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SOME CHARACTERIZATIONS OF LEFT(RIGHT) HOM-ALTERNATIVE ALGEBRAS

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Abstract

Some characterizations of left(or right) Hom-alternative algebras are given. Using a Hom-Bol algebra structure on a left(or right) Hom-alternative algebra, a necessary and sufficient condition for Hom-Malcev admissibility of a left(or right) Hom-alternative algebra is obtained.

1. Introduction

The theory of Hom-algebras originated from Hom-Lie algebras introduced by J.T. Hartwig, D. Larsson, and S.D. Silvestrov in [6] in the study of quasi-deformations of Lie algebras of vector fields,

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including q-deformations of Witt algebras and Virasoro algebras. An elementary but important property of Lie algebras is that each associative algebra A gives rise to a Lie algebra Lie(A) via the commutator bracket. In [12], Makhlouf and Silvestrov introduced the notion of a Hom-associative algebra (A, μ, α) and they proved that these Hom-algebras give rise to a Hom-Lie algebra HLie(A) via the commutator bracket. Since then, other types of Hom-algebras such are Hom-alternative algebras, Hom-Malcev algebras, Hom-Novikov algebras, · · ·, are introduced in the literarture.

Recall that a Hom-alternative algebra is a Hom-algebra whose Hom-associator is an alternating function . In particular, all Hom-associative algebras are Hom-alternative, but there is a non-Hom-associative Hom-alternative algebra [2]. Hom-lternative algebras are related to Hom-Maltsev algebras as Hom-associative algebras are related to Hom-Lie algebras. Indeed, the commutator Hom-algebra A^- of any Hom-alternative algebra is a Hom-Maltsev algebra i.e., every Hom-alternative algebra A is Hom-Maltsev-admissible.

This work is devoted to the study of some properties of Hom-alternative algebras. Based on the link between left (right) Hom-alternative and Hom-Bol algebras, we obtain some properties which allow to deduce a necessary and sufficient condition for Hom-Malcev admissibility of a left(or right) Hom-alternative algebra.

The rest of the paper is organised as follows. In section two, we recall some facts about Hom-algebras. Section three is devoted to the main results of this paper. Throughout this paper, all vector spaces and algebras are meant over a ground field K of characteristic 0.

2. Some basis on Hom-algebras

In this paper, we only consider multiplicative Hom-algebras.

Definition 2.1. [15]

(1) *A Hom-module is a pair (V, α) consisting of a module V and a*

linear endomorphism α ∈ End(V). A morphism f : (V, α) → (V ', α') of

Hom-modules is a linear map $f : V \to V$ *such that* $\alpha' \circ f = f \circ \alpha$.

(2) A Hom-algebra is a triple (A, μ, α) *in which* (A, α) *is a Hom-module and* $\mu : A^{\otimes 2} \to A$ *is a bilinear map (the multiplication).*

(3) A morphism $f : (V, \mu, \alpha) \to (V', \mu', \alpha')$ *of Hom-algebras is a*

morphism of underlying Hom-modules $f : (V, \alpha) \to (V', \alpha')$ *such that* $f \circ \mu = \mu' \circ f^{\otimes 2}$.

The binary Hom-algebras which we need are Hom-alternative al-

gebras and Hom-Malcev algebras. In fact Hom-Malcev algebras are

introduced in [16] to study Hom-alternative algebras studied in [11].

Definition 2.2. *A Hom-algebra* (A, μ, α) *is said*

(1) a left Hom-alternative algebra if

$$asa(x, x, y) = 0 \text{ for all } x, y, z \in A, \qquad (1)$$

(2) a right Hom-alternative algebra if

$$asa(x, y, y) = 0 \text{ for all } x, y, z \in A, \qquad (2)$$

(3) a Hom-alternative algebra if (1) and (2) hold where

$asa(x, y, z) := \mu(\mu(x, y), \alpha(a)) - \mu(\alpha(x), \mu(y, z))$ *is the Hom-associator of*

(A, μ, α).

