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SIMPLE CRITERIA FOR ALL *n*-TH ROOTS OF A NATURAL NUMBER BEING IRRATIONAL

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Abstract

This paper introduces a simple and straight-forward approach to determine for a given natural number x, $x \geq 2$, whether $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall\, n \in \mathbf{N}$, $n \geq 2$. Unlike earlier solutions, our approach does no longer rely on the knowledge of the prime factorization of x. If we just consider the value of the two last digits and the sum of digits of x, we are able to identify correctly more than 50% of all natural numbers, whose n-th roots are irrational $\forall n \in \mathbf{N}$, $n \geq 2$.

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1. Introduction

Already since ancient times, mathematicians have investigated the question whether the root of a given natural number x yields to a rational or irrational value, i.e., "$\sqrt[n]{x} \in \mathbf{Q}$ or $\sqrt[n]{x} \notin \mathbf{Q}$?". So, e.g., Euklid- already more than 2000 years ago - has proven that $\sqrt{2} \notin \mathbf{Q}$ [1]. In [2], we have answered the question stated above in a much more general form, namely we completely solved the question "$\sqrt[n]{x} \in \mathbf{Q}$ or $\sqrt[n]{x} \notin \mathbf{Q}$?" for all $x \in \mathbf{R}^+$, $n \in \mathbf{N}, n \geq 2$, where $\mathbf{R}^+ := \{x \in \mathbf{R} | x > 0\}$. Moreover, in [3], we have presented very simple tests which allow one to obtain an immediate answer for the cases that x is a natural number. In addition, we have also presented a very simple method to calculate the value of $\sqrt[n]{x}$, $n \in \mathbf{N}, n \geq 2, x \in \mathbf{N}, x \geq 2,$ for cases in which $\sqrt[n]{x} \in \mathbf{Q}$. However, an important precondition of the results of [3] is that the prime factorization of the natural number x considered is available.

In this contribution, we do no longer assume that the prime factorization [4] is available to us, because for extremely large values of x it is well-known that prime factorization might be practically infeasible. In particular, we will demonstrate that already by means of combining three extremely simple tests it is possible to determine more than 50% of the natural numbers x of an interval $[x_1, x_2]$ for which $\sqrt[n]{x} \notin \mathbf{Q}$ holds $\forall\ n \in \mathbf{N}, n \geq 2$. If, in addition, the prime numbers within $[x_1, x_2]$ are known to us, the probability of successful prediction of numbers with irrational roots only, can be increased to up to about 85% (as, e.g., our example in Section 4 demonstrates).

Our paper is structured as follows: Section 2 presents the three simple tests which allow one to determine natural numbers in an interval $[x_1, x_2]$, which satisfy the condition "$\sqrt[n]{x} \notin \mathbf{Q}$, $x \in \mathbf{N}$, $x \in [x_1, x_2]$, $\forall\ n \in \mathbf{N}$, $n \geq 2$". In Section 3, we shortly discuss how the knowledge of primes within the interval $[x_1, x_2]$ allows one to further increase the hit rate of natural numbers within $[x_1, x_2]$ the n-th roots of which are irrational for all n, $n \geq 2$. In Section 4, we will apply our results to the interval [2, 100] by way of example. The paper concludes with a short summary and outlook.

2. Simple Tests to Determine Natural Numbers x for which $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall\ n \in \mathbf{N}$, $n \geq 2$

To simplify our notation and argumentation throughout this paper let us introduce the following notations and abbreviations:

We denote by:

- $\mathbf{N}_{\geq 2} := \{x \in \mathbf{N} | x \geq 2\}$.
- $\{x_1, x_2\}_{\mathbf{N}} := \{x \in \mathbf{N} | x_1 \leq x \leq x_2\}$.
- $X_{\text{rat\_roots}}[x_1, x_2] := \{x \in [x_1, x_2]_{\mathbf{N}} | \exists\ n \in \mathbf{N}_{\geq 2}, \sqrt[n]{x} \in \mathbf{Q}, x_1 \geq 2\}$.
- $X_{\text{rat\_roots}} := \{x \in \mathbf{N}_{\geq 2} | \exists\ n \in \mathbf{N}_{\geq 2}, \sqrt[n]{x} \in \mathbf{Q}\}$.
- $X_{\text{irrat\_roots}}[x_1, x_2] := \{x \in [x_1, x_2]_{\mathbf{N}} | \sqrt[n]{x} \notin \mathbf{Q}, \forall\ n \in \mathbf{N}_{\geq 2}, x_1 \geq 2\}$.
- $X_{\text{irrat\_roots}} := \{x \in \mathbf{N}_{\geq 2} | \sqrt[n]{x} \notin \mathbf{Q}, \forall\ n \in \mathbf{N}_{\geq 2}\}$.

