

A MATLAB SIMULATION OF BELL'S INEQUALITY VIOLATION IN QUANTUM ENTANGLEMENT

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Abstract

This paper presents a MATLAB simulation to demonstrate the violation of the Bell's inequality in quantum entanglement. While the principle has always passed the test of time in laboratories, it may be intuitively useful if we demonstrate the case through a MATLAB simulation.

1. Introduction: The Bell's Inequality Theorem

Bell's inequality theorem ([1]) asserts that, given three sets A , B and C , the number of elements in the set A less than its intersection with the set B plus the number of elements in the set B less than its intersection

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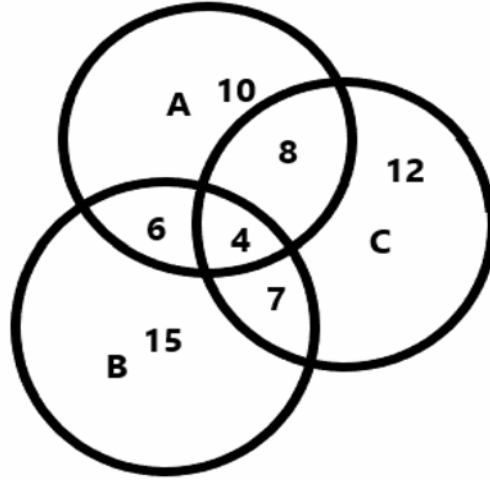
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with the set C is more than or equal to the number of elements in the set A less than its intersection with the set C .

$$N(A \setminus B) + N(B \setminus C) \geq N(A \setminus C).$$

As an example, following diagram illustrates the Bell's inequality:



$$N(A) = 10 + 6 + 4 + 8 = 28,$$

$$N(B) = 15 + 6 + 4 + 7 = 32,$$

$$N(C) = 12 + 8 + 4 + 7 = 31,$$

$$N(A \setminus B) = 10 + 8 = 18,$$

$$N(B \setminus C) = 6 + 15 = 21,$$

$$N(A \setminus C) = 10 + 6 = 16,$$

$$N(A \setminus B) + N(B \setminus C) \geq N(A \setminus C),$$

$$18 + 21 \geq 16.$$

A visual inspection of any such Venn diagram provides sufficient proof of the Bell's inequality. Though the theorem seems to be mathematically elementary, it is however an important theorem with

profound implication in quantum theory that merits a more formal proof. Giving a formal proof of the important Bell's inequality theorem provides a more rigorous explanation of the theorem.

2. A Set Theoretic Proof of the Bell's Inequality Theorem

Suppose we have three sets of events A , B and C . We have to prove that the number of times that event A occurs less than the number of times that events A and B occur together, plus the number of times that event B occurs less than the number of times that events B and C occur together is more than or equal to the number of times that event A occurs less than the number of times that events A and C occur together. In set notation, we have to prove the following:

$$N(A \setminus B) + N(B \setminus C) \geq N(A \setminus C).$$

We proceed algorithmically as below.

Step 1: We know the following equalities are true:

$$N(A \setminus B) = N(A) - N(A \cap B) = N(A) - N(B \cap A).$$

$$N(B \setminus C) = N(B) - N(B \cap C) = N(B) - N(C \cap B),$$

$$N(A \setminus C) = N(A) - N(A \cap C) = N(A) - N(C \cap A).$$

Step 2: Following is also true.

$$N(B) \geq N(B \cap A) + N(B \cap C) - N(A \cap B \cap C).$$

Step 3: Following is also true.

$$N(A \cap C) \geq N(B \cap A \cap C).$$

Step 4: We can replace $N(B \cap A \cap C)$ with $N(A \cap C)$ in the Step 2 inequality.