Proposition 2.3. *Let* (A, μ, α) *be a Hom-algebra. Then,* (A, μ, α) *is left Hom-alternative algebra if and only if* $A^{op} := (A, \mu^{op}, \alpha)$ *is a right Hom-alternative algebra where*

$$\mu^{op}(x, y) := \mu(y, x) \,. \qquad (3)$$

Proof. *Let* (A, μ, α) *be a Hom-algebra and* $x, y, z \in A$. *Then*

$asa^{op}(x, y, z) := \mu^{op}(\mu^{op}(x, y), \alpha(z)) - u^{op}(\alpha(x), \mu^{op}(y, z)) = \mu(\alpha(z), \mu(y, x)) - \mu(\mu(z, y), \alpha(x)) = asa(z, y, x)$.

Hence, $asa^{op}(x, x, z) = 0 \Longleftrightarrow asa(z, x, x) = 0$.

Remark 2.4. *By Proposition 2.3, we observe that all statements for* *left Hom-alternative algebras have their corresponding statements for right Hom-alternative algebras and conversely.*

Example 2.5. *(1) Let* (A, μ) *be an alternative algebra and* α *be a morphism of* (A, μ)*. Then* $A_\alpha := (A, \mu_\alpha := \alpha \circ \mu, \alpha)$ *is a Hom-alternative algebra called the twisted of* (A, μ)*.*

(2) The octonions algebra O also called Cayley Octaves or Cayley algebra is 8-dimensional with a basis $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$*, where* e_0 *is the identity for the multiplication. This algebra is twisted into the eight-dimensional Hom-alternative algebra* $O_\alpha = (O, \mu_1, \alpha)$ *[16] with the same basis* $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ *where* $\alpha(e_0) = e_0$, $\alpha(e_1) = e_5$, $\alpha(e_2) = e_6$, $\alpha(e_3) = e_7$, $\alpha(e_4) = e_1$, $\alpha(e_5) = e_2$, $\alpha(e_6) = e_3$, $\alpha(e_7) = e_4$ *and the multiplication table is:*

μ_1	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_5	e_6	e_7	e_1	e_2	e_3	e_4
e_1	e_5	$-e_0$	e_1	e_4	$-e_6$	e_3	$-e_2$	$-e_7$
e_2	e_6	$-e_1$	$-e_0$	e_2	e_5	$-e_7$	e_4	$-e_3$
e_3	e_7	$-e_4$	$-e_2$	$-e_0$	e_3	e_6	$-e_1$	e_5
e_4	e_1	e_6	$-e_5$	$-e_3$	$-e_0$	e_4	e_7	$-e_2$
e_5	e_2	$-e_3$	e_6	$-e_6$	$-e_4$	$-e_0$	e_5	e_1
e_6	e_3	e_2	$-e_4$	e_1	$-e_7$	$-e_5$	$-e_0$	e_6
e_7	e_4	e_7	e_3	$-e_5$	e_2	$-e_1$	$-e_6$	$-e_0$

and into the eight-dimensional Hom-alternative algebra $O_\beta = (O, \mu_2, \beta)$ *[11] with the same basis* $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ *where* $\beta(e_i) = -e_i$ *for all* $i \in \{0, \cdots, 7\}$ *and the multiplication table is:*

μ_2	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	$-e_1$	$-e_2$	$-e_3$	e_4	e_5	$-e_6$	e_7
e_1	$-e_1$	$-e_0$	e_4	e_7	e_2	$-e_6$	$-e_5$	e_3
e_2	$-e_2$	$-e_4$	$-e_0$	e_5	$-e_1$	e_3	e_7	e_6
e_3	$-e_3$	$-e_7$	$-e_5$	$-e_0$	$-e_6$	$-e_2$	$-e_4$	$-e_1$
e_4	e_4	$-e_2$	e_1	e_6	$-e_0$	e_7	$-e_3$	$-e_5$
e_5	e_5	e_6	$-e_3$	e_2	$-e_7$	$-e_0$	$-e_1$	e_4
e_6	$-e_6$	e_5	$-e_7$	e_4	e_3	e_1	$-e_0$	$-e_2$
e_7	e_7	$-e_3$	$-e_6$	e_1	e_5	$-e_4$	e_2	$-e_0$

Nor O_α , neither O_β , are alternative algebras. Moreover, both α and β are automorphisms of O.

Definition 2.6. *Let (A, [,], α) be a Hom-algebra.*

(1) The Hom-Jacobian of A is the the trilinear map $J\alpha : A^{\otimes 3} \to A$ defined by

$$J\alpha (x, y, z) := \sigma[[x, y], \alpha(z)] \qquad (4)$$

where σ designates the sum over cyclic permutation of x, y, z.

