Journal of Pure and Applied Mathematics: Advances and Applications
Volume 26, Number 1, 2023, Pages 49-63
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jpamaa_7100122275

PARAMETRIZATION OF ALGEBRAIC POINTS OF GIVEN DEGREE ON THE PARTICULARLY HYPERELLIPTIC CURVE II OF TOMASZ JEDRZEJAK

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Abstract

We explicitly define the set of algebraic points of any given degree over $\mathbb{Q}$ on the affine equation curve of $y^2 = x(x^4 + 1259712)$.

This note deals with a special case of a hyperelliptic curve of affine equation $\mathcal{C}_{5,A} : y^2 = x^5 + Ax$. These curves are described by Tomasz Jędrzejak in [7], who showed that the Mordell-Weil group is finite when $A = 1259712$ and explained the generators of the torsion group for this family of curves.

2020 Mathematics Subject Classification: 14L40, 14H40, 14C20.
Keywords and phrases: Mordell-Weil group, Jacobian, linear system.
Received March 31, 2023

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1. Introduction

Let $\mathcal{C}$ be a smooth projective plane curve defined on $\mathbb{Q}$. For all algebraic extension field $\mathbb{K}$ of $\mathbb{Q}$, we denote by $\mathcal{C}(\mathbb{K})$ the set of $\mathbb{K}$-rational points of $\mathcal{C}$ on $\mathbb{K}$ and by $\mathcal{C}^{(d)}(\mathbb{Q})$ the set of algebraic points of degree d over $\mathbb{Q}$. The degree of an algebraic point R is the degree of its field of definition on $\mathbb{Q}$, i.e., $\deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$. A famous theorem of Fatlings [6] shows that if $\mathcal{C}$ is a smooth projective plane curve defined over $\mathbb{K}$ of genus $g \geq 2$, then $\mathcal{C}(\mathbb{K})$ is finite. Fatling's proof is still ineffective in the sense that it does not provide an algorithm for computing $\mathcal{C}(\mathbb{K})$. Currently for curve $\mathcal{C}$ defined over a numbers field $\mathbb{K}$ of genus $g \geq 2$, there is no know algorithm for computing the set $\mathcal{C}(\mathbb{K})$ or for deciding if $\mathcal{C}(\mathbb{K})$ is empty. But there is a bag of strikes that can be used to show that $\mathcal{C}(\mathbb{K})$ is empty, or to determine $\mathcal{C}(\mathbb{K})$ if it is not empty. These include local method, Chabauty method [2], Descent method [5], Mordell-Weil sieves method [1]. These methods often succeed with less than full knowledge of the Jacobian of the curve. If it is finite it is not hard to determine $\mathcal{C}(\mathbb{Q})$ and to generalize for all number field $\mathbb{K}$. The purpose of this note is to determine a parametrization of the set $\mathcal{C}_{1259712}^{(\ell)}(\mathbb{Q})$ on the curve $\mathcal{C}_{1259712} : y^2 = x(x^4 + 1259712)$. The curve $\mathcal{C}_{1259712}$ studied in [7] has $\mathrm{rk}(\mathcal{J})(\mathbb{Q}) = 0$ when $A = 1259712$, so the Mordell-Weil group of the Jacobian $\mathcal{J}(\mathbb{Q})$ is finite.

1.1. Main result. Our main result is the following theorem:

Theorem 1. *The set of algebraic points of degree at most* ℓ (*with* $\ell \geq 5$) *on* $\mathbb{Q}$ *on the curve* $\mathcal{C}_{1259712}$ *of affine equation* $y^2 = x(x^4 + 1259712)$ *is given by*

$$\mathcal{F} = \mathcal{F}_1 \bigcup \mathcal{F}_2 \bigcup \left(\bigcup_{k \in \{2, 3\}} \mathcal{F}_3^k \right) \bigcup \left(\bigcup_{k \in \{2, 3\}} \mathcal{F}_4^k \right) \bigcup \left(\bigcup_{k \in \{0, \ldots, 3\}} \mathcal{F}_5^k \right),$$