2.1. Test using even natural numbers being no multiples of 4

Let X_2 denote the set of even numbers being no multiples of 4, i.e.,

$$X_2 := \{4 \bullet y - 2 \mid y \in \mathbf{N}\}.$$

So, $X_2 = \{2, 6, 10, 14, 18, \ldots\}$.

In [3], it has been proven that, if for $x \in \mathbf{N}_{\geq 2}$ the prime factorization of x is given by:

$$x = p_1 \bullet p_2^{k\_2} \bullet p_3^{k\_3} \bullet\bullet\bullet\, p_m^{k\_m} \tag{1}$$

p_i representing prime numbers $\forall i,\ p_i \neq p_j,\ \forall\ i \neq j,\ k\_i \in \mathbf{N}$ (where k_i to be read as k_i), then $x \in X_{\text{irrat\_roots}}$.

We observe that for $\forall x \in X_2$, prime factorization of x is given by equation (1) if we are setting $p_1 = 2$ and, therefore, $\forall x \in X_2 : x \in X_{\text{irrat\_roots}}$ holds. As X_2 covers 25% of all elements of $\mathbf{N}_{\geq 2}$, we see that already the elements of X_2 cover more than ¼ of all elements of $X_{\text{irrat\_roots}}$.

Moreover, the test whether a number $x \in \mathbf{N}_{\geq 2}$ satisfies exactly the properties assumed for the elements of X_2 is rather trivial because we only have to test that:

(1) x is a multiple of 2 (which is equivalent to: x is an even number);

(2) x is *not* an (integer) multiple of 4 (which is equivalent to: the last two digits of x are not an (integer) multiple of 4).

To summarize, already the set X_2 provides a rather dense coverage of $X_{\text{irrat\_roots}}$.

2.2. Tests using natural numbers being an (integer) multiple of 3 and no multiples of 9

Let X_3 denote the set $X_3 := \{x \in \mathbf{N}_{\geq 2} | x$ is an (integer) multiple of 3 and not an (integer) multiple of 9}. So, $X_3 = \{3, 6, 12, 15, 21, \ldots\}$.

Again, X_3 satisfies the assumptions of Equation (1), here, if we set $p_1 = 3$. And, X_3 covers roughly 2/9 of c successive natural numbers if c is chosen to be sufficiently large. So, in general, X_3 covers more than 20% of the elements being part of $X_{\text{irrat\_roots}}$.

Again, two (trivial) tests have to be carried out to prove that a number $x \in \mathbf{N}_{\geq 2}$ is an element of X_3:

(1) x is a multiple of 3 *iff.* the sum of the digits of x is a multiple of 3;

(2) x is *not* an (integer) multiple of 9 *iff.* the sum of the digits of x is not a multiple of 9.

2.3. Tests using natural numbers being an (integer) multiple of 5 and no multiples of 25

Let X_5 denote the set $X_5 := \{x \in \mathbf{N}_{\geq 2} | x$ is an (integer) multiple of 5 and not an (integer) multiple of 25}.

So, $X_5 = \{5, 10, 15, 20, 35, \ldots\}$.

Again, if we now set $p_1 = 5$ then X_5 satisfies the assumptions of equation (1). And, X_5 covers roughly 16% of c successive natural numbers if c is chosen to be sufficiently large.

The two (trivial) tests that have to be carried out, in this case, to prove that a number $x \in \mathbf{N}_{\geq 2}$ is an element of X_5 are as follows:

(1) x is a multiple of 5 *iff.* the last digit of x is 0 or 5;

(2) x is *not* an (integer) multiple of 25 *iff.* the two last digits of x are different from 00, 25, 50 and 75.

If we apply the tests whether an arbitrarily chosen number $x \in [2, c]$ is an element of $X_2 \cup X_3 \cup X_5$ we observe that $x \in X_2 \cup X_3 \cup X_5$ in more than 50% of all cases (if c is chosen sufficiently large). Evidently, if $x \in X_2 \cup X_3 \cup X_5$ then this implies that $x \in X_{\text{irrat\_roots}}$. Thus, we can conclude that the combined test – based on X_2, X_3 and X_5 – namely "$x \in X_2 \cup X_3 \cup X_5$?" for $x \in [2, c]_{\mathbf{N}}$ allows us to determine more than 50% of the elements being part of $X_{\text{irrat\_roots}}[2, c]$, which is a pleasingly high percentage, in particular, if we take into account that all tests applied are extremely simple ones.