(2) A Hom-algebra (A, [,], α) such that [,] is skew-symmetric is said to be

(a) a Hom-Lie algebra if the following identity

$$J\alpha (x, y, z)= 0, \qquad (5)$$

called the Hom-Jacobi identity holds for all $x, y, z \in A$.

(b) a Hom-Malcev algebra if the so-called following Hom-Malcev identity

$$J\alpha(\alpha(x), \alpha(y), [x, z]) = [J\alpha (x, y, z), \alpha^2(x)], \qquad (6)$$

holds for all $x, y, z \in A$.

Example 2.7. Let $(M, [,], \alpha)$, *be a Hom-algebra where non-zero products in a basis* (e_1, e_2, e_3, e_4) *are given by* $[e_1, e_2] = -e_2 = -[e_2, e_1]$, $[e_1, e_3] = -e_3 = [e_3, e_1]$, $[e_1, e_4] = e_4 = -[e_4, e_1]$, $[e_2, e_3] = 2e_4 = -[e_3, e_2]$ *and* α *is defined as* $\alpha(e_1) = e_1 + e_4$, $\alpha(e_2) = e_2 + e_3$, $\alpha(e_3) = e_3$, $\alpha(e_4) = e_4$. *Then* $(M, [,], \alpha)$ *is a Hom-Malcev algebra.*

Equivalent to (6) defining identities of Hom-Malcev algebras are found in [9] and [16] where, in particular, it is shown [9] that in any anti-commutative Hom-algebra $(A, [,], \alpha)$, the Hom-Malcev identity (6) is equivalent to

$$J\alpha(\alpha(x), \alpha(y), [u, v]) = [\alpha^2(u), J\alpha(x, y, v)] + [J\alpha(x, y, u), \alpha^2(v)] - 2J\alpha(\alpha(u), \alpha(v), [x, y]) \quad (7)$$

3. Characterizations

In this section, we prove the main results of this paper. After con-sidering the Hom-Bol structure on any left (right) Hom-alternative algebras, some properties of these Hom-algebras are obtained.

First, observe that Hom-algebras mentioned in the previous section are binary Hom-algebras. The first generalization of binary algebras was the ternary algebras introduced in [10]. Ternary algebraic structures also appeared in various domains of theoretical and mathematical physics (see, e.g., [14]). Likewise, binary Hom-algebras are generalized to n-ary Hom-algebra structures in [1] (see also [17]) and by binary-ternary Hom-algebras [8], [5], [3]. The class of binary-ternary Hom-algebras which that is of interest in our setting is the one of Hom-Bol algebras defined [3].

Definition 3.1. *A Hom-Bol algebra is a quadruple* $(B, [,], [,,], \alpha)$ *in which B is a K-module,* $[,] : B^{\otimes 2} \rightarrow B$ *is a bilinear map,* $[,,] : B^{\otimes 3} \rightarrow B$ *is a trilinear map such that:*

(HB1) $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, (HB2) $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$,

(HB3) $[x, y] = -[y, x]$,

(HB4) $[x, y, z] = -[y, x, z]$,

(HB5) $\sigma[x, y, z] = 0$,

(HB6) $[\alpha^2(x), \alpha^2(y), [u, v, w]] = [[x, y, u], \alpha^2(v), \alpha^2(w)]$

$+ [\alpha^2(u), [x, y, v], \alpha^2(w)] + [\alpha^2(u), \alpha^2(v), [x, y, w]]$,

(HB7) $[\alpha(x), \alpha(y), [u, v]] = [\alpha^2(u), [x, y, v]] + [[x, y, u], \alpha^2(v)] +$

$+ [\alpha(u), \alpha(v), [x, y]] - \alpha([[u, v], [x, y]])$.