Since $N(A \cap C)$ has a bigger or an equal magnitude than $N(B \cap A \cap C)$ the inequality remains unchanged. Hence:

$$N(B) \geq N(B \cap A) + N(B \cap C) - N(A \cap C).$$

Step 5: $N(A)$ is added to both sides of the inequality. The inequality remains unchanged.

$$N(A) + N(B) \geq N(A) + N(B \cap A) + N(B \cap C) - N(A \cap C).$$

Step 6: The terms $N(B \cap A)$ and $N(B \cap C)$ are moved to the left side of the inequality.

$$N(A) - N(B \cap A) + N(B) - N(B \cap C) \geq N(A) - N(A \cap C).$$

Step 7: Using the equality equations in Step1 the above inequality is modified to the following inequality:

$$N(A \setminus B) + N(B \setminus C) \geq N(A \setminus C).$$

Or in probability terms:

$$P(A \setminus B) + P(B \setminus C) \geq P(A \setminus C).$$

The Bell's inequality is hence proved.

Above indicates that this inequality holds no matter how the numbers are assigned in the Venn diagram. However, there is an underlying assumption in deploying the Venn diagram, that the relation of events A , B and C is local and static. If the relation of the events is not local, as in quantum entanglement, one may not be able to depict the numbers pertaining to the events and their intersections in the Venn diagram to start with, and the Bell's inequality may be violated.

Bell suggested that violation of this inequality in quantum entanglement disputes the presence of hidden variable and locality and hence validates the spooky action at distance as contested by Einstein ([2]), a suggestion that was proved after he sadly passed away.

3. Entanglement and Bell's Inequality

Suppose a stream of paired particles, with opposite spins is generated at a point. At each generation of paired particles, one particle is sent to Alice and the paired one to Bob, far away. Alice measures the spin of each particle she receives, in one of the three directions $A1$, $A2$ or $A3$ randomly with equal probability, with a 120 degrees angle between each two directions. Bob, subsequently, after receipt of his particles, measure their spin randomly again with equal probability in one of the three directions $B1$, $B2$ or $B3$, all having orientations as $A1$, $A2$ or $A3$.

Let $P(Ai)$ mean the probability of the direction Ai being observed by Alice, $i = 1$ to 3, $P(Bk)$ the probability of direction k being observed by Bob, $k = 1$ to 3, and $P(Ai, Bk)$ the probability of the directions i being observed by Alice and k being observed simultaneously by Bob.

We have the following.

$$P(Ai) = 1/3,$$

$$P(Bk) = 1/3,$$

$$P(Ai, Bk) = (1/3 * 1/3) = 1/9.$$

Let $P(U|Ai)$ means the probability of the spin observed by Alice on the direction Ai be "up" and $P(D|Ai)$ means the probability of the spin observed by Alice on the direction Ai be "down". Let also $P(U|Bk)$ means the probability of the spin observed by Bob on the direction Bk be "up" and $P(D|Bk)$ means the probability of the spin observed by Bob on the direction Bk be "down".

$$P(U|Ai) + P(D|Ai) = 1,$$

$$P(U|Bk) + P(D|Bk) = 1.$$

We can envisage two scenarios.

Scenario 1: No spooky action at a distance. Presence of locality and hidden variables.

Any particle generated is determined to fall under any one of the following types. (Adopted from the Brian Greene's ([3]) YouTube program "Your Daily Equation #21)

	$P(U A1)$,	$P(U A2)$,	$P(U A3)$	$P(D B1)$,	$P(D B2)$,	$P(D B3)$
Type 1:	[1	1	1]	[1	1	1]
Type 2:	[1	0	0]	[1	0	0]
Type 3:	[1	1	0]	[1	1	0]
Type 4:	[0	1	1]	[0	1	1]
Type 5:	[0	0	1]	[0	0	1]
Type 6:	[0	1	0]	[0	1	0]
Type 7:	[1	0	1]	[1	0	1]
Type 8:	[0	0	0]	[0	0	0]

Since Alice and Bob choose any of the three directions to measure the particles randomly, the number of cases on average where the pairs of particles measured by Alice and Bob have opposite or same orientations would be as follows:

Type 1: Opposite orientation 9, Same orientation 0,

Type 2: Opposite orientation 5, Same orientation 4,

Type 3: Opposite orientation 5, Same orientation 4,

Type 4: Opposite orientation 5, Same orientation 4,

Type 5: Opposite orientation 5, Same orientation 4,

Type 6: Opposite orientation 5, Same orientation 4,

Type 7: Opposite orientation 5, Same orientation 4,

Type 8: Opposite orientation 9, Same orientation 0.

