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ADEQUATE INTELLIGENT CONSTRUCTING PREDICTIVE OR OPTIMAL STATISTICAL DECISIONS UNDER PARAMETRIC UNCERTAINTY OF APPLIED STOCHASTIC MODELS

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Abstract

The method used here focuses on the basic quantities and supporting statistics needed to construct prediction limits or optimal solutions for expected outcomes under the parametric uncertainty of applied stochastic models. It is applicable whenever the statistical problem is invariant under a group of transformations acting transitively on the parameter space. This

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method does not require any tabulation and is applicable whether the statistics are complete or Type II censored. The exact prediction limits of the order statistics associated with a sample of basic distributions can be found quickly and easily, making tables, simulations, Monte Carlo estimated percentiles, special computer programs, and approximations unnecessary. The proposed technique is based on the transformation of probabilities and the averaging of reference values. It is conceptually simple and easy to use. The discussion is limited to one-sided prediction limits. Finally, we provide practical numerical examples where the proposed analytical methodology is illustrated in terms of the one-parameter (or two-parameter) exponential distribution. Applications to other log-location-scale distributions can follow directly.

Keywords: anticipated outcomes, parametric uncertainty, unknown (nuisance) parameters, elimination, pivotal quantities, ancillary statistics, prediction limits, tolerance limits, optimal decisions.

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1. Introduction

Statistical forecasting and optimization (under parametric uncertainty) based on past and present knowledge is a fundamental problem in statistics that arises in many contexts and leads to different solutions. Statistical forecasting is the process by which values of unknown observables (potential observations yet to be made or past observations that are no longer available) are inferred from current observations and other available information. The approach used here is a special case of more general considerations applicable whenever the statistical problem is invariant under the group of transformations acting transitively on the parameter space [1-11].

1.1. Computational intelligence method for removing unknown (nuisance) parameters from the base model

Removing unknown (nuisance) parameters from the base model includes the following 3 steps: (1) invariant embedding of sample statistics into the base model, (2) averaging the base model over the probability distribution of the pivotal quantity, and (3) finding an effective statistical decision.

2. Preliminaries

2.1. Adequate transformation of the distribution of the order statistic

Theorem 1 (Adequate transformation of the distribution of the order statistic)**.** *Let us assume that there is a random sample of m ordered observations* $Y_1 \leq \ldots \leq Y_m$ *from a known distribution with probability density function* $f_\theta(y)$, *probability distribution function* $F_\theta(y)$, *where* θ *is the parameter* (*in general, vector*), *then the adequate transformation of the cumulative distribution function of the l-th order statistic* Y_l, $l \in \{1, 2, \ldots, m\}$, *is given by*

$$G_\theta(y_l|m) = P_\theta(Y_l \leq y_l|m) = \sum_{j=l}^{m} \binom{m}{j} [F_\theta(y_l)]^j [1 - F_\theta(y_l)]^{m-j}$$

$$= \int_0^{F_\theta(y_l)} f_{l,m-l+1}(u)du, \tag{1}$$

where

$$f_{l,m-l+1}(u) = \frac{1}{B(l, m-l+1)} u^{l-1}(1-u)^{(m-l+1)-1}, \; 0 < u < 1, \tag{2}$$

is the probability density function (*PDF*) *of the beta distribution* $(Beta(l, m-l+1))$ *with shape parameters* l *and* $m-l+1$.

Proof. On the one hand, it follows from (1) that

$$\frac{d}{dy_l} G_\theta(y_l|m) = \frac{d}{dy_l} P_\theta(Y_l \leq y_l|m) = \frac{d}{dy_l} \sum_{j=l}^{m} \binom{m}{j} [F_\theta(y_l)]^j [1 - F_\theta(y_1)]^{m-j}$$

$$= \sum_{j=l}^{m} \binom{m}{j} \frac{d}{dy_l} [F_\theta(y_l)]^j [1 - F_\theta(y_1)]^{m-j}$$

$$= \sum_{j=l}^{m} \binom{m}{j} \Big[jF_\theta(y_l)^{j-1}(1 - F_\theta(y_l))^{m-j} F'_\theta(y_l)$$

