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THE INVESTIGATION OF MULTISTEP MULTIDERIVATIVE METHODS WITH CONSTANT COEFFICIENTS AND ITS APPLICATION

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Abstract

Specialists have always dealt with approximate calculations. However, official recognition was right in Newton's era. For example, Newton-Cottes method for calculation of definite integrals. Followers of Newton, have constructed approximate methods for solving some mathematical problems. Leonard Euler has constructed his famous direct method for solving ODEs (Ordinary Differential Equations), which is refined and developed in the works of Adams-Moulton, Adams-Bashford, Cowell and others. All these methods are generalize in the results of which has constructed multistep methods with the constant coefficients. We have described here the chronological development of numerical methods. Noted that multistep methods with constant coefficients have investigated by many authors as Shura-Bura, Bakhalov, Dahlquist, Lambert, Ibrahimov, Henricy, Brunner, Imanova and etc. This method has generalized by some authors: Dahlquist, Enrite, Kobza, Ibrahimov and etc. In the works of named authors have shown that, if in the multistep methods participate the derivatives of the desired functions, then these methods will be more exact. By using this idea, some authors proposed to reduce the differential equation of the first order to a differential equation of any order and then use the appropriate method using high derivatives of desired solutions. By taking into account these many authors decided to use multistep multiderivative methods. Noted that these methods can be used in different form depending from the order of differential equations. In the construction of multistep methods in usually is used the presentation of the right-hand side of the given differential equation. Consequently, here for each of which it turns out numerous special cases, needed to construct and explore multistep multiderivative methods. Therefore, here consider to investigation of Multistep Multiderivative Methods (MMM) explicit, implicit and advanced types and their application for solving of initial value problem for ODEs and for the Volterra integro-differential equation and also application of the above mentioned methods for solving of the Volterra integral equation of the second kind.

1. Introduction

As is known, the initial value problem for the Ordinary Differential Equations (ODEs) in more general form can be presented as follows:

$$y^{(r)} = f(x,\ y,\ \delta_1,\ y;\ \ldots \delta_{r-1} y^{(r-1)}),$$

$$y_{(x_0)}^{(l)} = y_0^{(l)}\ (l = 0, 1, \ldots, r-1),\ x_0 \le x \le X. \qquad (1)$$

Here $\delta_j = 0$ or $1 (j = 1, 2, \dots, r-1)$.

Consequently if $r = 1$, then from (1) receive the following initial-value problem:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 \le x \le X, \tag{2}$$

which has been studied by many authors using the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \; n = 0, 1, \dots, N-k. \tag{3}$$

Here $0 < h$-is step size, N-amount of the subintervals and $Nh = X$. Suppose that the problem (1) has the unique solution, which has defined on the interval $[x_0, X]$. Noted that the problem (2) by using method (3) has been investigated by many others (see, for example, [1]-[16]). It is easy to verify that problem (1) is the generalization of the problem (2), which is investigated less than problem (2) (see, for example, [3]-[5], [7], [8]). For that would show how useful study problem (1), let's consider the following special case:

$$y''_{(x)} = f\left(x, y_{(x)}, y'_{(x)}\right), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \; x \in [x_0, X]. \tag{4}$$

As is known, one of the popular method for solving this problem can be presented as the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\delta_1 \sum_{i=0}^{k} \beta_i^{(1)} y'_{n+i}$$

$$+ h^2 \delta_2 \sum_{i=0}^{k} \beta_i^{(2)} f(x_{n+i}, y_{n+i}, \delta_1 y'_{n+i}) \; (n = 0, 1, \dots, N-k). \tag{5}$$

If $\delta_1 = 0$ and $\delta_2 = 1$, then from (5) is receiving following:

$$\sum_{i=0}^{k} \overline{\alpha}_i y_{n+i} = h^2 \sum_{i=0}^{k} \overline{\beta}_i f(x_{n+i}, y_{n+i}), \tag{6}$$

which is called as the general Störmer-Verlet method. And also it is known that if method (5) is stable, then that will be convergent. The conseption stability for the method (5) can defined as follows:

Definition 1. Method (5) is called as the stable, if the roots of the polynomial $\rho(\lambda) \equiv \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k-1} + \ldots + \alpha_1 \lambda + \alpha_k$ located in the unite circle on the boundary of which, there is not multiple root.

Given that, method (6) is the special case of the method (5), so one can be think that if the method (5) converges, then method (6) also converges. However, it does not. Let us recall the classic definition for the second order derivative for continuous function:

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x),$$

which can be presented as special case for the method (6) in the following form:

$$f(x+h) - 2f(x) + f(x-h) = h^2 f''(x).$$

From here receive that characteristic polynomial of the method (6), has a double root $\lambda = 1$.

Therefore method (6) is not stable in the above sense. Based on this, the conception of stability defined as following.

Definition 2. Method (6) is called stable, if the roots of the polynomial $\bar{\rho}(\lambda) \equiv \bar{\alpha}_k \lambda^k + \bar{\alpha}_{k-1}\lambda^{k-1} + \ldots + \bar{\alpha}_1 \lambda + \bar{\alpha}_0$ are located in the unite circle on the boundary of which, there is not multiple root except for the double root $\lambda = 1$. It follows from here that methods (5) and (6) are independent objects of research. Let us generalize the above using methods. Then we get:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{r-1} h^j \delta_j \sum_{i=0}^{k} \beta_i^{(1)} y_{n+i}^{j} + h^r \delta_r \sum_{i=0}^{k} \beta_i^{(r)} f_{n+i}. \tag{7}$$

This method is a generalization of the well-known linear multistep methods with constant coefficients, which are usually called as the multistep multiderivative methods (MMM) with constant coefficients. Let us consider the application of some partial cases of this method to solve some initial-value problem for ODEs.

