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PERMANENT SOLUTIONS FOR TWO MIXED INITIAL-BOUNDARY VALUE PROBLEMS WHICH DESCRIBE MOTIONS OF BURGERS FLUIDS BETWEEN PARALLEL PLATES: APPLICATIONS

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Abstract

Two isothermal unsteady motions of the incompressible Burgers fluids between infinite horizontal parallel plates are analytically and numerically investigated when a differential expression of the shear stress is prescribed on the lower plate. Closed-form expressions are established for the dimensionless permanent velocity fields and the corresponding shear stresses. They seem to be the first exact solutions for such motions of Burgers fluids. The corresponding permanent solutions for Oldroyd-B, Maxwell, second grade and Newtonian fluids performing similar motions can immediately be obtained as limiting cases of general solutions. The obtained velocity fields are used to determine the needed time to reach the steady state for the respective motions. It was found that the steady state for such isothermal motions of the incompressible non-Newtonian fluids is earlier obtained for the Burgers fluids in comparison with Oldroyd-B or Maxwell fluids.

1. Introduction

One-dimensional viscoelastic fluid model proposed by Burgers [1] was often used to describe the behaviour of different materials such as food products (like cheese), soil and asphalt [2, 3]. Saal and Labout [4] shown that the mechanical behaviour of asphalt of different compositions can be well enough described by this model. A good agreement between the predictions of Burgers' model and the behaviour of asphalt and sand-asphalt specimens was noted by Lee and Markwick [5]. The transient creep properties of the earth's mantle and high temperature viscoelasticity of fine-grained polycrystalline olivine were modelled by Peltier et al. [6] and Yuen and Peltier [7], respectively Chopra [8] and Tan et al. [9] using the same model.

The extension of Burgers linear model to a frame-indifferent three-dimensional form was realized by Krishnan and Rajagopal [10] and the first exact steady solutions for the motion of such a fluid seem to be those of Ravindran et al. [11] in an orthogonal rheometer. Along the time many interesting solutions corresponding to isothermal motions of the incompressible Burgers fluids have been determined. Among them we remember the most recent results of Akram et al. [12], Fetecau et al. [13], Hussain et al. [14] and Fetecau et al. [15] whose solutions are related to the present results. However, none of the above mentioned studies refers to motions of incompressible Burgers fluids between parallel plates when a differential expression of the shear stress is given on a part of the boundary.

Relatively recent, Renardy [16, 17] noted that boundary conditions containing differential expressions of stresses must be considered for rate type fluids in order to get well-posed boundary value problems for some motions of such fluids. In this note, the first dimensionless exact permanent (steady state or long time) solutions for some isothermal unidirectional motions of incompressible Burgers fluids between infinite horizontal parallel plates are provided when a differential expression of the non-trivial shear stress is given on a part of the boundary. They can easily be particularized to give permanent solutions for incompressible Oldroyd-B, Maxwell, second grade and Newtonian fluids performing similar motions. The obtained solutions, which are presented in simple forms, are used to determine the required time to reach the steady or permanent state for the respective motions. It was found that the steady state for such motions of incompressible fluids is rather obtained for Newtonian as compared with non-Newtonian fluids.

2. Problem Presentation and Governing Equations

Consider an incompressible Burgers fluid (IBF) at rest between two infinite horizontal parallel flat plates at the distance d apart. Its constitutive equations are given by the following relations [11]:

$$\boldsymbol{T} = -\hat{p}\boldsymbol{I} + \boldsymbol{S}, \quad \boldsymbol{S} + p\frac{\delta \boldsymbol{S}}{\delta t} + q\frac{\delta^2 \boldsymbol{S}}{\delta t^2} = \mu\left(\boldsymbol{A} + r\frac{\delta \boldsymbol{A}}{\delta t}\right), \tag{1}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{S}$ is the extra-stress tensor, $\boldsymbol{I}$ is the unit tensor, $\boldsymbol{A}$ is the first Rivlin-Ericksen tensor, $\hat{p}$ is the hydrostatic pressure, μ is the fluid viscosity, p, q and $r\,(\leq p)$ are material constants and $\delta/\delta t$ denotes the well known time upper-convected derivative. Since the incompressible fluids undergo isochoric motions only, the continuity equation

$$\text{tr}\boldsymbol{A} = 0 \text{ or equivalent } \text{div}\,\boldsymbol{w} = 0, \tag{2}$$

has to be identically satisfied. In the last equality $\boldsymbol{w}$ is the velocity vector.

It is worth to point out the fact that the fluids characterized by the constitutive equations (1) contain as special cases the incompressible Oldroyd-B, Maxwell and Newtonian fluids if $q = 0$, $q = r = 0$, respectively, $p = q = r = 0$. For some motions, like those to be here considered, the governing equations corresponding to the incompressible second grade fluids can also be obtained as particular cases of the present equations. In the following, as well as in the references [12-15], we shall consider isothermal motions of IBFs for which the velocity field $\boldsymbol{w}$ and the extra-stress tensor $\boldsymbol{S}$ are given by the relations

$$\boldsymbol{w} = \boldsymbol{w}(y, t) = w(y, t)\mathbf{e}_x, \quad \boldsymbol{S} = \boldsymbol{S}(y, t), \tag{3}$$

where $\mathbf{e}_x$ is the unit vector along the x-direction of a convenient Cartesian coordinate system x, y and z having the y-axis perpendicular to plates.