$$- (m-j)F_\theta(y_l)^j (1 - F_\theta(y_l))^{m-j-1} F'_\theta(y_l) \Big]$$

$$
\begin{aligned}
&= \sum_{j=l}^{m} \binom{m}{j} \Big[j F_\theta(y_l)^{j-1} (1 - F_\theta(y_l))^{m-j} \\
&\quad - (m-j) F_\theta(y_l)^{j} (1 - F_\theta(y_l))^{m-j-1} \Big] f_\theta(y_l) \\
&= \sum_{j=l}^{m} \frac{m!}{(j-1)!\,(m-j)!} F_\theta(y_l)^{j-1} (1 - F_\theta(y_l))^{m-j} F_\theta(y_l) \\
&\quad - \sum_{j=l}^{m-1} \frac{m!}{j!(m-j-1)!} F_\theta(y_l)^{j} (1 - F_\theta(y_l))^{m-j-1} f_\theta(y_l) \\
&= \frac{m!}{(l-1)!(m-l)!} F_\theta(y_l)^{l-1} (1 - F_\theta(y_l))^{m-l} f_\theta(y_l) \\
&\quad + \sum_{j=l+1}^{m} \frac{m!}{(j-1)!\,(m-j)!} F_\theta(y_l)^{j-1} (1 - F_\theta(y_l))^{m-j} f_\theta(y_l) \\
&\quad - \sum_{j=l}^{m-1} \frac{m!}{j!(m-j-1)!} F_\theta(y_l)^{j} (1 - F_\theta(y_l))^{m-j-1} f_\theta(y_l) \\
&= \frac{m!}{B(l,\, m-l+1)!} F_\theta(y_l)^{l-1} (1 - F_\theta(y_l))^{m-l} f_\theta(y_l) = g_\theta(y_l|m).
\end{aligned}
\tag{3}
$$

If $j = i + 1$, we have that

$$
\begin{aligned}
&\sum_{j=l+1}^{m} \frac{m!}{(j-1)!\,(m-j)!} F_\theta(y_l)^{j-1} (1 - F_\theta(y_l))^{m-j} f_\theta(y_l) \\
&\qquad = \sum_{i=l}^{m-1} \frac{m!}{i!(m-i-1)!} F_\theta(y_l)^{i} (1 - F_\theta(y_l))^{m-i-1} f_\theta(y_l).
\end{aligned}
\tag{4}
$$

Thus, it follows from (3) that

$$\frac{d}{dy_l} G_\theta(y_l|m) = \frac{d}{dy_l} P_\theta(Y_l \le y_l|m)$$

$$= \frac{1}{B(l,\ m-l+1)!} F_\theta(y_l)^{l-1}(1 - F_\theta(y_l))^{(m-l+1)-l} f_\theta(y_l)$$

$$= g_\theta(y_l|m). \tag{5}$$

On the other hand, it follows from (1) that

$$\frac{d}{dy_l} G_\theta(y_l|m) = \frac{d}{dy_l} P_\theta(Y_l \le y_l|m) = \frac{d}{dy_l} \int_0^{F_\theta(y_l)} f_{l,\,m-l+1}(u)du$$

$$= \frac{d}{dy_l} \int_0^{F_\theta(y_l)} \frac{1}{B(l,\ m-l+1)} u^{l-1}(1-u)^{(m-l+1)-l} du$$

$$= \frac{1}{B(l,\ m-l+1)} F_\theta(y_l)^{l-1}(1 - F_\theta(y_l))^{(m-l+1)-l} f_\theta(y_l)$$

$$= g_\theta(y_l|m). \tag{6}$$

Thus, $F_\theta(y_l)$ is the generalized pivotal quantity:

$$F_\theta(y_l) = u \sim f_{l,\,m-l+1}(u) = \frac{1}{B(l,\ m-l+1)} u^{l-1}(1-u)^{(m-l+1)-1}, \quad 0 < u < 1. \tag{7}$$

This ends the proof.

