Journal of Algebra, Number Theory: Advances and Applications
Volume 25, Number 2, 2023, Pages 47-51
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jantaa_7100122284

CORRIGENDUM TO: ZERO DIVISOR GRAPH OF A LATTICE WITH RESPECT TO AN IDEAL

MONA TAREK, M. A. SEOUD and AHMED GABER

Department of Mathematics
Faculty of Science
Ain Shams University
Cairo
Egypt
e-mail: Mona.Saad@sci.asu.edu.eg
m.a.seoud@hotmail.com
a.gaber@sci.asu.edu.eg

Abstract

In this paper, we point out several errors in Afkhami et al. [1]. Moreover, we reform many proofs in Afkhami's article.

1. Introduction

Among the results that Afkhami et al. show in [1], we state the following. For a lattice $\mathfrak{L}$, $\Gamma(\mathfrak{L})$ is defined to be the graph associates the following set of vertices:

$$\{\alpha \in \mathfrak{L};\ \alpha \wedge \beta = 0 \text{ for some non-zero element } \beta \in \mathfrak{L}\}.$$

The vertices α and β are adjacent provided that $\alpha \wedge \beta = 0$.

2020 Mathematics Subject Classification: 05C25, 06B99.
Keywords and phrases: bounded distributive lattice, ideal, graph, zero divisor graph.
Received June 30, 2023; August 31, 2023

Throughout the paper, the symbol $\mathfrak{L}$ stands for a lattice. Also, we denote the set of all ideals in $\mathfrak{L}$ as $\mathfrak{I}(\mathfrak{L})$.

Definition 1.1 ([1], Definition 3.1)**.** Let $\kappa \in \mathfrak{I}(\mathfrak{L})$. Define the zero divisor graph of $\mathfrak{L}$ with respect to κ denoted by $\Gamma_\kappa(\mathfrak{L})$, as follows:

$$\{\alpha \in \mathfrak{L}\backslash\kappa;\ \alpha \wedge \beta \in \kappa \text{ for some non-zero element } \beta \in \mathfrak{L}\backslash\kappa\}.$$

For distinct $\alpha, \beta \in \mathfrak{L}$. The vertices α and β are adjacent providing that $\alpha \wedge \beta \in \kappa$.

Afkhami et al. claim that if $\kappa = \{0\}$. Then $\Gamma_\kappa(\mathfrak{L})$ is isomorphic to $\Gamma(\mathfrak{L})$. Regard the next example:

Example 1.2. Consider $\mathfrak{L}$ with the Hasse diagram in Figure 1 and $\kappa = \{0\}$. Then $(\mathfrak{L}, {}^\vee, {}^\wedge)$ is a bounded lattice. Obviously, the set of vertices of $\Gamma(\mathfrak{L})$ is $\{0, \alpha, \delta, \epsilon, \eta\}$ as $0 \wedge \beta = \alpha \wedge \delta = \alpha \wedge \eta = \epsilon \wedge \delta = \epsilon \wedge \eta = 0$. Also, $V(\Gamma_\kappa(\mathfrak{L})) = \{\alpha, \delta, \epsilon, \eta\}$. Then $|V(\Gamma(\mathfrak{L}))| = 5 \neq |V(\Gamma_\kappa(\mathfrak{L}))| = 4$. Hence $\Gamma_\kappa(\mathfrak{L})$ is not isomorphic to $\Gamma(\mathfrak{L})$.

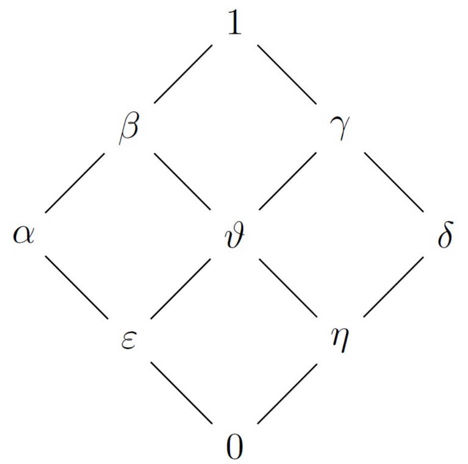

Figure 1.

Afkhami presents the following results:

Proposition 1.3 ([1], Proposition 3.2)**.** *Let* $\kappa \in \Im(\mathfrak{L})$, $\Gamma_\kappa(\mathfrak{L})$ *is connected with a diameter less than or equal to* 3. *Moreover, the girth of* $\Gamma_\kappa(\mathfrak{L})$ *is less than* 7 *provided that* $\Gamma_\kappa(\mathfrak{L})$ *has a cycle.*

For $\kappa \in \Im(\mathfrak{L})$ and $\alpha \in \mathfrak{L}$. We set

$$(\kappa : \alpha) = \{\delta \in \mathfrak{L} : \delta \wedge \alpha \in \kappa\}.$$

Obviously, we have that $(\kappa : \alpha) \in \Im(\mathfrak{L})$ only if $\mathfrak{L}$ has the distributive property.

Lemma 1.4 ([1], Lemma 3.3)**.** *For* $\kappa \in \Im(\mathfrak{L})$ *with a distributive* $\mathfrak{L}$. *Let* $x - \alpha - y$ *is a path in* $\Gamma_\kappa(\mathfrak{L})$. *Hence either* $\kappa \cup \{\alpha\} \in \Im(\mathfrak{L})$, *or* $x - \alpha - y$ *is part of a cycle of length does not exceed* 4.

In ([1], Theorem 3.4), Afkhami constructs the proof by cases. In fact, case (4) is impossible to happen and an extra condition $(|V(\Gamma_\kappa(\mathfrak{L}))| \geq 3)$ is used as a necessary condition. In Theorem 1.5, we present Theorem 3.4 in [1] without this condition and a reformulation of the proof.