Substituting $\boldsymbol{w}(y, t)$ and $\boldsymbol{S}(y, t)$ from Equations (3) in the second equality from (1) and bearing in mind the fact that the fluid has been at rest up to the initial moment $t = 0,$ it can be proven that the components $S_{yy}, S_{yz}, S_{zz},$ and S_{zx} of the extra-stress tensor $\boldsymbol{S}$ are zero while the non-trivial shear stress $\eta(y, t) = S_{xy}(y, t)$ has to satisfy the next partial differential equation

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta(y, t) = \mu\left(1 + r\frac{\partial}{\partial t}\right)\frac{\partial w(y, t)}{\partial y};\ 0 < y < d,\ t > 0. \tag{4}$$

For these motions the continuity equation is identically satisfied and the balance of linear momentum, in the presence of conservative body forces but in absence of a pressure gradient in the flow direction, reduces to the next partial differential equation [13, 15]

$$\rho\frac{\partial w(y, t)}{\partial t} = \frac{\partial \eta(y, t)}{\partial y};\ 0 < y < d,\ t > 0, \tag{5}$$

where ρ is the constant density of the fluid. The appropriate initial conditions are

$$w(y, 0) = \left.\frac{\partial w(y, t)}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 w(y, t)}{\partial t^2}\right|_{t=0} = 0; \quad \eta(y, 0) = \left.\frac{\partial \eta(y, t)}{\partial t}\right|_{t=0} = 0;$$

$$0 \le y \le d. \tag{6}$$

In the following we investigate motions of IBFs with the next boundary conditions

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta(0, t) = \mu\left(1 + r\frac{\partial}{\partial t}\right)\left.\frac{\partial w(y, t)}{\partial y}\right|_{y=0} = S\cos(\omega t),$$

$$w(d, t) = 0;\ t > 0, \tag{7}$$

or

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta(0, t) = \mu\left(1 + r\frac{\partial}{\partial t}\right)\frac{\partial w(y, t)}{\partial y}\bigg|_{y=0} = S\sin(\omega t),$$

$$w(d, t) = 0;\ t > 0. \tag{8}$$

In above relations S is a constant shear stress and ω is the frequency of the oscillations.

Introducing the next dimensionless functions, variables and parameters

$$w^* = w\sqrt{\frac{\rho}{S}},\quad \eta^* = \frac{\eta}{S},\ y^* = \frac{y}{d},\ t^* = \frac{S}{\mu}t,$$

$$p^* = \frac{S}{\mu}p,\ q^* = \frac{S^2}{\mu^2}q,\ r^* = \frac{S}{\mu}r,\ \omega^* = \frac{\mu}{S}w, \tag{9}$$

and abandoning the star notation, the governing equations (4) and (5) take the non-dimensional forms

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta(y, t) = \frac{1}{\sqrt{\mathrm{Re}}}\left(1 + r\frac{\partial}{\partial t}\right)\frac{\partial w(y, t)}{\partial y};\ 0 < y < 1,\ t > 0, \tag{10}$$

$$\frac{\partial w(y, t)}{\partial t} = \frac{1}{\sqrt{\mathrm{Re}}}\frac{\partial \eta(y, t)}{\partial y};\ 0 < y < 1,\ t > 0, \tag{11}$$

in which the Reynolds number Re is defined by

$$\mathrm{Re} = \frac{Sd^2}{\mu\nu} = \frac{Vd}{\nu}, \tag{12}$$

and $V = Sd/\mu$ is a characteristic velocity.

The corresponding dimensionless initial and boundary conditions are

$$w(y,\,0) = \left.\frac{\partial w(y,\,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 w(y,\,t)}{\partial t^2}\right|_{t=0} = 0;\ \eta(y,\,0) = \left.\frac{\partial \eta(y,\,t)}{\partial t}\right|_{t=0} = 0;$$

$$0 \le y \le 1, \tag{13}$$

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta(0,\,t) = \frac{1}{\sqrt{\text{Re}}}\left(1 + r\frac{\partial}{\partial t}\right)\left.\frac{\partial w(y,\,t)}{\partial y}\right|_{y=0} = \cos(\omega t),$$

$$w(1,\,t) = 0;\ t > 0, \tag{14}$$

or again the initial conditions (13) together with the following boundary conditions:

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta(0,\,t) = \frac{1}{\sqrt{\text{Re}}}\left(1 + r\frac{\partial}{\partial t}\right)\left.\frac{\partial w(y,\,t)}{\partial y}\right|_{y=0} = \sin(\omega t),$$

$$w(1,\,t) = 0;\ t > 0. \tag{15}$$

Dimensionless starting solutions $w_c(y,\,t)$, $\eta_c(y,\,t)$ and $w_s(y,\,t)$, $\eta_s(y,\,t)$ corresponding to the two motion problems in discussion have to satisfy the governing equations (10) and (11) and the initial and boundary conditions (13), (14), respectively (13), (15). The form of the boundary conditions (14) and (15) and the fact that the fluid was at rest at the initial moment $t = 0$, suggest us the fact that both motions becomes steady in time. Generally, the starting solutions corresponding to such motions of incompressible fluids can be written as sums of their steady state (permanent or long time) and transient components, namely,

$$w_c(y,\,t) = w_{cp}(y,\,t) + w_{ct}(y,\,t), \quad \eta_c(y,\,t) = \eta_{cp}(y,\,t) + \eta_{ct}(y,\,t), \tag{16}$$

respectively,

$$w_s(y,\,t) = w_{sp}(y,\,t) + w_{st}(y,\,t), \quad \eta_s(y,\,t) = \eta_{sp}(y,\,t) + \eta_{st}(y,\,t). \tag{17}$$

Of course, some time after the motion initiation, the fluid moves according to the starting solutions $w_c(y, t)$, $\eta_c(y, t)$ or $w_s(y, t)$, $\eta_s(y, t)$. After this time, when the transients disappear or can be neglected (the magnitude of the transient components $w_{ct}(y, t)$, $\eta_{ct}(y, t)$, $w_{st}(y, t)$, $\eta_{st}(y, t)$ being zero or small enough), the fluid behaviour can be characterized by the steady state or permanent solutions $w_{cp}(y, t)$, $\eta_{cp}(y, t)$, $w_{sp}(y, t)$, $\eta_{sp}(y, t)$. This is the time to touch the steady or permanent state. From mathematical point of view, it is the time after which the diagrams of starting solutions superpose over those of their steady state components. In practice, this time is very important for the experimental researchers that have to know the moment after which the fluid moves according to the steady state solutions. In order to determine it, it is sufficient to know the steady state solutions which are independent of the initial conditions but satisfy the boundary conditions and governing equations. This is the reason, that in the following section, we shall determine closed-form expressions only for the steady state solutions $w_{cp}(y, t)$, $\eta_{cp}(y, t)$, $w_{sp}(y, t)$, and $\eta_{sp}(y, t)$.

3. Closed-form Expressions for the Steady State Solutions

In order to provide closed-form expressions for the dimensionless steady state solutions $w_{cp}(y, t)$, $\eta_{cp}(y, t)$ and $w_{sp}(y, t)$, $\eta_{sp}(y, t)$ of the two isothermal motions of IBFs we use the complex velocity and shear stress fields $w_p(y, t)$ and $\eta_p(y, t)$ defined by

$$w_p(y, t) = w_{cp}(y, t) + i w_{sp}(y, t),$$

$$\eta_p(y, t) = \eta_{cp}(y, t) + i\eta_{sp}(y, t); \; y > 0, \, t \in R, \qquad (18)$$

where i is the complex unit. The two complex entities have to satisfy the following partial differential equations (see Equations (10) and (11))

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\eta_p(y, t) = \frac{1}{\sqrt{\mathrm{Re}}}\left(1 + r\frac{\partial}{\partial t}\right)\frac{\partial w_p(y, t)}{\partial y};\ 0 < y < 1,\ t \in R, \tag{19}$$

$$\frac{\partial w_p(y, t)}{\partial t} = \frac{1}{\sqrt{\mathrm{Re}}}\frac{\partial \eta_p(y, t)}{\partial y};\ 0 < y < 1,\ t \in R. \tag{20}$$

Eliminating $\eta_p(y, t)$ between Equations (19) and (20) and bearing in mind the boundary conditions (14) and (15), it result that $w_p(y, t)$ has to satisfy the next boundary value problem

$$\left(1 + p\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}\right)\frac{\partial w_p(y, t)}{\partial t} = \frac{1}{\mathrm{Re}}\left(1 + r\frac{\partial}{\partial t}\right)\frac{\partial^2 w_p(y, t)}{\partial y^2};$$

$$0 < y < 1,\ t \in R, \tag{21}$$

$$\left(1 + r\frac{\partial}{\partial t}\right)\frac{\partial w_p(y, t)}{\partial y}\bigg|_{y=0} = \sqrt{\mathrm{Re}}\,\mathrm{e}^{i\omega t},\ w_p(1, t) = 0;\ t \in R. \tag{22}$$