2.2. Conditional probability density function of the order statistic

Theorem 2. *Let* $Y_1 \le \ldots \le Y_k$ *be the first k ordered observations* (*order statistics*) *in a sample of size m from a continuous distribution with some probability density function* $f_\theta(y)$ *and distribution function* $F_\theta(y)$, *where* θ *is a parameter* (*in general, vector*). *Then the conditional probability density function of* $Y_l (1 \le k < l \le m)$ *given* $Y_k = y_k$ *is determined as follows.*

The joint density of $Y_1 \leq \ldots \leq Y_k$ and Y_l is given by

$$g_\theta(y_1, \ldots, y_k, y_l) = \frac{m!}{(l-k-1)!\,(m-l)!} \prod_{i=1}^{k} f_\theta(y_i)[F_\theta(y_l)$$

$$- F_\theta(y_k)]^{l-k-1} f_\theta(y_l)[1 - F_\theta(y_l)]^{m-l}$$

$$= \left(\frac{m!}{(m-k)!} \prod_{i=1}^{k} f_\theta(y_i)[1 - F_\theta(y_l)]^{m-k} \right) \left(\frac{(m-k)!}{(l-k-1)!\,(m-l)!} \right.$$

$$\times \left[\frac{F_\theta(y_l) - F_\theta(y_k)}{1 - F_\theta(y_k)} \right]^{l-k-1} \left[\frac{1 - F_\theta(y_l)}{1 - F_\theta(y_k)} \right]^{m-l} \frac{f_\theta(y_l)}{1 - F_\theta(y_k)}$$

$$= (g_\theta(y_1, \ldots, y_k;\, m))\,(g_\theta(y_l | y_k;\, m)). \tag{8}$$

It follows from (8) that

$$g_\theta(y_l | y_1, \ldots, y_k;\, m) = \frac{g_\theta(y_1, \ldots, y_k, y_l;\, m)}{g_\theta(y_1, \ldots, y_k;\, m)}$$

$$= \frac{g_\theta(y_1, \ldots, y_l;\, m) g_\theta(y_l | y_k;\, m)}{g_\theta(y_1, \ldots, y_k;\, m)} = g_\theta(y_l | y_k;\, m)$$

$$= \frac{(m-k)!}{(l-k-1)!\,(m-l)!} \left[\frac{F_\theta(y_l) - F_\theta(y_k)}{1 - F_\theta(y_k)} \right]^{l-k-1} \left[\frac{1 - F_\theta(y_l)}{1 - F_\theta(y_k)} \right]^{m-1} \frac{f_\theta(y_l)}{1 - F_\theta(y_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!\,(m-l)!} \left[\frac{1 - F_\theta(y_k) - (1 - F_\theta(y_l)}{1 - F_\theta(y_k)} \right]^{l-k-1} \left[\frac{1 - F_\theta(y_l)}{1 - F_\theta(y_k)} \right]^{m-1}$$

$$\times \frac{f_\theta(y_l)}{1 - F_\theta(y_k)} = \frac{(m-k)!}{(l-k-1)!\,(m-l)!} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)} \right]^{l-k-1} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)} \right]^{m-1} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}. \tag{9}$$

$$G_\theta(y_l|y_k;\, m) = P_\theta(Y_l \le y_l|Y_k = y_k;\, m) = \int_{y_k}^{y_l} g_\theta(y|y_k;\, m)dy$$

$$= \int_{y_k}^{y_l} \frac{(m-k)!}{(l-k-1)!\,(m-l)!}\left[1-\frac{\overline{F}_\theta(y)}{\overline{F}_\theta(y_k)}\right]^{l-k-1}\left[\frac{\overline{F}_\theta(y)}{\overline{F}_\theta(y_k)}\right]^{m-l}\frac{f_\theta(y)}{\overline{F}_\theta(y_k)}\,dy$$

$$= -\int_{y_k}^{y_l} \frac{(m-k)!}{(l-k-1)!\,(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j\left[\frac{\overline{F}_\theta(y)}{\overline{F}_\theta(y_k)}\right]^{m-l+j} d\left(\frac{\overline{F}_\theta(y)}{\overline{F}_\theta(y_k)}\right)$$

