ for any $a \in K$. Let R be a ring. We denote the Jacobson radical of R by $J(R)$ and the units of R by $U(R)$. Set

$$\hat{R} = \{a_0 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} | r_i > 0, a_0 \in V, a_{r_i} \in K \text{ for any } i\},$$

and

$$\hat{S} = \{1 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} | r_i > 0, a_{r_i} \in K \text{ for any } i\}.$$

We can easily get the following lemma by [1].

Lemma 2.5 ([1]). *For any $t > 0$, set $\hat{R}_t = \{a_0 + a_1x^t + \cdots + a_nx^{nt} | a_0 \in V, a_i \in K$ for any $i\}$ and $\hat{S}_t = \{1 + a_1x^t + \cdots + a_nx^{nt} | a_i \in K$ for any $i\}$. Then $\hat{S}_t$ is an Ore system in $\hat{R}_t$ and $\hat{S}_t^{-1}\hat{R}_t$ is a total valuation ring of $K(x^t, \sigma(t))$.*

Definition 2.6. Let K be a skew field and V be a total valuation ring of K. Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism. We say that σ is compatible with V if for any $r \in \mathbb{Q}$ and $a \in K$, $\sigma(r)(a) \in V$ if and only if $a \in V$.

Lemma 2.7 ([2]). *For any $t > 0$, set $R_t = \{\sum_{i=0}^{n} a_i x^{ti} | a_i \in V$ for any $i\}$ and $S_t = \{\sum_{i=0}^{n} a_i x^{ti} | a_i \in V$ and at least one $a_j \in U(V)\}$. Assume that σ is compatible with V. Then S_t is an Ore system in R_t and $S_t^{-1}\hat{R}_t$ is a total valuation ring of $K(x^t, \sigma(t))$.*

It is easy to get the following lemma.

Lemma 2.8. *For any $f_1, f_2, \cdots, f_l \in R_0 = \{\sum_{i=1}^{n} a_{r_i} x^{r_i} | r_i \geq 0, a_{r_i} \in K\}$, there exists a natural number m such that $f_1, f_2, \cdots, f_l \in K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]$.*

Proof. Let $f_1 = a_{r_1} x^{r_1} + \cdots + a_{r_k} x^{r_k}$. Assume that $r_i = \frac{s_i}{n_i}$, $s_i \in \mathbb{Z}$, $n_i \in \mathbb{N}$ for any i. Let $m_1 = n_1 n_2 \cdots n_k$. Then $f_1 \in K[x^{\frac{1}{m_1}}, \sigma(\frac{1}{m_1})]$. Similarly, we can get $m_2, \cdots, m_l$, such that $f_i \in K[x^{\frac{1}{mi}}, \sigma(\frac{1}{m_i})]$, $i = 2, \cdots, l$. Set $m = m_1 m_2 \cdots m_l$. Then $f_i \in K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]$ for all i. □

The following lemma is obtained in [6], we give a proof for reader's convenience.

Lemma 2.9 ([6]). *R_0 is an Ore ring and $K(\mathbb{Q}, \sigma)$ is its quotient skew field.*

Proof. For any $f \in R_0$, $g \in R_0 \setminus \{0\}$, there exists an $m \in \mathbb{N}$ such that $f, g \in K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]$. Since $K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]$ is an Ore ring, there exist

$f_1, f_2 \in K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]$ and $g_1, g_2 \in K[x^{\frac{1}{m}}, \sigma(\frac{1}{m})]\setminus\{0\}$, such that $g_1 f = f_1 g$, $fg_2 = gf_2$. Hence R_0 is an Ore ring. If $\alpha = g^{-1}f \in K(\mathbb{Q}, \sigma)$, then $\alpha \in K(x^{\frac{1}{m}}, \sigma(\frac{1}{m}))$. Hence $K(\mathbb{Q}, \sigma)$ is the quotient skew field of R_0.