2. About Some Applications of the Method (7)

It is obvious that if to consider application of the method (7) in the case $r = 3,$ then receive that the investigation of the method (7) is more difficult than the investigation of the method (5). Noted that method (7) and problem (1) are fundamentally investigated in [17] and [18]. By simple comparison receive that if the $\lambda = 1$ is l-multiple root, then method (7) can be presented as following:

$$\sum_{i=0}^{k} \alpha_i^l y_{n+i} = \sum_{j=l}^{s} h^j \delta_j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)}. \tag{8}$$

From here, it follows, that $\lambda = 1$ is l multiples root for the polynomial $\rho(\lambda) \equiv \alpha_k^l \lambda^k + \alpha_{k-1}^l \lambda^{k-1} + \ldots + \alpha_1^l \lambda + \alpha_0^l.$ Let us for the investigation of the method (8) to reduce the conception of l-stability.

Definition 3. Method (8) is called as the l-stable, if the roots of the polynomial $\rho(\lambda) \equiv \alpha^l{}_k \lambda^k + \alpha^l{}_{k-1} \lambda^{k-1} + \ldots + \alpha_0$ are located in the unit circle on the boundary of which there is not multiple root except the l-multiple root $\lambda = 1.$ As is known, the second conception for the comparison of the multistep method is the degree, which for the method (7) can be defined as:

Definition 4. The integer p is called as the degree for method (7), if the following asymptotic equality is perfomed:

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - \sum_{j=1}^{r} h^j \delta_j \beta_i^{(j)} y^j (x + ih)) = O(h^{p+1}),\ h \to 0. \tag{9}$$

But the conception of degree for the method (8) can be given as:

$$\sum_{i=0}^{k}\left(\alpha_i y(x+ih)-\sum_{j=l}^{s}h^j\delta_j\beta_i^{(j)}y^{(j)}(x+ih)\right)=O(h^{p+l}),\ h\to 0. \qquad (10)$$

As is known to compare multistep methods in mostly have used the conception of stability and degree. In 1952, Shura-Bura proved that if method (3) is stable, then method (3) converges and vice versa, if the method (3) is convergence, then that is stable. For the first time Bakhvalov found a connection between order and degree for the explicit methods (if $k \le 10$) receiving from the (3). And in 1956, Dahlquist fundamentally investigated method (3). The second Dahlquists work has published in 1959 in which, he had fundamentally investigated method (5). Basically, he investigated the explicit and implicit methods by receiving from the method (5). Ibrahimov has investigated the following advanced (forward-jumping) methods:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{l}\beta_i y'_{n+i}+h^2\sum_{i=0}^{s}\gamma_i f_{n+i}. \qquad (11)$$

The method (5) follows from the method (11) in the case $\beta_i=\delta_1\beta_i^{(1)},\ \gamma_i=\delta_2\beta_i^{(2)}.$ Method (3) can also be received from the formula (11). Let us study of the method (11). In first let us to consider the case $\gamma_i=0(i=0,1,\ldots,l),$ then receive following:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{l}\beta_i y'_{n+i}. \qquad (12)$$

If $l>k$ then receive the new class methods, since there are no such methods in the class methods (3). It follows that class method (12) is the separate class method, which is called the forward-jumping or advanced methods, which was fundamentally investigated by Ibrahimov. He has defined the basic disadvantages of these methods and given any way to remedy this disadvantage. And also He prove that in the class of methods

(12) there are stable methods of advanced type with the degree $p > 2[l/2] + 2$. And constructed concrete stable methods with the degree $p = 5$ for the $k = 2$ and $l = 3$. And also constructed special predictor corrector method for using advanced methods of type (12).

Ibrahimov proved that if method (12) has the degree p, then there are stable method of type (12) with maximum degree $p = k + m + 1$ $(m = s - k)$ $k \geq 3m$. Note that method (11) in the case $k < \max(s, l)$ has been fundamentally investigated by Ibrahimov, who has constructed some more specific methods for solving problem (4). For the define the maximum value of the degree of the method (11) have proved some theorems, one of which can be presented as:

Theorem 1. *Ibrahimov suppose that the method* (11) *is stable*, *have the degree of* p, $\alpha_k \neq 0$ *and* $k < l$ (*more precisely* $l = k + m$). *Then there are methods of type* (11) *with the degree* $p \leq l + s + m + 1$ *for* $k = 2r - 1 \geq 2m + 1$ *aid* $s = 2j$. *In other cases there are stable methods of type* (11) *with the degree*: $p \leq l + s + m + 2$.