with

$$\mathcal{F}_1 = \left\{ \left(x, -\frac{\sum_{i=0}^{\frac{\ell}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell-5}{2}} a_j x^j} \right) \middle| \begin{array}{c} a_0 \textit{ and } b_0 \textit{ not simultaneousy zero,} \\ a_{\frac{\ell}{2}} \neq 0 \textit{ if } \ell \textit{ is even, } b_{\frac{\ell-5}{2}} \neq 0 \textit{ if } \ell \\ \textit{is odd and } x \textit{ is a solution} \\ \textit{of the equation:} \\ \left(\sum_{i=0}^{\frac{\ell}{2}} a_i x^i + \right)^2 = \left(\sum_{j=0}^{\frac{\ell-5}{2}} a_j x^j \right)^2 x(x^4 + 1259712) \end{array} \right\},$$

$$\mathcal{F}_2 = \left\{ \left(x, -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \middle| \begin{array}{c} a_{\frac{\ell+2}{2}} \neq 0 \textit{ if } \ell \textit{ is even, } b_{\frac{\ell-3}{2}} \neq 0 \\ \textit{if } \ell \textit{ is odd and } x \textit{ solution of} \\ \textit{the equation:} \\ \left(\sum_{i=1}^{\frac{\ell+1}{2}} a_i x^{i-\frac{1}{2}} \right)^2 = \left(\sum_{j=0}^{\frac{\ell-4}{2}} b_j x^j \right)^2 (x^4 + 1259712) \end{array} \right\},$$

$$\mathcal{F}_3^k = \left\{ \left(x, -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i (x^i + \mu^i)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \middle| \begin{array}{c} a_{\frac{\ell+2}{2}} \neq 0 \textit{ if } \ell \textit{ is even, } b_{\frac{\ell-3}{2}} \neq 0 \\ \textit{if } \ell \textit{ is odd and } x \textit{ solution of} \\ \textit{the equation:} \\ \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(\frac{x^i + \mu^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x^4 + 1259712) \\ \textit{with} \qquad \mu^i = -\frac{1}{2}(\eta_{k-2}^i + \eta_k^i) \end{array} \right\},$$

$$\mathcal{F}_4^k = \left\{ \begin{array}{c} \left(x, -\dfrac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i (x^i + \mu^i)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \Bigg| \begin{array}{c} a_{\frac{\ell+2}{2}} \neq 0 \ \textit{if } \ell \textit{ is even, } b_{\frac{\ell-3}{2}} \neq 0 \\ \textit{if } \ell \textit{ is odd and } x \textit{ solution of} \\ \textit{the equation}: \end{array} \\ \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(\dfrac{x^i + \nu^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x^4 + 1259712) \ \textit{with} \\ \nu^i = -\dfrac{1}{2}(\eta^i_{\kappa-2} + \eta^i_{\kappa}),\ \kappa = k+1 \ \textit{if } k = 2 \ \textit{and } \kappa = k-1 \ \textit{if } k = 3 \end{array} \right\},$$

$$\mathcal{F}_5^k = \left\{ \begin{array}{c} \left(x, -\dfrac{\sum_{i=1}^{\frac{\ell+5}{2}} a_i (x^i + \omega^i)}{\sum_{j=0}^{\frac{\ell}{2}} b_i x^j} \right) \Bigg| \begin{array}{c} a_{\frac{\ell+5}{2}} \neq 0 \ \textit{if } \ell \textit{ is even, } b_{\frac{\ell}{2}} \neq 0 \\ \textit{if } \ell \textit{ is odd and } x \textit{ solution of} \\ \textit{the equation}: \end{array} \\ \left(\sum_{i=1}^{\frac{\ell+5}{2}} a_i \left(\dfrac{x^i + \omega^i}{x^{\frac{5}{2}}} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell}{2}} b_i x^{j-2} \right)^2 (x^4 + 1259712) \\ \textit{with} \quad \omega^i = -\dfrac{1}{4} \left(\sum_{k=0}^{3} \eta^i_k \right) \end{array} \right\}.$$