Now, bearing in mind the linearity of the governing equations (19)-(21) and the nature of the boundary conditions (22), we are looking for solutions of the forms

$$w_p(y, t) = W(y)\mathrm{e}^{i\omega t},\ \eta_p(y, t) = T(y)\mathrm{e}^{i\omega t};\ 0 < y < 1,\ t \in R, \tag{23}$$

where $W(\cdot)$ and $T(\cdot)$ are complex functions. Direct computations show that

$$w_p(y, t) = \sqrt{\mathrm{Re}}\,\frac{\sinh[\gamma(y - 1)]}{\cosh(\gamma)}\frac{\mathrm{e}^{i\omega t}}{\gamma(1 + ir\omega)};\ 0 < y < 1,\ t \in R, \tag{24}$$

$$\eta_p(y, t) = \frac{\cosh[\gamma(y - 1)]}{\cosh(\gamma)}\frac{\mathrm{e}^{i\omega t}}{1 - q\omega^2 + ip\omega};\ 0 < y < 1,\ t \in R. \tag{25}$$

Simple computations show that $w_p(y, t)$ and $\eta_p(y, t)$ given by Equations (24), respectively (25) satisfy the governing equations (19)-(21) and the boundary conditions (22).

Consequently, the dimensionless steady state solutions $w_{cp}(y, t)$, $\eta_{cp}(y, t)$, $w_{sp}(y, t)$, and $\eta_{sp}(y, t)$ corresponding to the two motions of IBF are given by the simple relations

$$w_{cp}(y, t) = \sqrt{\mathrm{Re}}\, \Re e\left\{\frac{\sinh[\gamma(y-1)]}{\cosh(\gamma)} \frac{e^{i\omega t}}{\gamma(1 + ir\omega)}\right\}; \; 0 < y < 1, \; t \in R, \tag{26}$$

$$\eta_{cp}(y, t) = \Re e\left\{\frac{\cosh[\gamma(y-1)]}{\cosh(\gamma)} \frac{e^{i\omega t}}{1 - q\omega^2 + ip\omega}\right\}; \; 0 < y < 1, \; t \in R, \tag{27}$$

$$w_{sp}(y, t) = \sqrt{\mathrm{Re}}\, \mathrm{Im}\left\{\frac{\sinh[\gamma(y-1)]}{\cosh(\gamma)} \frac{e^{i\omega t}}{\gamma(1 + ir\omega)}\right\}; \; 0 < y < 1, \; t \in R, \tag{28}$$

$$\eta_{sp}(y, t) = \mathrm{Im}\left\{\frac{\cosh[\gamma(y-1)]}{\cosh(\gamma)} \frac{e^{i\omega t}}{1 - q\omega^2 + ip\omega}\right\}; \; 0 < y < 1, \; t \in R, \tag{29}$$

where $\Re e$ and Im denote the real, respectively the imaginary part of that which follows and the complex constant γ is given by the relation

$$\gamma = \sqrt{\frac{i\omega\, \mathrm{Re}(1 - q\omega^2 + ip\omega)}{1 + ir\omega}}. \tag{30}$$

Finally, it is worth to point out that the corresponding solutions for incompressible Oldroyd-B, Maxwell and Newtonian fluids performing similar motions can immediately be obtained making $q = 0$, $q = r = 0$, respectively, $p = q = r = 0$ in the relations (26)-(29). Taking $q = r = 0$ in these equalities, for instance, we find the corresponding solutions for the upper-convected Maxwell fluids performing similar motions, namely,

$$w_{Mcp}(y,\, t) = \sqrt{\mathrm{Re}}\ \Re e\left\{\frac{\sinh[\delta(y-1)]}{\cosh(\delta)}\,\frac{\mathrm{e}^{i\omega t}}{\delta}\right\};\ 0 < y < 1,\ t \in R, \tag{31}$$

$$\eta_{Mcp}(y,\, t) = \Re e\left\{\frac{\cosh[\delta(y-1)]}{\cosh(\delta)}\,\frac{\mathrm{e}^{i\omega t}}{1 + ip\omega}\right\};\ 0 < y < 1,\ t \in R, \tag{32}$$

$$w_{Msp}(y,\, t) = \sqrt{\mathrm{Re}}\ \mathrm{Im}\left\{\frac{\sinh[\delta(y-1)]}{\cosh(\delta)}\,\frac{\mathrm{e}^{i\omega t}}{\delta}\right\};\ 0 < y < 1,\ t \in R, \tag{33}$$

$$\eta_{Msp}(y,\, t) = \mathrm{Im}\left\{\frac{\cosh[\delta(y-1)]}{\cosh(\delta)}\,\frac{\mathrm{e}^{i\omega t}}{1 + ip\omega}\right\};\ 0 < y < 1,\ t \in R, \tag{34}$$

where $\delta = \sqrt{i\omega\,\mathrm{Re}(1 + ip\omega)}$. The solutions $w_{Mcp}(y, t)$, $\eta_{Mcp}(y, t)$ and $w_{Msp}(y, t)$, $\eta_{Msp}(y, t)$, given by Equations (31), (32) and (33), (34), respectively satisfy the boundary conditions

$$\left(1 + p\frac{\partial}{\partial t}\right)\eta(0,\, t) = \frac{1}{\sqrt{\mathrm{Re}}}\,\frac{\partial w(y,\, t)}{\partial y}\bigg|_{y=0} = \cos(\omega t),\ w(1,\, t) = 0;\ t \in R, \tag{35}$$

respectively,

$$\left(1 + p\frac{\partial}{\partial t}\right)\eta(0,\, t) = \frac{1}{\sqrt{\mathrm{Re}}}\,\frac{\partial w(y,\, t)}{\partial y}\bigg|_{y=0} = \sin(\omega t),\ w(1,\, t) = 0;\ t \in R. \tag{36}$$

As it was to be expected, the dimensional forms of $w_{Mcp}(y, t)$, $\eta_{Mcp}(y, t)$ $w_{Msp}(y, t)$, and $\eta_{Msp}(y, t)$ from Equations (31)-(34) are identical to the dimensional forms of the corresponding solutions obtained by Fetecau et al. [18, Equations (48)-(51)].