Theorem 1.5 ([1], Theorem 3.4)**.** *Consider a distributive* $\mathfrak{L}$, *let* $\kappa \in \Im(\mathfrak{L})$. *If* $\Gamma_\kappa(\mathfrak{L})$ *contains a cycle, hence the core K of* $\Gamma_\kappa(\mathfrak{L})$ *is a union of* 3-*cycles or* 4-*cycles. Furthermore, each element in* $\Gamma_\kappa(\mathfrak{L})$ *is a member of K or a vertex of degree* 1.

Proof. Suppose that x is a member of K and x is not a member of any 3-cycles or 4-cycles contained in $\Gamma_\kappa(\mathfrak{L})$. Let x be in a cycle $x - y - z - w - \ldots - \alpha - x$ whose length exceeds 4. In the virtue of Lemma 1.4, $\kappa \cup \{x\} \in \Im(\mathfrak{L})$. Obviously $x \wedge w \in \kappa \cup \{x\}$ and $x \wedge w \notin \kappa$ as x and w are not adjacent, then $x \wedge w = x$. Similarly, $x \wedge z = x$. Thus $x \wedge (w \wedge z) = x \in \kappa$. Which is a contradiction. Moreover, we have that $|V(\Gamma_\kappa(\mathfrak{L}))| \geq 3$ as $\Gamma_\kappa(\mathfrak{L})$ contains a cycle. Let α be an element in $V(\Gamma_\kappa(\mathfrak{L}))$ such that x does not belong to K nor a vertex of degree 1.

Suppose that α is of degree n for a natural number n. Hence α is adjacent to n distinct vertices $x, y, \epsilon, f, \ldots$. Since $\Gamma_{\kappa}(\mathfrak{L})$ contains a cycle and by Proposition 1.3, $\Gamma_{\kappa}(\mathfrak{L})$ is connected. Thus the path $x - \alpha - y - z - w - y$ is in $\Gamma_{\kappa}(\mathfrak{L})$. By Lemma 1.4, $\kappa \cup \{\alpha\} \in \mathfrak{I}(\mathfrak{L})$ and by using a similar manner we get that $(z \wedge w) \wedge \alpha = \alpha \in \kappa$. Which is a contradiction. □

Afkhami mentions in the proof of the previous theorem that the case (4): the path $x - \alpha - \beta - y$ is in $\Gamma_{\kappa}(\mathfrak{L})$, with x is of degree 1 and y in the core. Afkhami et al. said that case (4) can be reduced to the case (3): the path $x - \alpha - \beta$ is in $\Gamma_{\kappa}(\mathfrak{L})$, with x is of degree 1 and $\beta \in K$. This is not true. In fact, case (4) is impossible. Assume case (4), by the first part of Theorem 1.5, y at least in a cycle of length 3. Then $x - \alpha - \beta - y - w - e - y$. Hence the length between α and w is more than 3. Which contradicts Proposition 1.3. Afkhami also presents the previous theorem with a trivial condition $|V(\Gamma_{\kappa}(\mathfrak{L}))| \geq 3$. Indeed, this condition follows directly from the condition that $\Gamma_{\kappa}(\mathfrak{L})$ contains a cycle. (As the smallest cycle is the 3-cycle, then we have at least 3 vertices in $V(\Gamma_{\kappa}(\mathfrak{L}))$).

For every $x \in \mathfrak{L}$, the symbol $[x]^u$ is defined to be the set $\{\alpha \in \mathfrak{L}; x \leq \alpha\}$.

Proposition 1.6 ([1], Proposition 3.6)**.** *Let* $\mathfrak{L}$ *be distributive and* $\kappa \in \mathfrak{I}(\mathfrak{L})$. *If* $\bigcap_{k \in \kappa} [k]^u = \{1\}$, *hence* $\Gamma_{\kappa}(\mathfrak{L})$ *contains no cut points.*

Afkhami constructs an example ([1], Example 3.7) to show that the distributivity of $\mathfrak{L}$ is a necessity in the previous proposition. Unfortunately, his example does not satisfy the condition $\bigcap_{k \in \kappa} [k]^u = \{1\}$.

Example 1.7 ([1], Example 3.7)**.** Consider $\kappa = \{\phi, \{4\}, \{4, 5\}, \{4, 5, 6\}, \ldots\}$. Consider $\mathfrak{L} = \kappa \cup \{\{3\}, \phi, \{1\}, \mathbb{N}, \{1, 2\}\}$. We have that $\mathfrak{L}$ is a non-distributive lattice under inclusion with $\phi = 0$ and $\mathbb{N} = 1$. However, $\bigcap_{k \in \kappa} [k]^u = \{\mathbb{N} - \{1, 2, 3\}, \mathbb{N}\}$.

Moreover, we can show the necessity of $\mathfrak{L}$ to be distributive in Proposition 1.6 by the following example. Take the lattice $\mathfrak{L}$ stated in Example 1.7. Consider the ideal $J = \{\phi, \{1\}, \{3\}\}$ of $\mathfrak{L}$. Obviously, $\mathfrak{L}$ is not distributive and $\bigcap_{j \in J} [j]^u = \{\mathbb{N}\}$. However, the vertex $\{1, 2\}$ is a cut point of $\Gamma_J(\mathfrak{L})$.

References

[1] M. Afkhami, K. Khashyarmanes and K. Nafar, Zero divisor graph of a lattice with respect to an ideal, Beiträge zur Algebra und Geometrie 56(1) (2015), 217-225.

DOI: https://doi.org/10.1007/s13366-013-0172-4

■