$$= \frac{1}{B(l-k,\, m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\left[\frac{\overline{F}_\theta(y_k)}{\overline{F}_\theta(y_k)}\right]^{m-l+1+j}$$

$$-\frac{1}{B(l-k,\, m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l+1+j}$$

$$= 1-\frac{1}{B(l-k,\, m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+j+1}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l+j+1}$$

$$= \sum_{j=l-k}^{m-k}\binom{m-k}{j}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^j\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}, \tag{10}$$

where

$$\frac{1}{B(l-k,\, m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\left[\frac{\overline{F}_\theta(y_k)}{\overline{F}_\theta(y_k)}\right]^{m-l+1+j} = 1. \tag{11}$$

2.3. Adequate transformation of the conditional cumulative distribution function of the order statistic

Theorem 3 (Adequate transformation of the conditional cumulative distribution function (cdf) of the l-th order statistic)**.** *Let us assume that there is a random sample of m ordered observations* $Y_1 \le \ldots \le Y_m$ *from a known distribution with probability density function* $f_\theta(y)$, *cumulative distribution function* $F_\theta(y)$, *where* θ *is the parameter* (*in general, vector*), *then the adequate transformation of the conditional cumulative distribution function of the l-th order statistic* $Y_l (1 \le k < l \le m)$ *given* $Y_k = y_k$ *is determined as*

$$G_\theta(y_l|y_k;\, m) = P_\theta(Y_l \le y_l|Y_k = y_k;\, m)$$

$$= \sum_{j=l-k}^{m-k} \binom{m-k}{j} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^j \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}$$

$$= \int_0^{1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}} f_{l-k,\, m-l+1}(u)\,du, \tag{12}$$

where $\overline{F}_\theta(y) = 1 - F_\theta(y)$,

$$f_{l-k,\, m-l+1}(u) = \frac{1}{B(l-k,\, m-l+1)} u^{l-k-1}(1-u)^{(m-l+1)-1} du,\ \ 0 < u < 1, \tag{13}$$

is the probability density function (*PDF*) *of the beta distribution* $(Beta(l-k,\, m-l+1))$ *with shape parameters* $l-k$ *and* $m-k+1$.

Proof. On the one hand, it follows from (12) that

$$\frac{d}{dy_l} G_\theta(y_l|y_k;\, m) = \frac{d}{dy_l} P_\theta(Y_l \le y_l|Y_k = y_k;\, m)$$

$$= \frac{d}{dy_l} \sum_{j=l-k}^{m-k} \binom{m-k}{j} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^j \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}$$

$$= \sum_{j=l-k}^{m-k} \binom{m-k}{j} \frac{d}{dy_l} \left(\left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^j \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}\right)$$

$$= \sum_{j=l-k}^{m-k} \binom{m-k}{j} \left(j\left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j-1} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}\right.$$

$$\left. - (m-k-j)\left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^j \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j-1}\right) \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= \sum_{j=l-k}^{m-k} \frac{(m-k)!}{(j-1)!\,(m-k-j)!} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j-1} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$- \sum_{j=l-k}^{m-k-1} \frac{(m-k)!}{j!\,(m-k-j-1)!} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j+1} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!\,(m-l)!} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{l-k-1} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$+ \sum_{j=l-k+1}^{m-k} \frac{(m-k)!}{(j-1)!\,(m-k-j)!} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j-1} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$- \sum_{j=l-k}^{m-k-1} \frac{(m-k)!}{j!\,(m-k-j-1)!} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j-1} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!\,(m-l)!}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{l-k-1}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l}\frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= \frac{1}{B(l-k,\ m-l+1)}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{l-k-1}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l}\frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}. \tag{14}$$

If $j = i+1$, we have that

$$\sum_{j=l-k+1}^{m-k}\frac{(m-k)!}{(j-1)!\,(m-k-j)!}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j-1}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}\frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= \sum_{j=l-k}^{m-k-1}\frac{(m-k)!}{i!\,(m-k-i-1)!}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{i}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-i-1}\frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}. \tag{15}$$