Example 3.2. *Let $(B, [,], [, ,], \alpha)$ be a binary-ternary Hom-algebra where $[,]$ is skew-symmetric, $[, ,]$ is left skew-symmetric and non-zero products on a given basis (e_1, e_2, e_3, e_4) are $[e_1, e_2] = -b_2 e_2 - b_3e_3 - a_3b_2e_4$, $[e_1, e_3] = -ce_3$, $[e_1, e_4] = b_2 ce_4$, $[e_2, e_3] = 2b_2 ce_4$, $[e_1, e_2, e_1] = -b_2^2 e_2 - (b_2 b_3 + cb_3)e_3 - (b_2^2a_3 + b_2^2a_3 c)e_4$, $[e_1, e_3, e_1] = -c^2e_3$, $[e_1, e_4, e_1] = -b_2^2c^2e_4$ and the linear map α is $\alpha(e_1) = e_1 + a_3 e_3 + a_4e_4$, $\alpha(e_2) = b_2 e_2 + b_3 e_3 + a_3 b_2 e_4$, $\alpha(e_3) = ce_3$, $\alpha(e_4) = b_2ce_4$, with $a_3, a_4, b_2, b_3, c \in R$. Then, $(B, [,], [, ,], \alpha)$ is a Hom-Bol algebra.*

In [3], it is proved that Hom-Bol algebras are closed by self-morphism and therefore, method for constructing Hom-Bol algebras from either Bol algebras or Malcev algebras are given. Using Corolloray 3.3 in [3], we get the following.

Proposition 3.3. *Let $(A, \cdot, \alpha)$ be a left (or a right) alternative algebras and $\alpha \in End(A)$. Then $(A, [,]', [, ,]', \alpha)$ is a Hom-Bol algebra where $[x, y]' = \alpha(x \cdot y - y \cdot x)$ and the ternary operation is given by*

$$[x, y, z]' = \alpha^2 (y \circ (z \circ x) - y \circ (z \circ x)) \forall x, y, z \in A, \quad (8)$$

where $x \circ y = x \cdot y + y \cdot x$.

Proof. *If consider on A, the commutator $[x, y] := x \cdot y - y \cdot x$ and the ternary operation $[x, y, z] = y \circ (z \circ x) - y \circ (z \circ x)$, $\forall x, y, z \in A$, then $(A, [,],$*

$[,\,,\,])$ is a Bol algebra [13], [7]. Moreover, since α is a morphism of $(A, \cdot)$, we have $\alpha([x, y]) = \alpha(x) \cdot \alpha(y) - \alpha(y) \cdot \alpha(x) = [\alpha(x), \alpha(y)]$ and $\alpha([x, y, z]) = \alpha(y) \circ(\alpha(z) \circ \alpha(x)) - \alpha(y) \circ (\alpha(z) \circ \alpha(x)) = [\alpha(x), \alpha(y), \alpha(z)]$. Hence, α is also a morphism of $(A, [,\,], [,\,,\,])$. Therefore, Corollary 3.3 in [3] implies that $(A, [,\,]', [,\,,\,]', \alpha)$ is a Hom-Bol algebra.

In [4], we also know that Hom-Bol algebra can be obtained from left(or right) Hom-alternative algebras and a Hom-Malcev algebra. More precisely, consider in any Hom-Malcev algebra $(A, [,\,], \alpha)$, the ternary operation by

$$[x, y, z] =(1/3)(2[[x, y], \alpha(z)] - [[y, z], \alpha(x)] - [[z, x], \alpha(y)]) \tag{9}$$

or equivalently,

$$[x, y, z] = -(1/3)J\alpha(x, y, z) + [[x, y], \alpha(z)]. \tag{10}$$

Then, we get the following:

Proposition 3.4. *[4] Let $(A, [,\,], \alpha)$ be a Hom-Malcev algebra. Then*

$(A, [,\,], [,\,,\,], \alpha)$ is a Hom-Bol algebra where $[,\,,\,]$ is defined as (9).

Example 3.5. *Consider the Hom-Malcev algebra $(M, [,\,], \alpha)$ given in Example 2.7 and define the ternary operation $[,\,,\,]$ on M as in (9). Then, $(M, [,\,], [,\,,\,]\alpha)$ is a Hom-Bol algebra where the non-zero products for $[,\,,\,]$ are given by $[e_1, e_2, e_1] = -e_2 - 2e_3= -[e_2, e_1, e_1]$, $[e_1, e_3, e_1] = -e_3 = -[e_3, e_1, e_1]$, $[e_1, e_4, e_1] = -e_4 = -[e_4, e_1, e_1]$.*

The main purpose of the following result is to show the converse of Proposition 3.4 i.e., when a Hom-Bol algebra reduces to a Hom-Malcev algebra. This consideration is based on ternary operation ”[, ,]” of a given Hom-Bol algebra $(B, [,\,], [,\,,\,], \alpha)$ that could be expressed throughout its binary operation ”[,]” as (9).