Putting the Type 1 and Type 8 particles where all cases are of opposite orientation aside, we can see that at least 5 out of 9 cases are of opposite orientations. If the stream of particles generated and observed are very large we may safely assume that at least in 5/9 of cases the particles measured by Alice and Bob have opposite orientation.

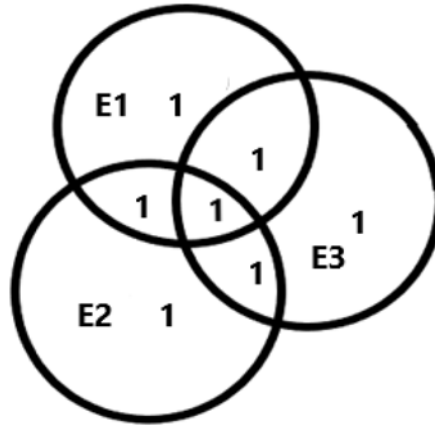
To see if the Bell's inequality is prevailing here, let's define the following sets of events for Alice. We could do the same for Bob.

E1: The set of types were the *A1* orientation is *U*.

E2: The set of types were the *A2* orientation is *U*.

E3: The set of types were the *A3* orientation is *U*.

In case we have one of each segment we can construct the following Venn diagram:



We can see that the Bell's inequality prevails here.

$$N(E1 \setminus E2) + N(E2 \setminus E3) >= N(E1 \setminus E3),$$

$$N(E1) = 4,$$

$$N(E2) = 4,$$

$$N(E3) = 4,$$

$$N(E1 \setminus E2) = 2,$$

$$N(E2 \setminus E3) = 2,$$

$$N(E1 \setminus E3) = 2,$$

$$2 + 2 > 2.$$

We can construct similar Venn diagram for any number of each type that we may have. Only the entries in the Venn diagram will change. The Bell's inequality would again prevail.

It consequently follows that if the number of cases, measured by Alice and Bob, having opposite orientations falls below the $5/9$ fraction, the Bell's inequality is violated, indicating the non-locality of the entanglement phenomena.

Scenario 2: Spooky action at a distance. Non-locality.

Alice randomly selects a direction to measure each particle she receives at her location with probability $P(Ai) = 1/3$. Bob also randomly selects a direction to measure the particle he receives at his location with probability $P(Bk) = 1/3$. There would be two cases.

If the directions randomly selected by Alice and Bob happens to be the same, the orientation of the particles they measure would be opposite. Otherwise, because of the quantum entanglement effect, it would be a function of the angle between the directions they have chosen, i.e., 120 degrees, with the probability being equal to $(\cos 120 \text{ deg})^2 = (1/2)^2 = 1/4$.

Case 1: Alica and Bob happen to randomly select same measurement direction. $P(i = k) = 3/9$.

$$P[(U|Ai) \text{ and } (D|Bk)] = 1.$$

Case 2: Alica and Bob happen to randomly select different measurement directions. $P(i \neq k) = 6/9$.

$$P[(U|Ai) \text{ and } (D|Bk)] = (\cos 120 \text{ deg})^2 = 1/4.$$

Hence in total:

$$P[(U|Ai) \text{ and } (D|Bk)] = (3/9) \cdot 1 + (6/9) \cdot 1/4 = 1/2.$$

Comparing the two scenarios, we can conclude that if the assumption of locality and the presence of hidden variables is valid we shall experience opposite orientations of spins measured by Alice and Bob in at least 5/9 of the cases. On the other hand, if the assumption of locality and the presence of hidden variables is not valid this ratio will be less than 5/9, i.e., 1/2 on average. This will refute the assumption of locality and the presence of hidden variables and proves the non-locality of quantum entanglement, which has been indeed frequently ratified in laboratory experiments. Following MATLAB simulation was carried out to illustrate the phenomena.