□

3. Extensions of V in $K(\mathbb{Q}, \sigma)$

Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism. In this section, we will study the extensions of V in $K(\mathbb{Q}, \sigma)$. Let $\mathbb{Q}^+ = \{r \in \mathbb{Q} | r > 0\}$ and $\mathbb{Q}^- = \{r \in \mathbb{Q} | r < 0\}$. Set $\hat{R} = \{a_0 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} | r_i \in \mathbb{Q}^+, a_0 \in V, a_{r_i} \in K$ for any $i\}$. It is a subring of $K[\mathbb{Q}, \sigma]$. Set $\hat{S} = \{1 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} | r_i \in \mathbb{Q}^+, a_{r_i} \in K$ for any $i\}$. We will show that $\hat{S}$ is an Ore system in $\hat{R}$ and $\hat{S}^{-1}\hat{R}$ is a total valuation ring of $K(\mathbb{Q}, \sigma)$.

Theorem 3.1. *Let V be a total valuation ring of a skew field K. Then V has at least the following two standard extensions $R_{(1)}$ and $R_{(-1)}$ in $K(\mathbb{Q}, \sigma)$. The valuation ring $R_{(1)}$ of $K(\mathbb{Q}, \sigma)$ with the property that $ax^r \in J(R_{(1)})$ for all $a \in K, r \in \mathbb{Q}^+$. The valuation ring $R_{(-1)}$ of $K(\mathbb{Q}, \sigma)$ with the property that $ax^r \in J(R_{(-1)})$ for all $a \in K, r \in \mathbb{Q}^-$.*

Proof. Set $\hat{S} = \{1 + a_{r_1}x^{r_1} + \cdots + a_{r_k}x^{r_k} | r_i \in \mathbb{Q}^+, a_{r_i} \in K$ for any $i\}$. It is trivial that $\hat{S}$ is a multiplicatively closed set. Let $f \in \hat{R}$ and $g \in \hat{S}$. Then there exists an $n \in \mathbb{N}$ such that $f, g \in K[x^{\frac{1}{n}}, \sigma(\frac{1}{n})]$ by Lemma 2.8. Let $\hat{R}_{\frac{1}{n}} = \{a_0 + a_1x^{\frac{1}{n}} + \cdots + a_s x^{\frac{s}{n}} | a_0 \in V, a_i \in K$ for any $i\}$, $\hat{S}_{\frac{1}{n}} =$

$\{1 + b_1 x^{\frac{1}{n}} + \cdots + b_l x^{\frac{l}{n}} | b_i \in K$ for any $i\}$. By Lemma 2.5, $\hat{S}_{\frac{1}{n}}$ is an Ore system in $\hat{R}_{\frac{1}{n}}$. Hence there exist $f_1, f_2 \in \hat{R}_{\frac{1}{n}}, g_1, g_2 \in \hat{S}_{\frac{1}{n}}$, such that $g_1 f = f_1 g$, $fg_2 = gf_2$, $\hat{R}_{\frac{1}{n}} \subseteq \hat{R}$ and $\hat{S}_{\frac{1}{n}} \subseteq \hat{S}$. Therefore $\hat{S}$ is an Ore system.

For any $\alpha \in K(\mathbb{Q}, \sigma)$, let $\alpha = g^{-1} f$, $f \in \hat{R}_0$, $g \in \hat{R}_0 \setminus \{0\}$. Then there exists an $n \in \mathbb{N}$ such that $f, g \in K[x^{\frac{1}{n}}, \sigma(\frac{1}{n})]$. Then $\alpha \in K(x^{\frac{1}{n}}, \sigma(\frac{1}{n}))$. By Lemma 2.5, $\hat{S}^{-1}_{\frac{1}{n}} \hat{R}_{\frac{1}{n}}$ is a total valuation ring of $K(x^{\frac{1}{n}}, \sigma(\frac{1}{n}))$, either $\alpha \in \hat{S}^{-1}_{\frac{1}{n}} \hat{R}_{\frac{1}{n}}$ or $\alpha^{-1} \in \hat{S}^{-1}_{\frac{1}{n}} \hat{R}_{\frac{1}{n}}$. We note that $\hat{S}^{-1}_{\frac{1}{n}} \hat{R}_{\frac{1}{n}} \subseteq \hat{S}^{-1} \hat{R}$. Hence $\hat{S}^{-1} \hat{R}$ is a valuation ring. If $\alpha \in \hat{S}^{-1} \hat{R} \cap K$, then $\alpha \in \hat{S}^{-1}_{\frac{1}{n}} \hat{R}_{\frac{1}{n}} \cap K = V$. Therefore, $\hat{S}^{-1} \hat{R}$ is an extension of V in $K(\mathbb{Q}, \sigma)$. We denote it by $R_{(1)}$. The construction of $R_{(1)}$ implies that $ax^r \in J(R_{(1)})$ for all $a \in K$ and $r \in \mathbb{Q}^+$.