As is known Gear and Butcher for the construction more exact methods are suggested to use hybrid methods, which for solving problem (2), in more general form can be presented as follows (see, for example, [25]-[32]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i+\nu_i},$$

$$n = 0, 1, \dots, N - k(|\nu_i| \leq 1, i = 0, 1, 2, \dots, k). \tag{13}$$

To construct methods with improved properties, Ibrahimov suggested constructing methods an intersection of methods (2) and (13). He proves that this method is more exact than the methods (2) and (11). Prove that if these methods are stable, then there are methods with the degree $p \leq 3k + 3$. Now let us to comparison the maximum values of the degree for the methods (3), (5) and (13). As was shown, if method (3)- is stable and has the degree of p, then $p \leq 2[k/2] + 2$ and if method (5) stable and has the degree of p, then $p \leq 2k + 2$ for the case $\delta_1 \neq 0, \delta_2 \neq 0$. Noted that if method (13) has the degree of p and stable, then $p \leq 2k + 2$ for $k = l = s$. The stable methods of type (5) and (13) have the same order of accuracy, but in the application of these methods arises different difficulty. For example in the application of the method (5) to solve problem (2) arises difficult in the calculation of values y''_m, but application that to solve problem (4) are not arises any difficult. Therefore, let us consider to determine the values of the maximum degree for the method (7).

3. Any Information About the Maximal Values of the Degree for the Method (7)

As is known, Euler himself, when he showed the shortcomings of his method, to eliminate which he proposed to use the calculation of the next members in the Taylor expansion of the solution, investigated problem. By using this, some specialists suggested to use the following method:

$$y_{n+1} = y_n + hy'_n + h^2 y''_n/2. \tag{14}$$

In the application of this method to solve problem (2), receive:

$$y_{n+1} = y_n + hf_n + h^2(f'_x(x_n, y_n) + f'_y(x_n, y_n)f(x_n, y_n))/2. \tag{15}$$

By comparison the method (15) with the explicit Euler method, receive that method (15) is more complex. But, if method (15) to use in solving of the problem (4), then receive:

$$y_{n+1} = y_n + hy'_n + h^2 f_n. \tag{16}$$

Application of the method (16) to solving problem (4) is very simple, than the application of the method (15). If method of (16), generalize then one receives method of (5). By using this properties of MMM, here recommending to apply that, to solving of the following problems:

$$y^{(s)} = g(x, y', y'', \dots, y^{(s-1)}), \; y^{(j)}_{(x_0)} = y^{(j)}_0 \; (j = 0, 1, \dots, s-1). \tag{17}$$

However, in some special cases, it becomes necessary to consider construction special methods for their solving. Based on this, it is proposed to take into account the above proposed in the construction of the suitable methods if the number of multistep methods more than 1 (one) and which are applies to solving some problems, then one can be used the results receiving in the [17] and [18] for the choosing suitable methods. Note that in these cases to find the maximum values for the stable methods of type (7) are actually. Therefore, let us consider following theorem.

Theorem 2. *Suppose that method* (7) *is stable*, *have degree of p and* $\alpha_k \neq 0$, *then* $p \leq (\delta_1 + \delta_2 + \dots + \delta_s)(k+1)+1$. *There are stable methods with the degree* $p = (\delta_1 + \delta_2 + \dots + \delta_s)(k+1)+1$ *for the* $r = 2j-1$ *and* $k = 2i$, *but in other cases there are stable methods of type* (7) *with the degree* (*see*, *for example*, [17], [18]):

$$p \leq (\delta_1 + \delta_2 + \dots + \delta_s)(k+1). \tag{18}$$

It follows from here that by the help of the above receiving relation one can find the rate of accuracy of the method, which can be received from the (18) as partial case if $\delta_1 \neq 0$. If $\delta_1 = 0$, then the estimation of (18) will be incorrect. But if the definition for the degree of the method (7) to use the asymptotic equality of (10), then one can be used the estimation of (18). Note that by, using results from the works [17] and [18], one can be found the maximum value for the degree of the MMM in the case its l-stability $1 \leq l \leq k-1$ for the method (7). By the above description way, receive that by the help of differentiating some equalities one can be constructed initial-value problem for ODEs of higher order ordinary differential equations to using initial-value problem for the ODEs of the first order. Then by using method (7), one can solve the receiving initial-value problem for ODEs with the high accuracy. However, it is possible to construct a more accurate method, with help of which, one can be to solved the above investigated problem, by the required accuracy. For the construction more exact methods, have been suggested to use advanced and hybrid methods. In the first section have been constructed the advanced methods with the first and the second derivative. As is known the method (12) has the degree of p and stable, then there are advanced methods with degree $p \leq k + m + 1\,(k \geq 3m;\ m = s - k(s > 0))$. By using this estimation and the Dahlquist's rool, receive that the advanced methods more exact than the multistep methods with the constant coefficients. A similar advantage exists also for the method (11).

By the generalization of the methods (11) and (12), receive the following:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = \sum_{j=1}^{s} \delta_j h^j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)} (n = 0, 1, \ldots, N-k). \tag{19}$$

It seems that method (19) can be received from the method (7) as a special case. However, this is not the case, because in the study of the method (7), the following condition is used: $\alpha_k \neq 0$. By using the condition $\alpha_k \neq 0$, receive that advanced methods is the separate class of methods for the investigation. Let us noted that the known scientists as Laplas, Steklov constructed advanced methods, which obeyed the laws of the Dahlquist. The advanced methods have constructed by the Kowell. However, the advantage of these methods has been proved by Ibrahimov, which has constructed predictor corrector methods for the application of advanced methods. And he proved that stable advanced methods are exactness than the stable explicit and implicit methods of multistep types. For the construction more exact methods, let us consider following section:

4. The Way to Construction of Hybrid Methods

In 1965, Butcher and Gear proved that the hybrid methods are exact than the explicit and implicit methods of multistep type. By taking into account of this, Ibrahimov constructed methods on the intersection of the methods (3) and (13), in which receive:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\sum_{i=0}^{k} \gamma_i f_{n+i+\nu_i} \quad (|\nu_i| \le 1;\ i = 0, 1, \dots, k). \qquad (20)$$

This method has been fundamentally investigated by Ibrahimov and proves that if method (20) is stable then in the class (20), there are methods with the degree $p = 3k + 3$, here p-is the degree of the method (20), which can be define as the following:

$$\sum_{i=0}^{k} (\alpha_i y(x + ih - h\beta_i y'(x + ih) - h\gamma_i y'(x + (i + \nu_i)h))$$

$$= O(h^{p+1}),\ h \to 0. \qquad (21)$$

As is known, all the methods have own advantages and disadvantages. One of the disadvantages of the method (20) is the calculation the of values $y_{n+i+\nu_i}$ and one of advantages of the method (20) is that they are more accurate.

Noted that the region of stability for the hybrid methods does not expand. But by using some predictor corrector methods the region of the stability can be expanded.

Let us consider the following methods, which are received from the method (20) as the partial case:

$$y_{n+1} = y_n + h\left(f_{n+1/2+\alpha} + f_{n+1/2-\alpha}\right)/2,\ \alpha = \sqrt{3}/6, \qquad (22)$$

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/12 + 5h\left(f_{n+1/2+\beta} + f_{n+1/2-\beta}\right)/12,\ \beta = \sqrt{5}/10. \qquad (23)$$

Methods (22) and (23) are stable and have the degree $p = 4$ and $p = 6$, respectively. From here we get that the hybrid methods are promising. Method (22) and (23) are the hybrid types and are more accurate than the known methods. Noted that the order of accuracy for the method (20), is more than the order of accuracy for the method (5). If take $\beta_i = 0$ $(i = 0, 1, \dots, k)$ in the method (20), then the order of accuracy for the receiving method will be one and the same with the order of accuracy for the method (5). Hence the method (20) is more exact than the method (5) $(\delta_1 = \delta_2 = 1)$. Hybrid methods have some similarities with the Gauss method. For example, if $\beta_i = 0\,(i = 0, 1, \dots, k)$, then from the (20) it follows the hybrid methods. If hybrid methods have the maximum degree, the coefficients $\gamma_i (i = 0, 1, \dots, k)$ will located symmetrically and will be positive. Noted that the nonlinear of algebraic equations for finding the coefficients of the hybrid methods in one case can be presented as follows:

$$\sum_{i=0}^{k} \alpha_i = 0; \sum_{i=0}^{k} (i\alpha_i - \beta_i - \gamma_i) = 0;$$

$$\sum_{i=0}^{k} ((i^s/s!)\alpha_i - (i^{s-1}/(s-1)!)\beta_i - (l_i^{s-1}/(s-1)!)\gamma_i) = 0;$$

$$l_i = i + \nu_i;\ s = 1, 2, \dots, p. \tag{24}$$

If is system (24) compares this with the system of algebraic equations constructed for finding the values of the coefficients, receive that these systems almost the same.

If this system of algebraic equation solved for the value $k = 2$, then one can be constructed the following method:

$$y_{n+2} = y_n + h(9f_{n+2} + 64f_{n+1} + 9f_n)/90$$
$$+ 49h\left(f_{n+1+\beta} + f_{n+1-\beta}\right)/90,\ \beta = \sqrt{21}/7. \tag{25}$$

This method is stable and has the degree $p = 8$. This method can be presented in other form by using the value f_{n+1} in the second bracket. Presentation of this method is not the only one. But the coefficients in the second bracket satisfies the Gauss coefficients properties. Let us consider the following presentation of the method (25):

$$y_{n+2} = y_n + h(9f_{n+2} + 15f_{n+1} + 9f_n)/90$$
$$+ 49h\left(f_{n+1+\beta} + f_{n+1} + f_{n+1-\beta}\right)/90. \qquad (26)$$

If follows from here that the hybrid points are located symmetrically to mesh point x_{n+1}. Let us consider method (22) and write that as following;

$$y_{n+1} = y_n + h\left(f_{n+1/2-\alpha} + f_{n+1/2+\alpha}\right)/2, \ \alpha = \sqrt{3}/6. \qquad (27)$$

How it follows from here, method (27) can be taken as symmetrical. Noted that method (27) has the maximum degree. But one can construct method, which will be the type of (26) and more exact, than the method (26).

By using the properties of above receiving hybrid methods, one can be approved that the basic properties of the hybrid and Gauss methods are one and the same (the properties of coefficients and the hybrid pointes (or Gauss nodes)).

As was noted above if method (20) is stable and has the degree of, then there are methods of type (20) with the degree $p \leq 3k + 3$. If $\beta_i = 0\,(i = 0, 1, \ldots, k)$, then $p \leq 2k + 2$. By the above described way receive that hybrid methods are promising. Therefore, we have considered application the receiving here, results to solving the Volterra integral and the Volterra integro-differential equations. By using receiving results, one can approve that hybrid methods of type (20) more exact than Gauss method and for all the values of k, there is the family of hybrid methods. Therefore, the hybrid methods are considered promising.