The similar solutions for incompressible second grade fluids, namely,

$$w_{SGcp}(y,\, t) = \sqrt{\mathrm{Re}}\ \Re e\left\{\frac{\sinh[\beta(y-1)]}{\cosh(\beta)}\,\frac{\mathrm{e}^{i\omega t}}{\beta(1 + ir\omega)}\right\};\ 0 < y < 1,\ t \in R, \tag{37}$$

$$\eta_{SGcp}(y,\, t) = \Re e\left\{\frac{\cosh[\beta(y-1)]}{\cosh(\beta)}\,\mathrm{e}^{i\omega t}\right\};\ 0 < y < 1,\ t \in R, \tag{38}$$

$$w_{SGsp}(y,\,t) = \sqrt{\mathrm{Re}}\,\mathrm{Im}\left\{\frac{\sinh[\beta(y-1)]}{\cosh(\beta)}\,\frac{e^{i\omega t}}{\beta(1+ir\omega)}\right\};\ 0 < y < 1,\ t \in R, \quad (39)$$

$$\eta_{SGsp}(y,\,t) = \mathrm{Im}\left\{\frac{\cosh[\beta(y-1)]}{\cosh(\beta)}\,e^{i\omega t}\right\};\ 0 < y < 1,\ t \in R, \quad (40)$$

where $\beta = \sqrt{i\omega\,\mathrm{Re}/(1+ir\omega)}$ are immediately obtained making $p = q = 0$ in Equations (26)-(29). As expected, the dimensional forms of these solutions are identical to those obtained by Fetecau and Vieru [19, Equations (45) and (46)] in the absence of magnetic and porous effects. Unfortunately, the denominator of the constant γ from this last reference has been omitted. Interesting steady solutions for flows of incompressible second grade fluid in a plane channel have been recently obtained by Baranovskii and Artemov [20, 21].

The corresponding dimensionless steady state Newtonian solutions, namely,

$$w_{Ncp}(y,\,t) = \Re e\left\{\frac{\sinh[(y-1)\sqrt{i\omega\,\mathrm{Re}}]}{\cosh(\sqrt{i\omega\,\mathrm{Re}})}\,\frac{e^{i\omega t}}{\sqrt{i\omega}}\right\};\ 0 < y < 1,\ t \in R, \quad (41)$$

$$\eta_{Ncp}(y,\,t) = \Re e\left\{\frac{\cosh[(y-1)\sqrt{i\omega\,\mathrm{Re}}]}{\cosh(\sqrt{i\omega\,\mathrm{Re}})}\,e^{i\omega t}\right\};\ 0 < y < 1,\ t \in R, \quad (42)$$

$$w_{Nsp}(y,\,t) = \mathrm{Im}\left\{\frac{\sinh[(y-1)\sqrt{i\omega\,\mathrm{Re}}]}{\cosh(\sqrt{i\omega\,\mathrm{Re}})}\,\frac{e^{i\omega t}}{\sqrt{i\omega}}\right\};\ 0 < y < 1,\ t \in R, \quad (43)$$

$$\eta_{Nsp}(y,\,t) = \mathrm{Im}\left\{\frac{\cosh[(y-1)\sqrt{i\omega\,\mathrm{Re}}]}{\cosh(\sqrt{i\omega\,\mathrm{Re}})}\,e^{i\omega t}\right\};\ 0 < y < 1,\ t \in R, \quad (44)$$

are also obtained making $p = q = r = 0$ into Equations (26)-(29) or $p = 0$ in Equations. (31)-(34). They correspond to isothermal motions of incompressible Newtonian fluids induced by the lower plate that applies an oscillatory shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. We also mention the fact that the dimensional forms of the velocity fields

$w_{Ncp}(y, t)$ and $w_{Nsp}(y, t)$ given by Equations (41) and (43), respectively, have opposite signs to the dimensional forms of the similar solutions obtained by Maria Javaid et al. [22, Equations (42)]. This is due the fact that the first boundary conditions have opposed signs in the two works.