2. Auxiliary Results

Definition 1. For a divisor $\mathcal{D} \in Div(\mathcal{C})$, we define the $\mathbb{Q}$-vector space denoted $\mathcal{L}(\mathcal{D})$ by:

$$\mathcal{L}(\mathcal{D}) := \{ f \in \mathbb{K}(\mathcal{C})^* \,|\, \mathrm{div}(f) \geq -\mathcal{D} \} \cup \{0\}.$$

Remark 1. For two divisors $\mathcal{D}$ and $\mathcal{D}'$, we have

$$\mathcal{D} \equiv \mathcal{D}' \Rightarrow \mathcal{L}(\mathcal{D}) \simeq \mathcal{L}(\mathcal{D}') \Rightarrow \dim \mathcal{L}(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}').$$

Lemma 1. *According to Lemma* 3.1 (*see* [8, *page* 205]), *we have* $\mathcal{J}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The projective form of the equation of the curve $\mathcal{C}_{1259712}$ is $Z^3Y^2 = X\prod_{k=0}^{3}(X - \eta_k Z)$, we note P_k, P_4 and ∞ the points of $\mathcal{C}$, defined by $P_k = [\eta_k : 0 : 1]$, $P_4 = [0 : 0 : 1]$ and $\infty = [0 : 1 : 0]$ with $\eta_k = 6\sqrt{3}e^{\frac{2k\pi}{4}\imath}$ and $k \in \{0, \dots, 3\}$.

Lemma 2. *For curve* $C_{1259712} : y^2 = x(x^4 + 1259712)$, *we have*

• $\operatorname{div}(x) = 2P_4 - 2\infty$, • $\operatorname{div}(x - \eta_k) = 2P_k - 2\infty$, *where* $P_k = [\eta_k : 0 : 1]$ *and* $k \in \{0, \dots, 3\}$.

• $\operatorname{div}(y) = \sum_{k=0}^{3} P_k + P_4 - 5\infty.$

In fact, it is calculations of the type $\operatorname{div}(x - \varpi) = \operatorname{div}(X - \varpi Z) - \operatorname{div}(Z) = (X = \varpi Z)\cdot\mathcal{C}_{1259712} - (Z = 0)\cdot C_{1259712}$ (see [4, proof Lemma 2, page 154]).

Corollary 1. *The following results are the consequences of Lemma* 2:

• $\sum_{k=0}^{5} j(P_k) + j(P_4) = 0$, • $2j(P_k) = 0$, *where* $P_k = [\eta_k : 0 : 1]$ *and* $k \in \{0, \dots, 3\}$.

So the $j(P_i)$ generate the same subgroup $\mathcal{J}(\mathbb{Q})$.

Lemma 3. *According to Lemma* 3.1 (*see* [8, *page* 205]), *we have*

$$\mathcal{J}(\mathbb{Q}) = \langle [P_4 - \infty], [P_0 + P_2 - 2\infty], [P_1 + P_3 - 2\infty] \rangle$$

$$= \{\alpha_1 j(P_4) + \alpha_2(j(P_0) + j(P_2)) + \alpha_3(j(P_1) + j(P_3)),$$

$$\textit{with } \alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}\}.$$

Lemma 4. *A* $\mathbb{Q}$*-base of* $\mathcal{L}(m\infty)$ *is given by*

$$\mathcal{B}_m = \left\{x^i \middle| i \in \mathbb{N} \textit{ and } i \leq \frac{m}{2}\right\} \bigcup \left\{yx^j \middle| j \in \mathbb{N} \textit{ and } j \leq \frac{m-5}{2}\right\}.$$

Proof. See proof of Lemma 4 [4, page 154].