Conversely, let R be any extension of V in $K(\mathbb{Q}, \sigma)$ with $ax^r \in J(R)$ for all $a \in K$ and $r \in \mathbb{Q}^+$. It follows that all expressions of the form $1 + a_{r_1} x^{r_1} + \cdots + a_{r_k} x^{r_k}$ with $r_i \in \mathbb{Q}^+$ are units in R. Since $\hat{R} \subseteq R$, $R_{(1)} \subseteq R$. For any non-zero element $\alpha = g^{-1} f \in K(\mathbb{Q}, \sigma)$, $f, g \in R_0 = \{\sum a_{r_i} x^{r_i} | a_{r_i} \in K, r_i \geq 0\}$. Let $f = a_{r_1} x^{r_1} + a_{r_2} x^{r_2} + \cdots + a_{r_t} x^{r_t}$ with $r_1 < r_2 < \cdots < r_t$ and $g = b_{s_1} x^{s_1} + b_{s_2} x^{s_2} + \cdots + b_{s_l} x^{s_l}$ with $s_1 < s_2 < \cdots < s_l$. We have the equation

$$f = a_{r_1} x^{r_1} (1 + \sigma(-r_1)(a_{r_1}^{-1} a_{r_2}) x^{r_2 - r_1} + \cdots + \sigma(-r_1)(a_{r_1}^{-1} a_{r_t}) x^{r_t - r_1})$$

$$= a_{r_1} x^{r_1} f_1,$$

and

$$g = b_{s_1} x^{s_1} (1 + \sigma(-s_1)(b_{s_1}^{-1} b_{s_2}) x^{s_2 - s_1} + \cdots + \sigma(-s_1)(b_{s_1}^{-1} b_{s_l}) x^{s_l - s_1})$$

$$= b_{s_1} x^{s_1} g_1,$$

with $f_1, g_1 \in \hat{S}$. Then $g^{-1} f = g_1^{-1} \sigma(-s_1)(b_{s_1}^{-1} a_{r_1}) x^{r_1 - s_1} f_1$. If $\alpha \in R$, then $r_1 - s_1 > 0$ or $r_1 = s_1$ and $\sigma(-s_1)(b_{s_1}^{-1} a_{r_l}) \in V$, i.e., $\alpha \in R_{(1)}$. Hence $R = R_{(1)}$.

Let $\hat{R}_{(-1)} = \{a_0 + a_{r_1} x^{r_1} + \cdots + a_{r_k} x^{r_k} | r_i \in \mathbb{Q}^-, a_0 \in V, a_{r_i} \in K$ for any $i\}$ and $\hat{S}_{(-1)} = \{1 + a_{r_1} x^{r_1} + \cdots + a_{r_k} x^{r_k} | r_i \in \mathbb{Q}^-, a_{r_i} \in K$ for any $i\}$. Similarly, we can get that $R_{(-1)} = \hat{S}_{(-1)}^{-1} \hat{R}_{(-1)}$ is an extension of V in $K(\mathbb{Q}, \sigma)$ with the property that $ax^r \in J(R_{(-1)})$ for all $a \in K$ and $r \in \mathbb{Q}^-$. □