5. Application of the Multistep Methods with Constant Coefficients to Solving Volterra Integral and Integro-Differential Equations

As is known the solution of initial-value problem for ODEs and Volterra integral equations has the direct relation. Therefor the specialist try to solve Volterra integral equations is reducing to solve initial-value problem for ODEs. It follows from here, that the quadrature method can be taking as the popular. Have shown that the value of the computational works increases, which is the main disadvantages of the quadrature formula (see, for example, [33], [34], [35]). Therefore, have constructed the new method in the application of which, the values of computational work does not increases. Noted that this method in one version can be presented as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i g_i + h \sum_{i=0}^{k} \sum_{j=i}^{k} \beta_i^{(j)} K(x_{n+j},\ x_{n+i},\ y_{n+i}). \tag{28}$$

It is easy to understand that this method remember the multistep methods with constant coefficients/Noted that the corresponding Volterra integral equation can be presented as:

$$y(x) = g(x) + \int_{x_0}^{x} K(x,\ s,\ y(s))ds,\ x_0 \le s \le X. \tag{29}$$

We have investigated method (28), fundamentally. It is to say that, has been found the maximum value for the degree of stable methods of type (28) and have been proven that the method of type (28) with the maximum degree is not unique. One of the main results can be presented as:

Theorem 3. *If the method* (28) *is stable and has the degree of*, *then in the class* (28) *there are methods with the degree* $p \le 2[k/2] + 2$. *And for all the values of* k, *there are stable methods of type* (28), *with the degree*

$p_{\max} \le 2k$. *By taking into account of the above described, receive that for the construction more exact stable methods, one can used the advanced or hybrid methods. It is easy to say that the hybrid methods can be taking as perspective, because the stable hybrid methods are more exact. For the presentation of application hybrid methods to solve Volterra integral equations, let us consider the following*:

$$\sum_{i=0}^{k} \alpha_i (y_{n+i} - g_{n+i}) = h \sum_{i=0}^{k}$$

$$\times \sum_{j=i}^{k} \gamma_i^{(j)} K\left(x_{n+j},\, x_{n+i+\nu_i},\, y_{n+i,\, \nu_i}\right) (|\nu_i| < 1;\; i = 0, 1, \dots, k). \quad (30)$$

In the application of the method (30) to solve problem (29) arises some difficult with the calculation of the initial values for the method (30) finding the solution of the nonlinear algebraic equations. One of the main disadvantage of hybrid method is the calculation of values of the solution for investigated problem at the hybrid points: $x + (i + \nu_i)h$ $(i = 0, 1, \dots, k)$, here x-fixed point.

For the construction more exact methods, one can be constructed the new methods on the intersection of the methods (28) and (30). In this case receive:

$$\sum_{i=0}^{k} \alpha_i (y_{n+i} - g_{n+i}) = h \sum_{i=0}^{k} \sum_{j=1}^{k} \beta_i^{(j)} K\left(x_{n+j},\, x_{n+i},\, y_{n+i}\right)$$

$$+ h \sum_{i=0}^{k} \sum_{j=i}^{k} \gamma_i^{(j)} K\left(x_{n+j},\, x_{n+i+\nu_i},\, y_{n+i,\, \nu_i}\right),\; |\nu_i| < 1;\; n = 0, 1, \dots, N - k. \quad (31)$$

By using above described results, it can be argued the degree of the method (31) satisfies the condition: $p \le 3k + 3$ for the stable methods and $p \leq 4k + 2$ for the others. By taking into account above mentioned, receive that hybrid methods are promising. By using this result, have been constructed predictor-corrector methods for the application of hybrid methods to solving of some problems. And also established some relationship between the values of the degree for the predictor and correctors methods. In some cases have been constructed special methods for the calculation of the values desired solutions at the hybrid methods (see, for example, [39], [40], [41]). By using some properties of method (11), have been considered application of this method to solve above mentioned problems for all the values of k, l, s (see, for example, [13], [17], [18]). Noted that, the number of methods with maximum accuracy here is always more than one. And also, noted that the number of terms in the methods, which are, applied to solving of Volterra integral and integro-differential equation, are always greater than the terms in the method, which are applied to solve of initial-value problem for ODEs. And now, let us comprise the method (11) with the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{l} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{s} \gamma_i y''_{n+i}. \tag{32}$$

The method (32) and (11) one and the same. But application of the method (32) to solving problems as the (3), (4), (29) and, the following:

$$y'(x) = f(x, y) + \int_{x_0}^{x} k(x, s, y(s)) ds, \; y(x_0) = y_0 \;\; x_0 \leq s \leq X,$$

is very easy than the application of method (11), which in the application to solve problem (2) can be written as the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i}, \; n = 0, 1, \ldots, N-k, \tag{33}$$

here $g(x, y) = f'_x(x, y) + f'_y(x, y) f(x, y)$.

And now let us applied, method (3) to solving of initial-value problem for integro differential equation:

$$y'(x) = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds, \; y(x_0) = y_0, \; x_0 \le s \le X. \tag{34}$$

Multistep methods with the constant coefficients for solving problem (34) can be presented as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\sum_{i=0}^{k} \gamma_i z_{n+i},$$

$$\sum_{i=0}^{k} \overline{\alpha}_i z_{n+i} = h\sum_{i=0}^{k}\sum_{j=i}^{k} \gamma_i K(x_{n+j}, x_{n+i}, y_{n+i}),$$

here $z(x)$-has defined as:

$$z(x) = \int_{x_0}^{x} K(x, s, y(s))ds, \; x_0 \le s \le x \le X.$$

The similar results have been established for solving the problem (34).