3. Proof of the Main Theorem

Let $R \in \mathcal{C}_{1259712}(\overline{\mathbb{Q}})$ to $[\mathbb{Q}(R):\mathbb{Q}] = \ell$ with $\ell \geq 5$ and $R \notin \{P_{k, k \in \{0, \dots, 3\}}, P_4, \infty\}$. Consider $R_1, \dots, R_\ell$ the Galois conjugates of R and let $t = [R_1 + \dots + R_\ell - \ell\infty] \in \mathcal{J}(\mathbb{Q})$. From Lemma 3, we have $t = -\alpha_1 j(P_4) - \alpha_2(j(P_0) + j(P_2)) - \alpha_3(j(P_1) + j(P_3))$, $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$ and hence $[R_1 + \dots + R_\ell - \ell\infty] = [(\alpha_1 + 2\alpha_2$. This gives the following formula:

$$[R_1 + \dots + R_\ell + \alpha_1 P_4 + \alpha_2(P_0 + P_2) + \alpha_3(P_1 + P_3)$$
$$- (\ell + \alpha_1 + 2\alpha_2 + 2\alpha_3)\infty] = 0.$$

According to Abel Jacobi's theorem ([3, page 156]), there exists a rational function f of efinite on $\mathbb{Q}$ such that

$$\operatorname{div}(f) = R_1 + \dots + R_\ell + \alpha_1 P_4 + \alpha_2(P_0 + P_2) + \alpha_3(P_1 + P_3)$$
$$- (\ell + \alpha_1 + 2\alpha_2 + 2\alpha_3)\infty. \qquad (\star)$$

Four cases are possible:

Case 1: $\alpha_k = 0, \forall k \in \{1, 2, 3\}$.

The formula $(\star)$ becomes: $\mathrm{div}(f) = R_1 + \ldots + R_\ell - \ell\infty$, donc $f \in \mathcal{L}(\ell\infty)$.

According to Lemma 4, we have $f = \sum_{i=0}^{\frac{\ell}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-5}{2}} a_j y x^j$ with a_0 and a_0 not simultaneously all zero (otherwise one of the R_i should be equal P_0, which would be absurd), $a_{\frac{\ell}{2}} \neq 0$ if ℓ is even (otherwise one of R_i should be equal ∞, which would be absurd) and $b_{\frac{\ell-5}{2}} \neq 0$ if ℓ is odd (otherwise one of R_i should be equal ∞, which would be absurd). At point R_i, we have $f = 0$, implying that, $y = -\frac{\sum_{i=0}^{\frac{\ell}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell-5}{2}} a_j x^j}$. By replacing the value of y in the expression of the equation of the curve, we obtain

$$\left(\sum_{i=0}^{\frac{\ell}{2}} a_i x^i\right)^2 = \left(\sum_{j=0}^{\frac{\ell-5}{2}} a_j x^j\right)^2 x(x^4 + 1259712). \tag{1}$$

Expression (1) is an equation of degree ℓ in x. Indeed, whatever the parity of ℓ, the first member of Equation (1) has degree $2 \times \left(\frac{\ell}{2}\right) = \ell$ and the second member has degree $2 \times \left(\frac{\ell - 5}{2}\right) + 5 = \ell$. This gives a family of points of degree ℓ:

$$\mathcal{F}_1 = \left\{ \left(x, -\frac{\sum_{i=0}^{\frac{\ell}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell-5}{2}} a_j x^j} \right) \middle| \begin{array}{c} a_0 \text{ and } b_0 \text{ not simultaneously zero,} \\ a_{\frac{\ell}{2}} \neq 0 \text{ if } \ell \text{ is even, } b_{\frac{\ell-5}{2}} \neq 0 \text{ if } \ell \\ \text{is odd and } x \text{ is a solution} \\ \text{of the equation:} \end{array} \right. \\ \left. \left(\sum_{i=0}^{\frac{\ell}{2}} a_i x^i + \right)^2 = \left(\sum_{j=0}^{\frac{\ell-5}{2}} a_j x^j \right)^2 x(x^4 + 1259712) \right\}.$$

Case 2: only $\alpha_1 \neq 0$.