Theorem 3.2 *Let σ be compatible with V. Then we have the following:*

(1) *Set $R = \{\sum a_r x^r | r \geq 0, a_r \in V\}$ and $S = \{\sum a_r x^r \in R |$ at least one $a_r \in U(V)\}$. S is an Ore system in R.*

(2) *$S^{-1}R$ is an extension of V in $K(\mathbb{Q}, \sigma)$.*

Proof. (1) For any $f \in R$ and $g \in S$, there is an $n \in \mathbb{N}$ with $f, g \in K[x^{\frac{1}{n}}, \sigma(\frac{1}{n})]$ by Lemma 2.8. Let $R_{\frac{1}{n}} = \{\sum_{i=0}^{l} a_i x^{\frac{i}{n}} | a_i \in V\}$ and

$S_{\frac{1}{n}} = \{\sum_{i=0}^{l} a_i x^{\frac{i}{n}} | a_i \in V$ and at least one $a_i \in U(V)\}$. By Lemma 2.7, $S_{\frac{1}{n}}$ is an Ore system in $R_{\frac{1}{n}}$. Then there exist $f_1, f_2 \in R_{\frac{1}{n}}$ and $g_1, g_2 \in S_{\frac{1}{n}}$ such that $g_1 f = f_1 g,\ fg_2 = gf_2$. Since $R_{\frac{1}{n}} \subseteq R,\ S_{\frac{1}{n}} \subseteq S$. Hence S is an Ore system in R.

(2) For any $\alpha = g^{-1} f \in K(\mathbb{Q}, \sigma)$ with $f \in R_0,\ g \in R_0 \setminus \{0\}$, there exists an $n \in \mathbb{N}$ with $f, g \in K[x^{\frac{1}{n}}, \sigma(\frac{1}{n})]$ by Lemma 2.8. Then $\alpha \in K(x^{\frac{1}{n}}, \sigma(\frac{1}{n}))$, Since $S_{\frac{1}{n}}^{-1} R_{\frac{1}{n}}$ is a total valuation ring by Lemma 2.7, either $\alpha \in S_{\frac{1}{n}}^{-1} R_{\frac{1}{n}}$ or $\alpha^{-1} \in S_{\frac{1}{n}}^{-1} R_{\frac{1}{n}}$. We note that $S_{\frac{1}{n}}^{-1} R_{\frac{1}{n}} \subseteq S^{-1} R$. Hence $S^{-1} R$ is a total valuation ring of $K(\mathbb{Q}, \sigma)$. If $\alpha \in S^{-1} R \cap K$, then $\alpha \in S_{\frac{1}{n}}^{-1} R_{\frac{1}{n}} \cap K = V$. Hence $S^{-1} R$ is an extension of V in $K(\mathbb{Q}, \sigma)$. □

We denote $S^{-1} R$ by $V_{(1)}$. Set

$$A^{+} = \{\sum a_r x^r | r \in \mathbb{Q},\ a_r \in J(V) \text{ if } r < 0 \text{ and } a_r \in V \text{ if } r \geq 0\},$$

and

$$S_{(1)}^{+} = \{\sum a_r x^r | r \in \mathbb{Q},\ a_r \in J(V) \text{ if } r < 0,\ a_0 \in U(V),\ a_r \in V \text{ if } r > 0\}.$$

Using the result of [7], similar to the proof of Theorem 3.2, we can prove that $S_{(1)}^{+}$ is an Ore system of A^{+} and $V_{(1)}^{+} = (S_{(1)}^{+})^{-1} A^{+}$ is an extension of V in $K(\mathbb{Q}, \sigma)$. Set

$$A^{-} = \{\sum a_r x^r | r \in \mathbb{Q},\ a_r \in J(V) \text{ if } r > 0 \text{ and } a_r \in V \ \textit{if}\ r \leq 0\},$$

and

$$S^{-}_{(1)} = \{\sum a_r x^r | r \in \mathbb{Q},\ a_r \in J(V) \text{ if } r > 0,\ a_0 \in U(V),\ a_r \in V \text{ if } r < 0\}.$$

Similarly, we can prove that $S^{-}_{(1)}$ is an Ore system of A^{-} and $V^{-}_{(1)} = (S^{-}_{(1)})^{-1}A^{-}$ is an extension of V in $K(\mathbb{Q}, \sigma)$. The above result can be found in [9].