3. MATLAB Simulation

Scenario 1: locality and presence of hidden variables

Number of particle pairs generated and observed by Alice and Bob: 1000,000.

$$P(Ai) = 1/3, P(Bk) = 1/3.$$

A measurement is counted as one success whenever the orientations of the particles observed by Alice and Bob were of opposite orientations. Measurements follow one of the eight types discussed above. Only particle types 2 to 7 results are counted in the simulation. Hence $P(\text{type}) = 1/6$. Measurements by Alice and Bob would have been certainly opposite and always counted as success when observing Types 1 and 8. Hence the total success count of types 2 to 7 can be taken as the minimum success count figure.

MATLAB simulation success results

$$N(a1b1) = 110815$$

$$N(a1b2) = 37313$$

$$N(a1b3) = 36880$$

$$N(a2b1) = 37136$$

$$N(a2b2) = 111335$$

$$N(a2b3) = 36944$$

$$N(a3b1) = 36682$$

$$N(a3b2) = 37109$$

$$N(a3b3) = 111407$$

Total success number = 555621.

Scenario 2: Non-locality

Number of particle pairs generated and observed by Alice and Bob:
1000,000

$$P(A_i) = 1/3, P(B_k) = 1/3.$$

A measurement is counted as one success whenever the orientations of the particles observed by Alice and Bob were of opposite orientation.

MATLAB simulation success results

$$N(a1b1) = 110493$$

$$N(a1b2) = 27588$$

$$N(a1b3) = 27827$$

$$N(a2b1) = 27700$$

$$N(a2b2) = 111227$$

$$N(a2b3) = 27395$$

$$N(a3b1) = 27864$$

$$N(a3b2) = 28026$$

$$N(a3b3) = 111461$$

Total success number = 499681.

Comparison

Total number of successes, only counting types 2 to 7, when assuming locality = 555621.

Total number of successes when assuming non-locality = 499681 < 555621.

Bell's inequality is hence violated proving the non-locality in entanglement.

4. Conclusion

The paper first presented a set theoretic proof of the Bell's Inequality. The violation of the Bell's Inequality in quantum entanglement was then discussed and subsequently demonstrated through a MATLAB simulation.

References

- [1] J. Bell, On the Einstein–Podolsky–Rosen Paradox, *Physics* 1(3) (1964), 195-200.
- [2] A. Einstein, B. Podolsky and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, *Physical Review* 47(10) (1935), 777-780.
DOI: <https://doi.org/10.1103/PhysRev.47.777>
- [3] B. Greene, YouTube program “Your Daily Equation #21”, (2020).
- [4] L. Susskind and Art Friedman, *Quantum Mechanics: The Theoretical Minimum*, Basic Books: New York, 2014.