3. Improving Tests Regarding Irrational *n*-th Roots of Natural Numbers *x* if Additional Prime Number Knowledge is Available

If we are not sufficiently satisfied with the result of Section 2 already allowing us to determine more than 50% of the elements of $X_{\text{irrat\_roots}}[2, c]$, we could still improve this value in a straight-forward manner, if some or even all of the prime numbers $P_{[2, c]}$ within $[2, c]_{\mathbf{N}}$ are known. The reason for this simple improvement results from the fact that in [2] it has been proven that all prime numbers are elements of $X_{\text{irrat\_roots}}$. As moreover, $P_{[2, c]} \cap (X_2 \cup X_3 \cup X_5) = \{2, 3, 5\}$, if $c \geq 5$, we observe that the elements of $X_{\text{irrat\_roots}}[2, c]$, determined by the tests suggested in Section 2, are extended by nearly all the prime numbers being part of $P_{[2, c]}$. We will exemplify this in the next section in which we choose the interval [2, 100] by way of example.

Prime numbers can be determined in a relatively efficient manner, e.g., by using extended versions of the Agrawal-Kayal-Saxena algorithm for primality testing (AKS algorithm for short) [5], [6].

4. Example: Determination of Irrational n-th Roots of Natural Numbers $x, x, \in [2, 100]$

Let us now illustrate which $x \in X_{\text{irrat\_roots}}[2, 100]$ we are able to determine if we combine the tests suggested in Section 2 and in Section 3. The following elements of $X_{\text{irrat\_roots}}[2, 100]$ are determined by X_2 within $[2, 100]_{\mathbf{N}}$:

Result of X_2 : 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98.

The set of candidates produced by X_2 are complemented by candidates produced by X_3 (those ones being different from X_2 candidates):

- Additional candidates resulting from X_3 : 3, 12, 15, 21, 24, 33, 39, 48, 51, 57, 60, 69, 75, 84, 87, 93, 96.

- Additional candidates resulting from X_5 (being different from X_2 and X_3 candidates): 5, 20, 35, 40, 45, 55, 65, 80, 85, 95.

The number of all (different) candidates produced by X_2, X_3 and X_5 is 25 + 17 + 10 = 52.

The following prime numbers are part of $[2, 100]_{\mathbf{N}}$:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

As the prime numbers 2, 3, 5 are already part of $X_2 \cup X_3 \cup X_5$ we obtain 22 prime numbers which extend the elements of $X_{\text{irrat\_roots}}[2, 100]$ having already been determined by the tests related to X_2, X_3 and X_5. So, in total, the tests suggested in Section 2 and Section 3 allow one to determine 52 + 22 = 74 different elements of $X_{\text{irrat\_roots}}[2, 100]$.

To determine the cardinality of $X_{\text{irrat\_roots}}[2, 100]$ we consider the set $X_{\text{rat\_roots}}[2, 100] = \{4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100\}$.

Therefore, $|X_{\text{irrat\_roots}}[2, 100]| = 99 - |X_{\text{rat\_roots}}[2, 100]| = 99 - 12 = 87$. (Here, $|S|$ denotes the cardinality of set S).

To summarize, we see that the tests determined 52/87 $\approx$ 60% of all elements of $X_{\text{irrat\_roots}}[2, 100]$, and this success probability can be improved to 74/87 $\approx$ 85% if the prime numbers within $[2, 100]$ are taken into account, in addition.

5. Summary and Outlook

The primary aim of this paper was to recognize whether, for a natural number $x : \sqrt[n]{x} \notin \mathbf{Q}\ \forall\ n \in \mathbf{N}_{\geq 2}$, i.e., $x \in X_{\text{irrat\_roots}}$, and taking this decision only by means of looking at the basic properties of x (in particular, at the value of the last two digits and the sum of the digits of x). Very pleasingly, we are able to decide correctly in more than 50% of all cases whether a natural number x, chosen arbitrarily, is an element of $X_{\text{irrat\_roots}}$. So, even without the knowledge of the prime factorization of x, extremely simple tests are available which allow us – for the majority of natural numbers – to determine correctly whether $x \in X_{\text{irrat\_roots}}$. In addition to the possibilities resulting from the tests suggested in Section 2, the success probability in recognizing the elements of $X_{\text{irrat\_roots}}$ in the interval $[2, c]$ can be increased even more if knowledge about the prime numbers is available (e.g., those ones being part of the interval $[2, c]$).

To the best of our knowledge, the tests presented by us are the simplest and most elementary ones published up to now to determine natural numbers the n-th roots of which are irrational numbers $\forall\ n \in \mathbf{N}_{\geq 2}$.

The reason why it has been unnecessary in this paper to discuss the question “for which $x \in \mathbf{R}^{+} \setminus \mathbf{Q}$: $\sqrt[n]{x} \notin \mathbf{Q}\ \forall\ n \in \mathbf{N}_{\geq 2}$?” results from the fact that: $x \in X_{\text{irrat\_roots}}$, $\forall\, x \in \mathbf{R}^{+} \setminus \mathbf{Q}$, as it has been proven in [2].

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