Thus, it follows from (14) that

$$\frac{d}{dy_l}G_\theta(y_l|y_k;\ m) = \frac{d}{dy_l}P_\theta(Y_l \le y_l|Y_k = y_k;\ m)$$

$$= \frac{d}{dy_l}\sum_{j=l-k}^{m-k}\binom{m-k}{j}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{j}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-k-j}$$

$$= \frac{1}{B(l-k,\ m-l+1)}\left[1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{l-k-1}\left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l}\frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= g(y_l|y_k;\ m). \tag{16}$$

On the other hand, it follows from (12) that

$$\frac{d}{dy_l}G_\theta(y_l|y_k;\ m) = \frac{d}{dy_l}P_\theta(Y_l \le y_l|Y_k = y_k;\ m)$$

$$= \frac{d}{dy_l}\int_0^{1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}} f_{k,m-k+1}(u)du$$

$$= \frac{d}{dy_l} \int_0^{1-\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}} \frac{1}{B(l-k,\ m-l+1)} u^{l-k-1}(1-u)^{(m-l+1)-1} du$$

$$= \frac{1}{B(l-k,\ m-l+1)} \left[1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{l-k-1} \left[\frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}\right]^{m-l} \frac{f_\theta(y_l)}{\overline{F}_\theta(y_k)}$$

$$= g(y_l | y_k;\ m). \tag{17}$$

Thus, $1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)}$ is the generalized pivotal quantity:

$$1 - \frac{\overline{F}_\theta(y_l)}{\overline{F}_\theta(y_k)} = u \sim f_{l-k,\, m-l+1}(u)$$

$$= \frac{1}{B(l-k,\ m-l+1)} u^{l-k-1}(1-u)^{(m-l+1)-1} du, \quad 0 < u < 1. \tag{18}$$

This ends the proof.

2.4. One-parameter exponential distribution

This distribution is one of the most commonly used models in different situations of life-testing and reliability studies. Let $X = (X_1 \le \ldots \le X_k)$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution with the probability density function

$$f_\beta(x) = \beta^{-1} \exp(-x/\beta), \quad \beta > 0,\ x > 0, \tag{19}$$

and the cumulative probability distribution function

$$F_\beta(x) = 1 - \exp(-x/\beta), \tag{20}$$

where β is the scale parameter. It is assumed that the parameter β is unknown. In Type II censoring, which is of primary interest here, the

number of survivors is fixed and X_k is a random variable. It is known that

$$S_k = \sum_{j=1}^{k} X_j + (n-k)X_k, \tag{21}$$

is the complete sufficient statistic for β. The probability density function of S_k is given by

$$f_\beta(s_k|k) = \frac{1}{\Gamma(k)\beta^k} s_k^{k-1} \exp\left(-\frac{s_k}{\beta}\right), \; s_k \geq 0, \tag{22}$$

$$V_k = S_k/\beta, \tag{23}$$

is the pivotal quantity, the probability density function of which is given by

$$f(v_k|k) = \frac{1}{\Gamma(k)} v_k^{k-1} \exp(-v_k), \quad v_k \geq 0. \tag{24}$$

2.5. Statistical pivot-based estimation of probability distribution

Suppose X is a future observation from the same distribution (19), independent of $X = (X_1 \leq \ldots \leq X_k)$. Then the pivot-based estimate of (20) can be determined as follows:

Step 1. Invariant embedding of S_k in (20) to isolate unknown parameter β from the problem through V_k,

$$F_\beta(x) = 1 - \exp(-x/\beta) = 1 - \exp\left(-\frac{x}{s_k}\frac{s_k}{\beta}\right)$$

$$= 1 - \exp\left(-\frac{x}{s_k} v_k\right) = F_\beta\left(\frac{x}{s_k} v_k\right), \tag{25}$$

where

$$W = \frac{X}{S_k}, \tag{26}$$

is an ancillary statistic.