Proposition 3.6. *Let $(B, [,\,], [,\,,\,], \alpha)$ be a Hom-Bol algebra such* *that its ternary operation is expressed throughout its binary operation "[,]" as (9), for all $x, y, z \in B$. Then $(B, [,], \alpha)$ is a Hom-Malcev algebra.*

Proof. *Since [,] is skew-symmetric by definition and α is a morphism of $(B, [,], [,,], \alpha)$, it is also a morphism of $(B, [,], \alpha)$. It is therefore enough to prove the Hom-Malcev identity (6) or equivalently (7). Using (10) we have:*

$$J\alpha(x, y, z) = -3[x, y, z] + 3[[x, y], \alpha(z)]$$

and thus,

$$J\alpha(\alpha(x), \alpha(y), [u, v])$$

$$= -3[\alpha(x), \alpha(y), [u, v]] + 3[[\alpha(x), \alpha(y)], \alpha([u, v])]$$

$$= -3[[\alpha^2(u), [x, y; v]] - 3[[x, y; u], \alpha^2(v)] - 3[\alpha(u), \alpha(v), [x, y]]$$

$$-3[\alpha([x, y]), \alpha([u, v]) + 3[[\alpha(x), \alpha(y)], \alpha([u, v])](\text{ by } (HB7))$$

$$= -3[[\alpha^2(u), -(1/3)J\alpha(x, y, v) + [[x, y], \alpha(v)]] +$$

$$(1/3)[-J\alpha(x, y, u) + [[x, y], \alpha(u)], \alpha^2(v)]$$

$$-3(-(1/3)J\alpha(\alpha(u), \alpha(v), [x, y]) + [[\alpha(u), \alpha(v)], \alpha([x, y])])$$

$$+3[[\alpha(x), \alpha(y)], \alpha([u, v])] = [\alpha^2(u), J\alpha(x, y, v)]$$

$$+[J\alpha(x, y, u), \alpha^2(v)] + J\alpha(\alpha(u), \alpha(v), [x, y])$$

$$-3([[\alpha(u), \alpha(v)], \alpha([x, y])] + [[\alpha(v), [x, y]), \alpha^2(u)]$$

$$+[[x, y], [\alpha(u), \alpha^2(v)])$$

$$= [\alpha^2(u), J\alpha(x, y, v)] + [J\alpha(x, y, u), \alpha^2(v)]$$

$$+J\alpha(\alpha(u), \alpha(v), [x, y]) - 3J\alpha(\alpha(u), \alpha(v), [x, y])$$

$$= [\alpha^2(u), J\alpha(x, y, v)] + [J\alpha(x, y, u), \alpha^2(v)] - 2J\alpha(\alpha(u), \alpha(v), [x, y]).$$

Thus,

$$J\alpha(\alpha(x), \alpha(y), [u, v]) = [\alpha^2(u), J\alpha(x, y, v)]$$

$$+[J\alpha(x, y, u), \alpha^2(v)] - 2J\alpha(\alpha(u), \alpha(v), [x, y]).$$

Hence, we get the result by (7).

Theorem 3.7. *[4] Let $(A, \cdot, \alpha)$ be a left (or right) Hom-alternative algebra. Then $(A, [,], [,,], \alpha)$ is a Hom-Bol algebra where $[,]$ is the commutator and $[,,]$ is the ternary operation defined by*

$$[x, y, z] = as^J(y, z, x) \qquad (11)$$

where $As^J(x, y, z) := (x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z)$ is the Hom-associator of Jordan type defined in [4] with $x \circ y = x \cdot y + y \cdot x$. The ternary operation (11) very useful in this paper, is equivalent in the left Hom-alternative algebras case to

$$[x, y, z] = [[x, y], \alpha(z)] - 2as\alpha(x, y, z) \qquad (12)$$

and in the right Hom-alternative algebras case to

$$[x, y, z] = [[x, y], \alpha(z)] - 2as\alpha(z, y, x) \qquad (13)$$

(see [4], Proposition 4.3).