Now, taking the limit of Equations (41) and (42) when $\omega \to 0$, one obtains the solutions

$$w_{Np}(y) = \lim_{\omega \to 0} w_{Ncp}(y, t) = (y-1)\sqrt{\mathrm{Re}},$$

$$\eta_{Np} = \lim_{\omega \to 0} \eta_{Ncp}(y, t) = 1;\ 0 < y < 1,\ t \in R, \tag{45}$$

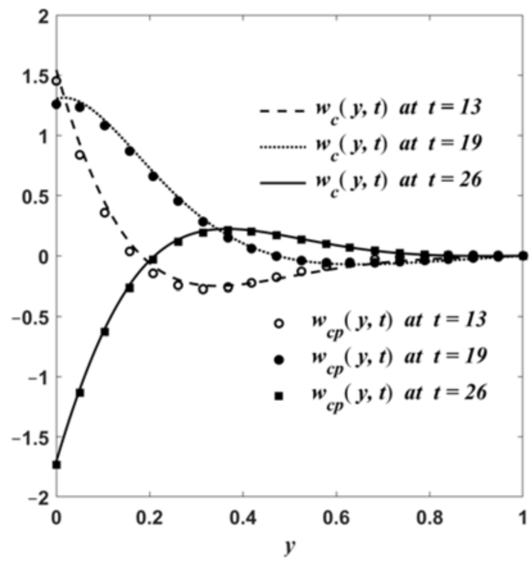
corresponding to the steady motion of incompressible Newtonian fluids produced by the lower plate that applies a constant shear stress S to the fluid. Here, a surprising result is the fact that the shear stress corresponding to such a motion of incompressible Newtonian fluids is constant on the entire flow domain although the fluid velocity is a function of the spatial variable y. Furthermore, this constant is just the shear stress applied by the lower plate to the fluid. In addition, the dimensional forms of these last steady solutions $w_{Np}(y)$ and η_{Np} are in accordance with the similar solutions resulting from the relations (28) of the reference [22].

4. Numerical Results and Applications

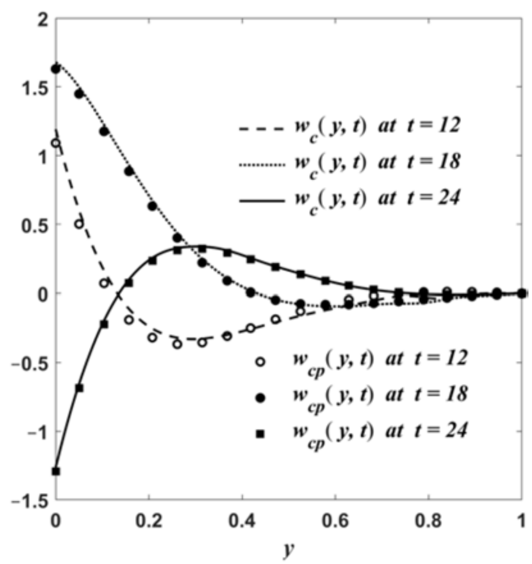
In this work, two mixed initial-boundary value problems are analytically and numerically investigated. Their solutions describe isothermal unsteady motions of IBFs between infinite horizontal parallel plates when a differential expression of the shear stress $\eta(y, t)$ different from zero is prescribed on a part of the boundary. Closed-form expressions are determined for the dimensionless steady state velocities $w_{cp}(y, t)$, $w_{sp}(y, t)$ and the corresponding non-trivial shear stresses $\eta_{cp}(y, t)$, $\eta_{sp}(y, t)$. The obtained results can easily be particularized to give dimensionless steady state solutions for incompressible Oldroyd-B, Maxwell, second grade and Newtonian fluids performing similar motions. For validation, the solutions corresponding to incompressible Maxwell, second grade and Newtonian fluids performing similar motions have been provided and compared with known solutions from the existing literature.

Now, as application, the dimensionless steady state solution $w_{cp}(y, t)$ given by Equation (26) is used in Figures 1-3 to determine the required time to touch the steady or permanent state for motions of the incompressible Burgers, Oldroyd-B and Maxwell fluids. As we already have mentioned, this is the time after which the diagrams of starting solution $w_c(y, t)$ superpose over those of its steady state component $w_{cp}(y, t)$.

From these graphical representations, which clearly show the convergence of the starting solutions $w_c(y, t)$, $w_{Oc}(y, t)$, $w_{Mc}(y, t)$ to their steady state components $w_{cp}(y, t)$, $w_{Ocp}(y, t)$, and $w_{Mcp}(y, t)$, respectively for increasing values of the time t, it clearly results that the required time to touch the steady state for such isothermal unsteady motions of the incompressible non-Newtonian fluids diminishes for increasing values of the material parameters p, q or r. Furthermore, bearing in mind the values that have been ascribed to these parameters, it also results that the steady state is earlier obtained for motions of Burgers fluids in comparison with Maxwell or Oldroyd-B fluids. It also results that this state is rather touched for motions of Maxell fluids in comparison with Oldroyd-B fluids.