The formula $(\star)$ becomes : $\operatorname{div}(f) = R_1 + \ldots + R_\ell + P_4 - (\ell + 1)P_\infty$, so $f \in \mathcal{L}((\ell + 1)P_\infty)$, according to Lemma 4, we have $f = \sum_{i=0}^{\frac{\ell+1}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-4}{2}} b_j y x^j$ and since $ord_{P_4} f = 1$, so $a_0 = 0$; implies that $f = \sum_{i=1}^{\frac{\ell+1}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-4}{2}} b_j y x^j$ with $a_{\frac{\ell+1}{2}} \neq 0$ if ℓ is even (otherwise one of the R_i should be equal ∞, which would be absurd) and $b_{\frac{\ell-4}{2}} \neq 0$ if ℓ odd (otherwise one of the R_i should be equal ∞, which would be absurd). At the points R_i, we have $f = 0$, which implies that $y = -\frac{\sum_{i=1}^{\frac{\ell+1}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell-4}{2}} b_j x^j}$. By replacing the value of y in the expression of the equation of the curve, we obtain: $\left(\sum_{i=1}^{\frac{\ell+1}{2}} a_i x^i\right)^2 =$ $\left(\sum_{j=0}^{\frac{\ell-4}{2}} b_j x^j\right)^2 x(x^4 + 1259712)$ which also corresponds to the equation:

$$\left(\sum_{i=1}^{\frac{\ell+1}{2}} a_i x^{i-\frac{1}{2}}\right)^2 = \left(\sum_{j=0}^{\frac{\ell-4}{2}} b_j x^j\right)^2 (x^4 + 1259712). \tag{2}$$

Expression (2) is an equation of degree ℓ in x.

Indeed, whatever the parity of ℓ, the first member of (2) is of degree $2 \times \left(\frac{\ell+1}{2} - \frac{1}{2}\right) = \ell$ and the second member is of degree $2 \times \left(\frac{\ell-4}{2}\right) + 4 = \ell$. This gives a family of points of degree ℓ:

$$\mathcal{F}_2 = \left\{ \left(x, -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i x^i}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}\right) \,\middle|\, \begin{array}{c} a_{\frac{\ell+1}{2}} \neq 0 \text{ if } \ell \text{ is even, } b_{\frac{\ell-4}{2}} \neq 0 \\ \text{if } \ell \text{ is odd and } x \text{ solution of} \\ \text{the equation:} \end{array} \right. \\ \left. \left(\sum_{i=1}^{\frac{\ell+1}{2}} a_i x^{i-\frac{1}{2}}\right)^2 = \left(\sum_{j=0}^{\frac{\ell-4}{2}} b_j x^j\right)^2 (x^4 + 1259712) \right\}.$$

Case 3: only one of $\alpha_k \neq 0$ with $k \in \{2, 3\}$.

The formula $(\star)$ becomes: $\mathrm{div}(f) = R_1 + \ldots + R_\ell + P_{k-2} + P_k - (\ell + 2)P_\infty$, so $f \in \mathcal{L}((\ell+2)P_\infty)$, according to Lemma 4, we have $f = \sum_{i=0}^{\frac{\ell+2}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j$ and since $ord_{P_{k-2}} f = ord_{P_k} f = 1$, so $a_0 = -\frac{1}{2}\sum_{i=1}^{\frac{\ell+2}{2}} a_i(\eta_{k-2}^i - \eta_k^i)$. By posing: $\mu^i = -\frac{1}{2}(\eta_{k-2}^i + \eta_k^i)$ implies that $f = \sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \mu^i) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j$ with $a_{\frac{\ell+2}{2}} \neq 0$ if ℓ is even (otherwise one of the R_i should be equal ∞, which would be absurd) and $b_{\frac{\ell-3}{2}} \neq 0$ if ℓ if l is odd (otherwise one of the R_i should be equal ∞, which would be absurd). At the points R_i, we have $f = 0$, which implies that $y = -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \mu^i)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}$. By replacing the value of y in the expression of the equation of the curve, we obtain equation $\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \mu^i)\right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j\right)^2 x(x^4 + 1259712)$ which also corresponds to the equation:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(\frac{x^i + \mu^i}{x}\right)\right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 (x^4 + 1259712). \tag{3}$$