Lemma 3.8. *Let $(A, \cdot, \alpha)$ be a left (or right) Hom-alternative algebra. Then the following identity*

$$2as\alpha(x, y, z) + 2as\alpha(y, z, x) + 2as\alpha(z, x, y) = \sigma[[x, y], \alpha(z)] \qquad (14)$$

holds for all $x, y, z \in A$.

Proof. *It is proved [16] (Lemma 3.16.) that in any Hom-algebra $(A, \cdot, \alpha)$, the following identity*

$$\sigma[[x, y], \alpha(z)] = as\alpha(x, y, z) + as\alpha(y, z, x) + as\alpha(z, x, y)$$

$$- as\alpha(y, x, z) - as\alpha(z, y, x) - as\alpha(x, z, y) \qquad (15)$$

always holds forall $x, y, z \in A$. If furthermore, $(A, \cdot, \alpha)$ is left(or right) Hom-altenative Hom-algebra then by (15), the desired equality (14) , now follows immediately from (1) (or (2)).

A Hom-algebra $(A, \cdot, \alpha)$ is said to be Hom-Lie (resp. Hom-Malcev) admissible algebra if and only if its commutator-Hom-algebra $A^- = (A, [,], \alpha)$ is a Hom-Lie (resp. Hom-Malcev) algebra.

The following result which gives a necessary and sufficent condition for Hom-Lie-admissibility of left Hom-alternative algebra is a consequence of Lemma 3.8.

Corollary 3.9. *Let $(A, \cdot, \alpha)$ be a left(or right) Hom-alternative algebra. Then A is Hom-Lie admissible if and only if*

$$2asa(x, y, z) + 2asa(y, z, x) + 2asa(z, x, y) = 0 \qquad (16)$$

for all x, y, z in A. In particulary, any left(or right) Hom-alternative Hom-algebras on a field of characteristic two is Hom-Lie admissible.

Corollary 3.10. *Let $(A, \cdot, \alpha)$ be a Hom-alternative algebra. Then A is Hom-Lie admissible if and only if*

$$6asa(x, y, z) = 0 \qquad (17)$$

for all x, y, z in A. In particulary, any Hom-alternative Hom-algebra on a field of characteristic two or three is Hom-Lie ad-missible.

Proposition 3.11. *A left Hom-alternative algebras $(A, \cdot, \alpha)$ is a Hom-Malcev admissible if and only if*

$$asa(x, y, z) = (1/6)\sigma[[x, y], \alpha(z)]. \qquad (18)$$

In the right Hom-alternative algebras case, the identity (18) reads as

$$asa(z, y, x) = (1/6)\sigma[[x, y], \alpha(z)], \qquad (19)$$

for all $x, y, z \in A$.

Proof. *We only give the proof in left Hom-alternative case. The second case can be done similarly. Relying on Proposition 3.6, it is enough to prove that the identity (18) is equivalent to the fact that the ternary operation of the associated Hom-Bol algebra $(A, [\,,\,], [\,,\,,\,], \alpha)$ as defined in (12) has the form (9). One can observe that*

$$(18) \Leftrightarrow -2asa(x, y, z) = -(1/3)\sigma[[x, y], \alpha(z)]$$

$$\Leftrightarrow [[x, y], \alpha(z)] - 2asa(x, y, z) = -(1/3)\sigma[[x, y], \alpha(z)] + [[x, y], \alpha(z)]$$

$$\Leftrightarrow [x, y, z] = (1/3)(2[[x, y], \alpha(z)] - [[y, z], \alpha(x)] - [[z, x], \alpha(y)]),$$

and the proof follows.

By straightforward computations we obtain the following another necessary and sufficient condition for Hom-Malcev-admissibility of a left Hom-alternative algebra:

Proposition 3.12. *Let (A, ·, α) be a left Hom-alternative algebra. Then (A, ·, α) is Hom- Malcev admissible if and only if*

$$2asa([x, z], \alpha(x), \alpha(y)) + 2asa(\alpha(y), [x, z], \alpha(x))$$

$$+2asa(\alpha(x), \alpha(y), [x, z]) = [\sigma[[x, y], \alpha(z)], \alpha^2(x)] \quad (20)$$

for all x, y, z ∈ A.

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