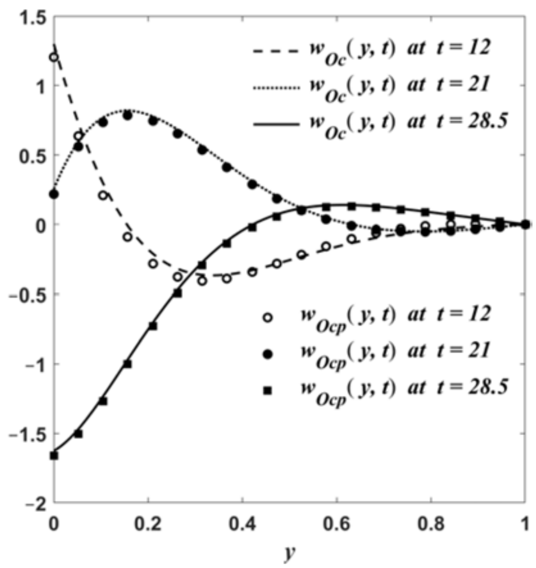


(a) $q = 0.1$

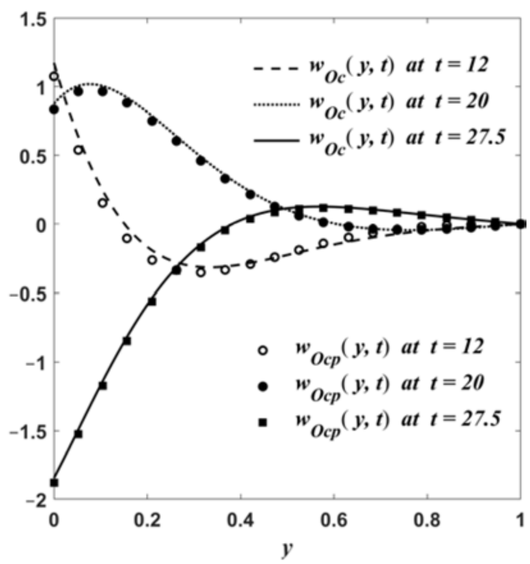


(b) $q = 0.9$

Figure 1. Required time to touch the steady state for the motion of incompressible Burgers fluids when a differential expression of the shear stress on the lower plate is equal to $S\cos(\omega t)$ at two distinct values of q and $p = 0.8$, $r = 0.7$, $\omega = \pi/12$, $\mathrm{Re} = 100$.

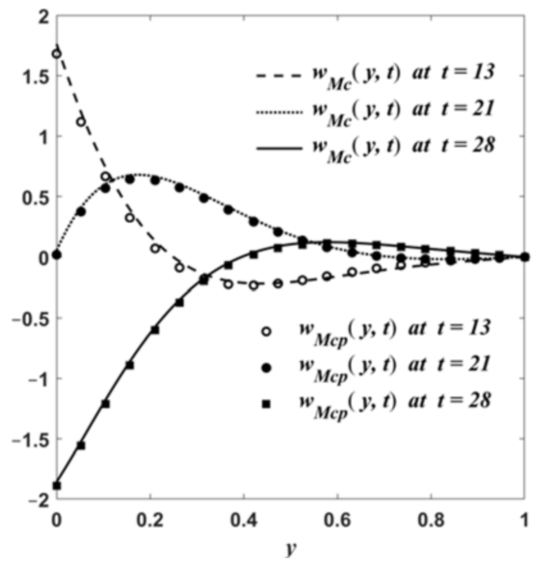


(a) $r = 0.1$

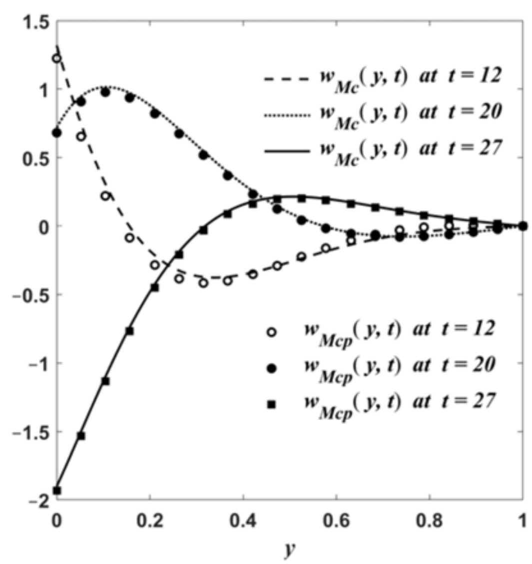


(b) $r = 0.7$

Figure 2. Required time to touch the steady state for the motion of the incompressible Oldroyd-B fluids when a differential expression of the shear stress on the lower plate is equal to $S\cos(\omega t)$ at two distinct values of r and $p = 0.8$, $\omega = \pi/12$, $\text{Re} = 100$.