Thus, here have been given optimal methods for solving initial-value problem for ODEs and the Volterra integro-differential equations, and also the Volterra integral equations.

Definition. The method is called as the optimal if is stable it has the maximum degree and uses of computational work that is almost constant amount, at all the mesh points.

Thus, here have been investigated the following methods: $\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{l} \beta_i y'_{n+i}$, the class of General multistep method with constant coefficients, $\sum_{i=0}^{k} \overline{\alpha}_i y_{n+i} = h\sum_{i=0}^{l} \overline{\beta}_i y'_{n+i} + h^2\sum_{i=0}^{s} \overline{\gamma}_i y''_{n+i}$, the class of General multistep second derivative method, $\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i+\nu_i}$ $(|\nu_i| < 1;\ i = 0, 1, \dots, k)$, The class of general hybrid method, $\sum_{i=0}^{s} \alpha'_i y_{n+i} = h \sum_{i=0}^{k} \beta'_i y'_{n+i} + h \sum_{i=0}^{k} \gamma'_i y'_{n+i+\nu_i}$, the class of multistep methods constructed on the intersection of multistep and hybrid methods, $\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h \sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i} + h^2 \sum_{i=0}^{k} \bar{\beta}_i y''_{n+i} + h^2 \sum_{i=0}^{k} \bar{\gamma}_i y''_{n+i+\nu_i}$. The class of multistep second derivative methods of hybrid type, $\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{s} h^j \delta_j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)}$ (MMM). The class of Multistep Multiderivative Methods.

As well as, has been defined the sign about some coefficients. For example, if $\beta_k^{(j)} \neq 0$ $(j = 1, 2, \dots, s)$, then $\beta_k^{(l)} \beta_k^{(l+1)} < 0$ $(l = 1, 2, \dots, s-1)$.

A similar relationship between for some similar coefficients has been receiving for the following advanced method: $\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i}$ $(m > 0)$. if $\beta_j \neq 0$ and $\beta_{j+1} \neq 0$, then $\beta_j \beta_{j+1} < 0$ $(k - m \leq j \leq k - 1)$.

6. Conclusion

As is known, one of the popular numerical method to solving of initial-value problem for Ordinary Differential Equations is the Multistep Multiderivative Methods with constant coefficient which was noted above in some partial cases has been fundamentally investigated by Dahlquist, but MMM in general form has been investigated by Ibrahimov. Here, have been shown that in which form has investigated MMM and how that can be applied to solving of initial-value problem for both ODEs of the r order and the Volterra integro-differential equation and also the Volterra integral equations. Noted that MMM has been investigated in more general form (by using different values of the both ODEs r order and the $\delta_j(j = 0, 1, \ldots r)$). And also have been determined the maximal value of the degree for the MMM in different cases. Thus, MMM have in fully formed investigated and constructed methods for their application to solving some suitable problems. Hence, MMM fundamentally investigated by V.Ibrahimov as the numerical method for solving above mentioned problems. Noted that he also fundamentally investigated Multistep Methods with first and second derivatives of advanced and hybrid types. For the obtain reliable information about the solution of investigated problem, here recommending to use the bilateral methods (see for example [42]-[45]). It should be noted that similar studies were carried out by other specialists (see for example [46]-[53]. As is known, in the application of these methods, arises some necessity for the define the sing of the non-zero coefficients $\beta_k^{(j)}$ (see, for example, [1], [17]). In view of what has been said, here the definition of sign for some coefficients was considered. Here have constructed approximately 20 methods of MMM type. In which, there are methods of advanced and hybrid types, having the different order of accuracy. And also, have been investigated and constructed symmetric methods having the different form (explicit, implicit, advanced and hybrid types). Given that many examples are solved with the methods which have been presented in the papers of the author and are published in the different journals, we decided do not repeat that here.

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References

[1] V. R. Ibrahimov, On the maximal degree of the k-step Obrechkoff's method, Bulletin of Iranian Mathematical Society 28(1) (2002), 1-28.

[2] G. Dahlquist, Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equation, Transitional of the Royal Institute of Technology, Stockholm, Sweden 130 (1959), 3-87.

[3] A. Iserles and S. P. Norset, Two-step methods and Bi-orthogonality, Mathematics of Computation 49(180) (1987), 543-552.

[4] V. R. Ibrahimov, Application of a numerical method to solve differential equations of higher orders, Reports of the Academy of Sciences Az.r. XXVII(5) (1971), 9-12.

[5] G. Dahlquist, Convergence and stability in the numerical integration of ordinary differential equations, Mathematica Scandinavica 4(1) (1956), 33-53.

DOI: https://doi.org/10.7146/math.scand.a-10454

[6] E. Hairer, C. Lubuch and G. Wanner, Geometric Numerical Integration, Second Edition, Springer (2004), 644.

DOI: https://doi.org/10.1007/3-540-30666-8

[7] J. Kobza, Second derivative methods of Adams type, Applikace Mathematicky 20 (1975), 389-405.

[8] G. Mehdiyeva and V. Ibrahimov, On the Investigation of Multistep Methods with Constant Coefficients, Lap Lambert, Academi Publising (2013), 314.

