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REFINEMENTS OF SINGULAR VALUE INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES

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Abstract

We obtain several inequalities relating the singular values of $AX - XB$ and $AX + XB$ for positive semidefinite matrices A, B. These results are refinement of Audeh's result.

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1. Introduction

We denote by M_n the vector space of all complex $n \times n$ matrices. The notation $A \geq 0$ is used to mean that A is positive semidefinite. The singular values of A are enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| := (A^*A)^{\frac{1}{2}}$.

Let $A, B \in M_n$, the singular value inequality [1]

$$s_j(A - B) \leq s_j\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{1}$$

for $A \geq 0$ and $B \geq 0$, aroused much interest and several alternate proofs were given. Of these the one germane to our discussion occurs in the paper of Audeh [3]. He obtained a singular value inequality for generalized commutator $AX - XB$.

For $A, B, X \in M_n$, a matrix of the form $AX + XA$ is called anticommutator, and a matrix of the form $AX + XB$ is called a generalized anticommutator. For recent studies and details for generalizations of singular value inequalities for generalized anticommutator, we refer to [3]-[5].

It is remarkable that generalized commutator and generalized anticommutator give striking results on many topics, including similarity, commutativity, hyperinvariant subspaces, spectral operators, and differential equations. Bhatia and Rosenthal [2] showed how these are useful in perturbation theory. In this paper, we refine Theorems 2.4 and 2.7 in [3].

2. Main Results

We begin this section with the following lemmas, which plays an important role in our discussion.

Lemma 1. *Let* $A, B \in M_n$. *Then*

$$s_j(AB^*) \leq s_j\left(\frac{|A| + |B|}{2}\right)^2. \tag{2}$$

Proof. By the definition of singular values and Theorem 2.1 in [6], we have

$$\begin{aligned} s_j(AB^*) &= \sqrt{\lambda_j(A^*AB^*B)} \\ &\leq \lambda_j\left(\frac{|A| + |B|}{2}\right)^2 \\ &= s_j\left(\frac{|A| + |B|}{2}\right)^2. \end{aligned}$$

Lemma 2 ([7])**.** *If* $0 \leq A \leq B$. *Then* $s_j(A) \leq s_j(B)$.

Our first result is the following singular value inequality for generalized commutator, which is a refinement of Theorem 2.4 in [3].

Theorem 3. *Let* $A, B, X \in M_n$ *with* $A \geq 0$, $B \geq 0$. *Then*

$$\begin{aligned} & s_j((AX - XB) \oplus 0) \\ & \leq \frac{1}{4} s_j\left(\left(\begin{pmatrix} A & A^{\frac{1}{2}} X B^{\frac{1}{2}} \\ \left(A^{\frac{1}{2}} X B^{\frac{1}{2}}\right)^* & B^{\frac{1}{2}} |X|^2 B^{\frac{1}{2}} \end{pmatrix}^{\frac{1}{2}} + \begin{pmatrix} A^{\frac{1}{2}} |X^*|^2 A^{\frac{1}{2}} & -A^{\frac{1}{2}} X B^{\frac{1}{2}} \\ \left(-A^{\frac{1}{2}} X B^{\frac{1}{2}}\right)^* & B \end{pmatrix}^{\frac{1}{2}}\right)^2\right) \\ & \leq \frac{1}{2} s_j\begin{pmatrix} A + A^{\frac{1}{2}} |X^*|^2 A^{\frac{1}{2}} & 0 \\ 0 & B + B^{\frac{1}{2}} |X|^2 B^{\frac{1}{2}} \end{pmatrix} \end{aligned}$$

for $j = 1, 2, \cdots, 2n$.

Proof. Let $C = \begin{pmatrix} A^{\frac{1}{2}} & XB^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$, $D^* = \begin{pmatrix} A^{\frac{1}{2}}X & 0 \\ -B^{\frac{1}{2}} & 0 \end{pmatrix}$.