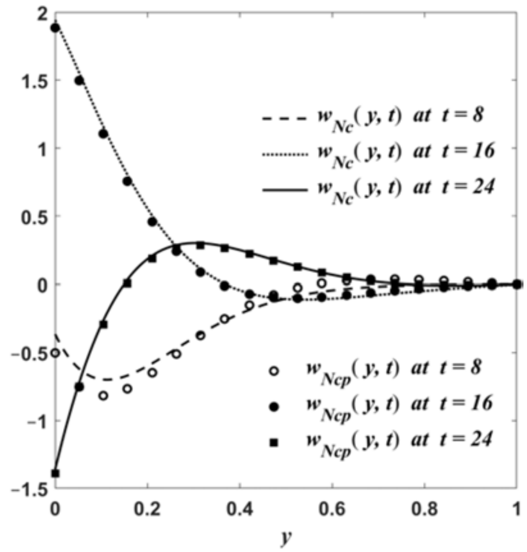
(a) $p = 0.1$



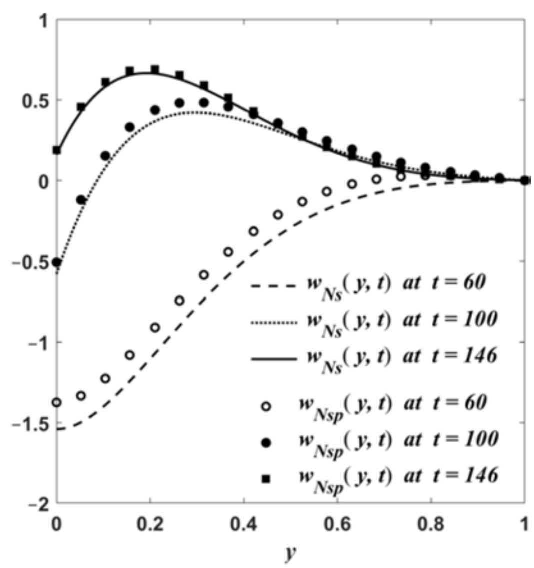
(b) $p = 0.8$

Figure 3. Required time to touch the steady state for the motion of incompressible Maxwell fluids when a differential expression of shear stress on the lower plate is equal to $S\cos(\omega t)$ at two distinct values of p and $\omega = \pi/12$, Re = 100.

Convergence of the dimensionless Newtonian starting solutions $w_{Nc}(y, t)$ and $w_{Ns}(y, t)$ to their steady state components $w_{Ncp}(y, t)$, respectively $w_{Nsp}(y, t)$ for increasing values of the time t is graphically proved in Figures 4(a) and 4(b).



(a) cosine oscillations



(b) sine oscillations

Figure 4. Required time to touch the steady state for motions of Newtonian fluids due to cosine or sine oscillations of the shear stress on the boundary when $\omega = \pi/12$ and $\mathrm{Re} = 100$.

From these figures it also results that need time to reach the steady state for such motions of incompressible Newtonian fluids is much shorter when the lower plate applies the shear stress $S\cos(\omega t)$ instead of $S\sin(\omega t)$ to the fluid. Consequently, the steady state for such motions of the incompressible Newtonian fluids is earlier obtained for cosine oscillations in comparison with sine oscillations of the shear stress on the boundary. This is obvious because, in the second case, the shear stress applied by the plate to the fluid is zero at the initial moment $t = 0$. In addition, the steady state for isothermal unsteady motions of the incompressible fluids is rather obtained for motions of Newtonian fluids in comparison with the non-Newtonian fluids.

5. Conclusions

Isothermal unsteady unidirectional motions of IBFs between infinite horizontal parallel plates are first time investigated when a differential expression of the non-trivial shear stress is prescribed on the lower plate as being equal to $S\cos(\omega t)$ or $S\sin(\omega t)$. Closed form-expressions are established for the non-dimensional steady state velocity fields $w_{cp}(y, t)$, $w_{sp}(y, t)$ and the corresponding shear stresses $\eta_{cp}(y, t)$ and $\eta_{sp}(y, t)$, respectively. In some particular cases, as expected, the dimensional forms of these solutions are identical to known results from the existing literature. The obtained solutions, presented in the simplest forms, are used to determine the required time to reach the steady state. This time, after which the fluid behaviour is characterized by the steady state solutions, is very important in practice for the experimental researchers.

The main conclusions of this study are:

(i) First exact expressions are provided for steady state solutions corresponding to two isothermal unidirectional motions of IBFs between infinite horizontal parallel plates when a differential expression of the non-trivial shear stress is prescribed on the lower plate.

(ii) These expressions can easily be particularized to give exact steady state solutions for Oldroyd-B, Maxwell, second grade and Newtonian fluids performing similar motions.

(iii) The convergence of the dimensionless starting velocities (numerical solutions) $w_c(y, t)$ and $w_s(y, t)$ to their steady state components $w_{cp}(y, t)$, respectively $w_{sp}(y, t)$ for increasing values of the time t could be a certainty for the correctness of obtained results.

(iv) Steady state for such isothermal motions of incompressible non-Newtonian fluids is rather obtained for the Burgers fluids in comparison with Oldroyd-B or Maxwell fluids.

(v) Steady state for motions of incompressible Newtonian fluids is much later obtained when the lower plate applies the shear stress $S \sin(\omega t)$ instead of $S \cos(\omega t)$ to the fluid.

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