Expression (3) is an equation of degree ℓ. Indeed, whatever the parity of ℓ, the first member of Equation (3) is of degree $2 \times \left(\frac{\ell+2}{2} - 1\right) = \ell$ and the second member is of degree $2 \times \left(\frac{\ell - 3}{2} - \frac{1}{2} \right) + 4 = \ell$. This gives a family of points of degree ℓ:

$$\mathcal{F}_3^k = \left\{ \begin{array}{c} \left(x, -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \mu^i)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \Bigg| \begin{array}{c} a_{\frac{\ell+2}{2}} \neq 0 \text{ if } \ell \text{ is even, } b_{\frac{\ell-3}{2}} \neq 0 \\ \text{if } \ell \text{ is odd and } x \text{ solution of} \\ \text{the equation:} \end{array} \\ \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(\frac{x^i + \mu^i}{x} \right) \right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}} \right)^2 (x^4 + 1259712) \\ \text{with} \qquad \mu^i = -\frac{1}{2}(\eta_{k-2}^i + \eta_k^i) \end{array} \right\}.$$

Case 4: $\alpha_1 \neq 0$ and one of $\alpha_k \neq 0$ with $k \in \{2, 3\}$.

The formula $(\star)$ becomes $\operatorname{div}(f) = R_1 + \ldots + R_\ell + P_4 + P_{k-2} + P_k - (\ell + 3)\infty$, from Corollary 1 the formula $(*)$ will be written $\operatorname{div}(f) = R_1 + \ldots + R_\ell + P_{\kappa-2} + P_\kappa - (\ell + 2)\infty$ with $\kappa = k + 1$ if $k = 2$ and $\kappa = k - 1$ if $k = 3$, so $f \in \mathcal{L}((\ell + 2)P_\infty)$, according to Lemma 4, we have $f = \sum_{i=0}^{\frac{\ell+2}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j$ and since $ord_{P_{\kappa-2}} f = ord_{P_\kappa} f = 1$, so $a_0 = -\frac{1}{2} \sum_{i=1}^{\frac{\ell+2}{2}} a_i(\eta_{\kappa-2}^i - \eta_\kappa^i)$. By posing: $\nu^i = -\frac{1}{2}(\eta_{\kappa-2}^i + \eta_\kappa^i)$ implies that $f = \sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \nu^i) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j$ with $a_{\frac{\ell+2}{2}} \neq 0$ if ℓ is even (otherwise one of the R_i should be equal ∞, which would be absurd) and $b_{\frac{\ell-3}{2}} \neq 0$ if ℓ is odd (otherwise one of the R_i should be equal ∞, which would be absurd). At the points R_i, we have $f = 0$, which implies that

$y = -\dfrac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \nu^i)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}$. By replacing the value of y in the expression of the equation of the curve, we obtain equation $\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \nu^i)\right)^2$ $= \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j\right)^2 x(x^4 + 1259712)$ which also corresponds to the equation:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(\frac{x^i + \nu^i}{x}\right)\right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 (x^4 + 1259712). \tag{4}$$

Expression (4) is an equation of degree ℓ in x.

Indeed, whatever the parity of ℓ, the first member of Equation (4) is of degree $2 \times \left(\frac{\ell+2}{2} - 1\right) = \ell$ and the second member is of degree $2 \times \left(\frac{\ell-3}{2} - \frac{1}{2}\right) + 4 = \ell$. This gives a family of points of degree ℓ:

$$\mathcal{F}_4^k = \left\{ \left(x, -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i(x^i + \mu^i)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}\right) \middle| \begin{array}{c} a_{\frac{\ell+2}{2}} \neq 0 \text{ if } \ell \text{ is even, } b_{\frac{\ell-3}{2}} \neq 0 \\ \text{if } \ell \text{ is odd and } x \text{ solution of} \\ \text{the equation:} \\ \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(\frac{x^i + \nu^i}{x}\right)\right)^2 = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 (x^4 + 1259712) \text{ with} \\ \nu^i = -\frac{1}{2}(\eta^i_{\kappa-2} + \eta^i_{\kappa}),\ \kappa = k+1 \text{ if } k = 2 \text{ and } \kappa = k-1 \text{ if } k = 3 \end{array} \right\}.$$

Remark 2. For the case where only $\alpha_1 = 0$, we find the second point.

Indeed, for $\alpha_1 = 0$, the formula $(\star)$ becomes $\operatorname{div}(f) = R_1 + \ldots + R_\ell + P_0 + P_1 + P_2 + P_3 + - (\ell + 4)\infty$, from Corollary 1 the formula $(*)$ will be written $\operatorname{div}(f) = R_1 + \ldots + R_\ell + P_4 + P_\kappa - (\ell + 1)\infty$, from which we obtain the second case.