Appendix

MATLAB script for the first scenario: Locality and hidden variables

```
%Simulating Quantum Entanglement
%Locality and hidden variables
%Alice and Bob randomly check observed states of particles
in three
%directions at their respective local positions
a1b1=0;
a1b2=0;
a1b3=0;
a2b1=0;
a2b2=0;
a2b3=0;
a3b1=0;
a3b2=0;
a3b3=0;
for n=1:1000000
    t=randperm(6);
    if t(1)==1;
        a=randperm(3);
        if a(1)==1;
            b=randperm(3);
            if b(1)==1;
                a1b1=a1b1+1;
            end
        end
    end
    if a(1)==2;
```

```
b=randperm(3);
if b(1)==2;
a2b2=a2b2+1;
end
if b(1)==3;
a2b3=a2b3+1;
end
end
if a(1)==3;
b=randperm(3);
if b(1)==2;
a3b2=a3b2+1;
end
if b(1)==3;
a3b3=a3b3+1;
end
end
end
if t(1)==2;
a=randperm(3);
if a(1)==1;
b=randperm(3);
if b(1)==1;
a1b1=a1b1+1;
end
if b(1)==2;
a1b2=a1b2+1;
end
end
```

```
end
if a(1)==2;
    b=randperm(3);
    if b(1)==1;
        a2b1=a2b1+1;
    end
    if b(1)==2;
        a2b2=a2b2+1;
    end
end
if a(1)==3;
    b=randperm(3);
    if b(1)==3;
        a3b3=a3b3+1;
    end
end
end
if t(1)==3 ;
    a=randperm(3);
    if a(1)==1;
        b=randperm(3);
        if b(1)==1;
            a1b1=a1b1+1;
        end
    end
end
if a(1)==2;
    b=randperm(3);
    if b(1)==2;
```

```
a2b2=a2b2+1;
end
if b(1)==3;
a2b3=a2b3+1;
end
end
if a(1)==3;
b=randperm(3);
if b(1)==2;
a3b2=a3b2+1;
end
if b(1)==3;
a3b3=a3b3+1;
end
end
end
if t(1)==4;
a=randperm(3);
if a(1)==1;
b=randperm(3);
if b(1)==1;
a1b1=a1b1+1;
end
if b(1)==2;
a1b2=a1b2+1;
end
end
end
if a(1)==2;
```

```
b=randperm(3);
if b(1)==1;
a2b1=a2b1+1;
end
if b(1)==2;
a2b2=a2b2+1;
end
end
if a(1)==3;
b=randperm(3);
if b(1)==3;
a3b3=a3b3+1;
end
end
end
if t(1)==5;
a=randperm(3);
if a(1)==1;
b=randperm(3);
if b(1)==1;
a1b1=a1b1+1;
end
if b(1)==3;
a1b3=a1b3+1;
end
end
end
if a(1)==2;
b=randperm(3);
```



```
        if b(1)==2;
            a2b2=a2b2+1;
        end
    end
end
if a(1)==3;
    b=randperm(3);
    if b(1)==1;
        a3b1=a3b1+1;
    end
    if b(1)==3;
        a3b3=a3b3+1;
    end
end
end
end
if t(1)==6
    a=randperm(3);
    if a(1)==1;
        b=randperm(3);
        if b(1)==1;
            a1b1=a1b1+1;
        end
        if b(1)==3;
            a1b3=a1b3+1;
        end
    end
end
end
if a(1)==2;
    b=randperm(3);
    if b(1)==2;
```

```
a2b2=a2b2+1;
end
end
if a(1)==3;
    b=randperm(3);
    if b(1)==1;
        a3b1=a3b1+1;
    end
    if b(1)==3;
        a3b3=a3b3+1;
    end
end
end
end
end
Sucesss= a1b1+a1b2+a1b3+a2b1+a2b2+a2b3+a3b1+a3b2+a3b3
a1b1
a1b2
a1b3
a2b1
a2b2
a2b3
a3b1
a3b2
a3b3
Sucesss
```

MATLAB script for the second scenario: Non-locality

```
%Simulating Quantum Entanglement
%Non-locality
%Alice and Bob randomly check observed states of particles
in three
%directions at their respective local positions
a1b1=0;
a1b2=0;
a1b3=0;
a2b1=0;
a2b2=0;
a2b3=0;
a3b1=0;
a3b2=0;
a3b3=0;
for n=1:1000000
    a=randperm(3);
    if a(1)==1;
        b=randperm(3);
        if b(1)==1;
            a1b1=a1b1+1;
        end
        if b(1)==2;
            p=randperm(4);
            if p(1)==1;
                a1b2=a1b2+1;
            end
        end
    end
end
```

```
if b(1)==3;
    p= randperm(4);
    if p(1)==1;
        a1b3=a1b3+1;
    end
end
end
if a(1)==2;
    b=randperm(3);
    if b(1)==2;
        a2b2=a2b2+1;
    end
    if b(1)==1;
        p= randperm(4);
        if p(1)==1;
            a2b1=a2b1+1;
        end
    end
end
if b(1)==3;
    p= randperm(4);
    if p(1)==1;
        a2b3=a2b3+1;
    end
end
end
if a(1)==3;
    b=randperm(3);
    if b(1)==3;
```