[9] G. Gupta, A polynomial representation of hybrid methods for solving ordinary differential equations, Mathematics of Computation 33(148) (1979), 1251-1256.

DOI: https://doi.org/10.2307/2006458

[10] I. Babushka, E. Vitasek and M. Prager, Numerical processes for solving differential equations, Mir (1969), 368.

[11] L. Skvortsov, Explicit two-step Runge-Kutta methods, Math. Modeling 21 (2009), 54-65.

[12] V. Ibrahimov, On a nonlinear method for numerical calculation of the cauchy problem for ordinary differential equation, Differential equation and applications, Proceeding of II International Conference Russe, Bulgarian (1982), 310-319.

[13] V. R. Ibrahimov, A relationship between order and degree for a stable formula with advanced nodes, Computational Mathematics and Mathematical Physics (USSR) 30 (1990), 1045-1056.

[14] J. Butcher, A modified multistep method for the numerical integration of ordinary differential equations, Journal of the ACM 12(1) (1965), 124-135.

DOI: https://doi.org/10.1145/321250.321261

[15] M. K. Kaabar, F. Martinez, J. F. Gomez-Aguilar, B. Ghanbari, M. Kaplan and H. Günerhan, New approximate analytical solutions for the nonlinear fractional schrödinger equation with second-order spatio-temporal dispersion via double laplace transform method, Mathematical Methods in the Applied Sciences 44(14) (2021), 11138-11156.

DOI: https://doi.org/10.1002/mma.7476

[16] D. A. Juraev, Cauchy problem for matrix factorizations of the helmholtz equation, Ukrainian Mathematical Journal 69(10) (2018), 1583-1592.

DOI: https://doi.org/10.1007/s11253-018-1456-5

[17] V. R. Ibrahimov, Relationship between of the order and the degree for a stable forward-jumping formula, Prib. Operator Methods, urav. Baku (1984), 55-63.

[18] V. R. Ibrahimov, About one relationship between order and degree for a stable forward-jumping formula, Az ANIINII, Russian, № 1025, p. 1-69.

[19] V. R. Ibrahimov, Application of k-step method to numerical solution of higher-order differential equations, №1241, Az NIINII, 17.03.89.

[20] M. R. Shura-Bura, Error estimates for numerical integration of ordinary differential equations, Prikl. Matem. and Mech., 1952, № 5, 575-588, (Russian).

[21] I. S. Mukhin, By the accumulation of errors in the numerical integration of differential-differential equations, Prikl. mat. and mech., V.6 (1952), 752-756 (Russian).

[22] V. R. Ibrahimov, One nonlinear method for the numerical solution of the Cauchy problem for ordinary differential equations ODEs and application proceedings of the report, Second International Conference Russia, Bulgaria (1982), 310-319.

[23] N. S. Bakhvalov, Some remarks on the question of numerical intefration of differential equation by the finit-difference method, Academy of Science report, USSA, N3 (1955), 805-808 (Russian).

[24] P. Henrici, Discrete variable methods in ODG, John Wiley and Sons, Inc., New York, London, 1962.

[25] V. R. Ibrahimov, A relationship between order and degree for a stable formula with advanced nodes, USSR Computational Mathematics and Mathematical Physics 30(4) (1990), 1045-1056.

DOI: https://doi.org/10.1016/0041-5553(90)90044-S

[26] G. Yu. Mehdiyeva, V. R. Ibrahimov and I. I. Nasirova, On some connections between Runge-Kutta and Adams methods, Transactions Issue Mathematics and Mechanics Series of Physical-Technical and Mathematical Science 5 (2005), 55-62.

[27] G. Yu. Mehdiyeva V. R. İbrahimov and M. N. Imanova, Application of multi step methods to the solving second order ordinary differential equation, The Third International Conference “Problems of Cybernetics and Informatics”, Baku, Azerbaijan (2010), 255-258.

[28] G. Mehdiyeva, V. Ibrahimov and M. Imanova, On a way for constructing numerical methods on the joint of multistep and hybrid methods, World Academy of Science, Engineering and Technology, Paris (2011), 240-243.

[29] G. Mehdiyeva, V. Ibrahimov and M. Imanova, An application of mathematical methods for solving of scientific problems, British Journal of Applied Science & Technology 14(2) (2016), 1-15.

DOI: https://doi.org/10.9734/BJAST/2016/22964

[30] G. Y. Mehdiyeva, M. N. Imanova, V. R. Ibrahimov, General hybrid method in the numerical solution for ODE of first and second order, in: Recent Advances in Engineering Mechanics, Structures and Urban Planning, Cambridge, UK, 2013, 175-180.

[31] G. Mehdiyeva, V. Ibrahimov, M. Imanova and A. Guliyeva, One a way for constructing hybrid methods with the constant coefficients and their applied, IOP Conference Series: Materials Science and Engineering 225 (2017); Article 012042.

DOI: https://doi.org/10.1088/1757-899X/225/1/012042

[32] V. R. Ibrahimov, G. Mehdiyeva, Xiao-Gunag Yue, Mohammmed K. A. Kaabar, Samad Noeiaghdam and D. Juraev, Novel symmetric numerical methods for solving symmetric mathematical problems, International Journal of Circuits, Systems and Signal Processing 15 (2021), 1545-1557.