Case 5: None of the n_k are zero

The formula $(\star)$ becomes: $\operatorname{div}(f) = R_1 + \ldots + R_\ell + P_4 + \sum_{k=0}^{3} P_k - (\ell + 5)\infty$,

so $f \in \mathcal{L}((\ell + 5)\infty)$, according to Lemma 4, we have $f = \sum_{i=0}^{\frac{\ell+5}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell}{2}} b_i y x^j$ and since $ord_{P_4} f = ord_{P_k} f = 1, \forall k \in \{0, \ldots, 3\}$, results in so : $a_0 = 0$ and $a_1 = -\frac{1}{4}\left(\sum_{i=1}^{\frac{\ell+5}{2}} a_i \left(\sum_{k=0}^{3} \eta_k^i\right)\right)$. By posing $\omega^i = -\frac{1}{4}\left(\sum_{k=0}^{3} \eta_k^i\right)$, we hence $a_1 = a\wp + \sum_{i=1}^{\frac{\ell+2}{2}} a_i \omega^i$ implies that $f = \sum_{i=1}^{\frac{\ell+5}{2}} a_i (x^i + \omega^i) + \sum_{j=0}^{\frac{\ell}{2}} b_i y x^j$ with $a_{\frac{\ell+5}{2}} \neq 0$ if ℓ is even (otherwise one of the R_i should be equal ∞, which would be absurd) and $b_{\frac{\ell}{2}} \neq 0$ is ℓ is odd (otherwise one of the R_i should be equal ∞, which would be absurd). At point R_i, we have $f = 0$, which implies that $y = -\frac{\sum_{i=1}^{\frac{\ell+5}{2}} a_i (x^i + \omega^i)}{\sum_{j=0}^{\frac{\ell}{2}} b_i x^j}$. By replacing the value of y in the expression of the equation of the curve, we obtain

$$\left(\sum_{i=1}^{\frac{\ell+5}{2}} a_i (x^i + \omega^i)\right)^2 = \left(\sum_{j=0}^{\frac{\ell}{2}} b_i x^j\right)^2 x(x^4 + 1259712).$$

This also corresponds to the equation

$$\left(\sum_{i=1}^{\frac{\ell+5}{2}} a_i\left(\frac{x^i+\omega^i}{x^{\frac{5}{2}}}\right)\right)^2=\left(\sum_{j=0}^{\frac{\ell}{2}} b_i x^{j-2}\right)^2(x^4+1259712). \tag{5}$$

Expression (5) is an equation of degree ℓ in x.

Indeed; whatever the parity of ℓ, the first member of Equation (5) is of degree $2\times\left(\frac{\ell+5}{2}-\frac{5}{2}\right)=\ell$ and the second member is of degree $2\times\left(\frac{\ell}{2}-2\right)+4=\ell$. This gives a point family of degree ℓ:

$$\mathcal{F}_5^k=\left\{\left(x,-\frac{\sum_{i=1}^{\frac{\ell+5}{2}} a_i(x^i+\omega^i)}{\sum_{j=0}^{\frac{\ell}{2}} b_i x^j}\right)\middle|\begin{array}{c} a_{\frac{\ell+5}{2}}\neq 0 \text{ if } \ell \text{ is even, } b_{\frac{\ell}{2}}\neq 0 \\ \text{if } \ell \text{ is odd and } x \text{ solution of} \\ \text{the equation:}\end{array}\right.$$
$$\left.\begin{array}{c}\left(\sum_{i=1}^{\frac{\ell+5}{2}} a_i\left(\frac{x^i+\omega^i}{x^{\frac{5}{2}}}\right)\right)^2=\left(\sum_{j=0}^{\frac{\ell}{2}} b_i x^{j-2}\right)^2(x^4+1259712)\\ \text{with } \omega^i=-\frac{1}{4}\left(\sum_{k=0}^{3}\eta_k^i\right)\end{array}\right\}.$$

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