Then for $j = 1, 2, \cdots, 2n$, we get

$$s_j((AX - XB) \oplus 0)$$

$$= s_j(CD^*)$$

$$\leq s_j\left(\frac{|C| + |D|}{2}\right)^2$$

$$= \frac{1}{4} s_j\left(\begin{pmatrix} A & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ \left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^* & B^{\frac{1}{2}}|X|^2 B^{\frac{1}{2}} \end{pmatrix}^{\frac{1}{2}} + \begin{pmatrix} A^{\frac{1}{2}}|X^*|^2 A^{\frac{1}{2}} & -A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ \left(-A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^* & B \end{pmatrix}^{\frac{1}{2}}\right)^2$$

$$\leq \frac{1}{2} s_j\begin{pmatrix} A + A^{\frac{1}{2}}|X^*|^2 A^{\frac{1}{2}} & 0 \\ 0 & B + B^{\frac{1}{2}}|X|^2 B^{\frac{1}{2}} \end{pmatrix},$$

where the first inequality follows from the inequality (2) and the second inequality is due to $\left(\frac{A + B}{2}\right)^2 \leq \frac{A^2 + B^2}{2}$ and Lemma 2. □

Next, we present the following refinement of Theorem 2.7 in [3], which is singular value inequality for generalized anticommutator. To reach our findings, we need the following lemmas:

Lemma 4 ([8]). *If* $0 \leq A \leq B$. *Then* $0 \leq A^r \leq B^r$ *for* $r \in [0, 1]$.

Lemma 5 ([8]). *Let* $A \in M_n$. *Then*

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \leq \begin{pmatrix} |A| & 0 \\ 0 & |A^*| \end{pmatrix}.$$

Theorem 6. *Let* $A, B, X \in M_n$ *with* $A \geq 0, B \geq 0$. *Then*

$$s_j((AX + XB) \oplus 0)$$

$$\leq \frac{1}{4} s_j\left(\left(\begin{pmatrix} A & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ \left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^* & B^{\frac{1}{2}}|X|^2B^{\frac{1}{2}} \end{pmatrix}^{\frac{1}{2}} + \begin{pmatrix} A^{\frac{1}{2}}|X^*|^2A^{\frac{1}{2}} & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ \left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^* & B \end{pmatrix}^{\frac{1}{2}}\right)^2\right)$$

$$\leq \frac{1}{2} s_j\begin{pmatrix} A + A^{\frac{1}{2}}|X^*|^2A^{\frac{1}{2}} + 2\left|\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^*\right| & 0 \\ 0 & B + B^{\frac{1}{2}}|X|^2B^{\frac{1}{2}} + 2\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right| \end{pmatrix},$$

for $j = 1, 2, \cdots, 2n$.

Proof. Let $C = \begin{pmatrix} A^{\frac{1}{2}} & XB^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$, $F^* = \begin{pmatrix} A^{\frac{1}{2}}X & 0 \\ B^{\frac{1}{2}} & 0 \end{pmatrix}$,

$$S = \begin{pmatrix} A + \left|\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^*\right| & 0 \\ 0 & B^{\frac{1}{2}}|X|^2B^{\frac{1}{2}} + \left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right| \end{pmatrix},$$

and

$$T = \begin{pmatrix} A^{\frac{1}{2}}|X^*|^2A^{\frac{1}{2}} + \left|\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^*\right| & 0 \\ 0 & B + \left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right| \end{pmatrix}.$$

Then for $j = 1, 2, \cdots, 2n,$ we obtain

$$s_j((AX + XB) \oplus 0)$$

$$= s_j(CF^*)$$

$$\leq s_j\left(\frac{|C| + |F|}{2}\right)^2$$

$$= \frac{1}{4} s_j\left(\begin{pmatrix} A & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ \left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^* & B^{\frac{1}{2}}|X|^2 B^{\frac{1}{2}} \end{pmatrix}^{\frac{1}{2}} + \begin{pmatrix} A^{\frac{1}{2}}|X^*|^2 A^{\frac{1}{2}} & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ \left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^* & B \end{pmatrix}^{\frac{1}{2}}\right)^2$$

$$\leq \frac{1}{4} s_j\left(S^{\frac{1}{2}} + T^{\frac{1}{2}}\right)^2$$

$$\leq \frac{1}{2} s_j\begin{pmatrix} A + A^{\frac{1}{2}}|X^*|^2 A^{\frac{1}{2}} + 2\left|\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^*\right| & 0 \\ 0 & B + B^{\frac{1}{2}}|X|^2 B^{\frac{1}{2}} + 2\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right| \end{pmatrix},$$

where the first inequality follows from the inequality (2), the second inequality is due to Lemmas 2, 4 and 5 and the third inequality is a direct result of matrix inequality $\left(\frac{A + B}{2}\right)^2 \leq \frac{A^2 + B^2}{2}$ and Lemma 2. □

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