DOI: https://doi.org/10.46300/9106.2021.15.167

[33] V. R. Ibrahimov and M. N. Imanova, Multistep methods of the hybrid type and their application to solve the second kind Volterra integral equation, Symmetry 13(6) (2021); Article 1087.

DOI: https://doi.org/10.3390/sym13061087

[34] G. Yu. Mehdiyeva, V. R. İbrahimov and M. N. Imanova, The application of the hybrid method to solving the Volterra integro-differential equation, World Congress on Engineering 2013, London, U.K., 3-5 July (2013), 186-190.

[35] G. Yu. Mehdiyeva, V. Ibrahimov and M. Imanova, Some refinement of the notion of symmetry for the Volterra integral equations and the construction of symmetrical methods to solve them, Journal of Computational and Applied Mathematics 306 (2016), 1-9.

DOI: https://doi.org/10.1016/j.cam.2016.03.026

[36] G. Yu. Mehdiyeva, V. R. Ibrahimov and M. N. Imanova, Application of hybrid methods to solving Volterra integral equation with the fixed boundares, International Journal of Advance Research and Innovative Ideas in Education 1(5) (2015), 491-502.

[37] Vagif Ibrahimov and Aliyeva Vusala, The construction of the finite-difference method and application, AIP Conference Proceedings 1648(1) (2015), 1-5.

DOI: https://doi.org/10.1063/1.4913104

[38] M. N. Imanova and V. R. Ibrahimov, The application of second derivative methods to solving Volterra integro-differential equations, AIP Conference Proceedings 1648(1) (2015), 1-5.

DOI: https://doi.org/10.1063/1.4913103

[39] G. Mehdiyeva, V. Ibrahimov and M. Imanova, On one application of hybrid methods for solving Volterra Integral equations, World Academy of Science, Engineering and Technology, Dubai (2012), 809-813.

[40] G. Yu. Mehdiyeva, V. R. İbrahimov and M. N. Imanova, Application of the hybrid methods to solving Volterre Integro-differential equations, World Academy of Science, Engineering and Technology Paris (2011), 1197-1201.

[41] G. Yu. Mehdiyeva, V. R. İbrahimov and M. N. Imanova, On the construction test equations and its applying to solving Volterra integral equation, Mathematical Methods for Information Science and Economics, Montreux, Switzerland, December 29-31 (2012), 109-114.

[42] G. Yu. Mehdyeva, V. R. Ibrahimov and M. N. Imanova, Finite difference methods with improved properties and their application to solving some model problems, 2022 International Conference on Computational Science and Computational Intelligence (CSCI), 2023.

[43] M. N. Imanova and V. R. Ibrahimov, The new way to solve physical problems described by ODE of the second order with the special structure, Wseas Transactions on Systems 22 (2023), 199-206.

DOI: https://doi.org/10.37394/23202.2023.22.20

[44] V. R. Ibrahimov, G. Yu. Mehdiyeva and M. N. Imanova, On the construction of symmetrical methods and step by step method for their application to solving initial-value problem for ODEs, Proceeding International Conference on Engineering, Science and Technology (October 13-16th 2022), 59–72.

[45] V. R. Ibrahimov, G. Yu. Mehdiyeva and M. N. Imanova, Construction of some algorithm for calculation of definite integrals by using advanced and hybrid methods, Proceeding International Conference on Engineering, Science and Technology (October 13-16th 2022), 84-99.

[46] M. Imanova and V. Ibrahimov, The application of Hybrid methods to solve some problems of mathematical biology, American Journal of Biomedical Science and Research 18(1) (2023), 793-799.

DOI: https://doi.org/10.34297/AJBSR.2023.18.002436

[47] D. A. Juraev, V. Ibrahimov and P. Agarwal, Regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation on a two-dimensional bounded domain, Palestine Journal of Mathematics 12(1) (2023), 381-403.

[48] D. A. Juraev, M. J. O. Jalalov and V. R. O. Ibrahimov, On the formulation of the Cauchy problem for matrix factorizations of the Helmholtz equation, Engineering Applications 2(2) (2023), 176-189.

[49] V. Ibrahimov and M. Imanova, About some applications multistep methods with constant coefficients to investigation of some biological problems, American Journal of Biomedical Science and Research 18(6) (2023), 531-542.

DOI: https://doi.org/10.34297/AJBSR.2023.18.002522

[50] D. A. Juraev, The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain, Palestine Journal of Mathematics 11(3) (2022), 604-613.

[51] D. A. Juraev, A. Shokri and D. Marian, On an approximate solution of the Cauchy problem for systems of equations of elliptic type of the first order, Entropy 24(7) (2022); Article 968.

DOI: https://doi.org/10.3390/e24070968

[52] Xiao-Guang Yue, Zeying Zhang, Arzu Akbulut, Mohammed K. A. Kaabar and Melike Kaplan, A new computational approach to the fractional-order Liouville equation arising from mechanics of water waves and meteorological forecasts, Journal of Ocean Engineering and Science (2022), 1-8.

DOI: https://doi.org/10.1016/j.joes.2022.04.001

[53] S. Deepa, A. Ganesh, V. Ibrahimov, S. S. Santra, V. Govindan, K. M. Khedher and S. Noeiaghdam, Fractional Fourier transform to stability analysis of fractional differential equations with prabhakar derivatives, Azerbaijan Journal of Mathematics 12(2) (2022), 131-153.

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