Corollary 3.3 ([9])**.** *Let* σ *be compatible with V. With the above notations,* $V^{+}_{(1)}$ *and* $V^{-}_{(1)}$ *are extensions of V in* $K(\mathbb{Q}, \sigma)$. *Furthermore,* $V^{+}_{(1)} \subset V_{(1)}$ *and* $V^{-}_{(1)} \subset V_{(1)}$.

It follows from the next result that an extension R of V exists in $K(\mathbb{Q}, \sigma)$ that contains x^r for all $r \in \mathbb{Q}$ if and only if σ is compatible with V.

Theorem 3.4. *There exists an extension R of V in* $K(\mathbb{Q}, \sigma)$ *with* $x^r \in U(R)$ *for all* $r \in \mathbb{Q}$ *if and only if* σ *is compatible with V.*

Proof. Assume that R is an extension of V in $K(\mathbb{Q}, \sigma)$ with $x^r \in U(R)$ for all $r \in \mathbb{Q}$. We have $x^r a = \sigma(r)(a)x^r$ for all $a \in K$. Hence $a \in U(R)$ if and only if $\sigma(r)(a) \in U(R)$. Therefore, $a \in U(V)$ if and only if $\sigma(r)(a) \in U(V)$. It implies that σ is compatible with V. Conversely, if σ is compatible with V, then $S^{-1}R$ is an extension of V in $K(\mathbb{Q}, \sigma)$ with $x^r \in U(R)$ for all $r \in \mathbb{Q}$ by Theorem 3.2. □

4. Example

In this section, we will provide a concrete example of $K(\mathbb{Q}, \sigma)$ that σ is compatible with V.

Example Let $K = F(y_r | r \in \mathbb{Q})$ be the rational function field over a field F in indeterminates $y_r (r \in \mathbb{Q})$. Let $\sigma : \mathbb{Q} \to Aut(K)$ be a group homomorphism defined by the following; for any $r \in \mathbb{Q}$, $\sigma(r)(a) = a$ for all $a \in F$, $\sigma(r)(y_s) = y_{r+s}$ for any y_s. Let $G = \mathbb{Z}^{(\mathbb{Q})}$ which is a totally ordered abelian group by lexicographical ordering. We define a valuation v of K as follows: $v(a) = 0$ for any non-zero $a \in F$ and for any non-zero homogeneous element $\alpha = y_{r_1}^{s_1} y_{r_2}^{s_2} \cdots y_{r_n}^{s_n} (r_1 < r_2 < \cdots r_2 < r_n), v(\alpha) = (s_k)_{(k \in Z)}$, where $(s_k)_{(k \in \mathbb{Z})}$ is the element in G such that the r_j-component of $v(\alpha)$ is $s_j (1 \leq j \leq n)$ and other components of it are all zeroes. Let $\beta = \beta_1 + \beta_2 + \cdots + \beta_m$ be any element in $F[y_r | r \in \mathbb{Q}]$, where β_i are non-zero homogeneous elements, we define $v(\beta) = \min\{v(\beta_i) | 1 \leq i \leq m\}$. As usual, we can extend the map v to $K \setminus \{0\}$. Let V be the valuation ring determined by v. Since $\sigma(t)$ is just shifting and for any $\alpha\beta^{-1} \in K$, $V\alpha\beta^{-1} = Vy_{r_1}^{s_1} y_{r_2}^{s_2} \cdots y_{r_n}^{s_n}$ for some $r_1, r_2, \cdots, r_n \in \mathbb{Q}, s_1, s_2, \cdots, s_n \in \mathbb{Z}$. Hence σ is compatible with V. Therefore, $R_{(1)}, R_{(-1)}, V_{(1)}, V_{(1)}^{+}$, and $V_{(1)}^{-}$ are extensions of V in $K(\mathbb{Q}, \sigma)$.

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