Step 2. Averaging (25) over the probability distribution of the pivotal quantity V_k to eliminate unknown parameter β from the problem. It follows from (25) and (24) that the pivot-based estimate of the cumulative distribution function (25) (through the pivot-based approach) is given by

$$F_{s_k}(x) = \int_0^\infty F_\beta\left(\frac{x}{s_k} v_k\right) f(v_k|k) dv_k$$

$$= \int_0^\infty \left[1 - \exp\left(-\frac{x}{s_k} v_k\right)\right] \frac{1}{\Gamma(k)} v_k^{k-1} \exp(-v_k) dv_k$$

$$= 1 - \frac{1}{\left(1 + \frac{x}{s_k}\right)^k}; \; F(w) = 1 - \frac{1}{(1+w)^k}. \tag{27}$$

The pivot-based estimate of the probability density function (19) is given by

$$f_{s_k}(x) = \frac{dF_{s_k}(x)}{dy} = \frac{k}{s_k}\left(1 + \frac{x}{s_k}\right)^{-k-1}, \; x \geq 0, \tag{28}$$

and

$$\overline{F}_{s_k}(x) = 1 - F_{s_k}(x) = \left(1 + \frac{x}{s_k}\right)^{-k}. \tag{29}$$

The probability density function of the l-th order statistic Y_l in a sample of size m is given by

$$g_\beta(y_l;\, m) = \frac{m!}{(l-1)!\,(m-l)!} \left[F_\beta(y_l)\right]^{l-1} \left[1 - F_\beta(y_l)\right]^{m-l} f_\beta(y_l)$$

$$= \frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \exp\left(-\frac{y_l}{\beta}(m - l + 1 + j)\right) \frac{1}{\beta},$$

$$y_l \in (0, \infty). \tag{30}$$

The cumulative probability distribution function of the l-th order statistic Y_l in a sample of size m is given by

$$G_\beta(y_l;\, m) = \int_0^{y_l} g_\beta(y)dy$$

$$= \int_0^{y_l} \frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \exp\left(-\frac{y}{\beta}(m-l+1+j)\right) \frac{1}{\beta}\, dy$$

$$= 1 - \frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \frac{\exp\left(-\frac{y_l}{\beta}(m-l+1+j)\right)}{m-l+j+1}. \tag{31}$$

The cumulative probability distribution function of the ancillary statistic

$$W_l = Y_l / S_k, \tag{32}$$

is given by

$$G(w_l) = \int_0^\infty G_\beta(y_l;\, m)\, f(v_k|k)dv_k$$

$$= \int_0^\infty G_\beta\left(\frac{y_l}{s_k}\frac{s_k}{\beta};\, m\right) f(v_k|k)dv_k = \int_0^\infty G_\beta(w_l v_k;\, m) f(v_k|k)dv_k$$

$$= 1 - \frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \frac{[1+(m-l+j+1)w_l]^{-k}}{m-l+j+1}, \tag{33}$$

where S_k is given by (21).

$$\frac{dG(w_l)}{dw_l} = g(w_l) = \frac{k}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \frac{\binom{l-1}{j}(-1)^j}{[1+(m-l+j+1)w_l]^{k+1}}, \tag{34}$$

is the probability density function of the ancillary statistic W_l.

Let $Y_1 \leq \ldots \leq Y_k$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution with the probability density function $f_\beta(y)$ and distribution function $F_\beta(y)$, where β is a parameter. Then it follows from (10) that the conditional probability distribution function of $Y_l (1 \leq k < l \leq m)$ given $Y_k = y_k$ is determined as

$$G_\beta(y_l | y_k;\, m) = P_\beta(Y_l \leq y_l | Y_k = y_k;\, m)$$

$$= 1 - \frac{1}{B(l-k,\, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+j+1} \left[\frac{\overline{F}_\beta(y_l)}{\overline{F}_\beta(y_k)} \right]^{m-l+j+1}$$

$$= 1 - \frac{1}{B(l-k,\, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j}$$

$$\times \exp\left(-\frac{y_l - y_k}{\beta} (m-l+1+j) \right). \tag{35}$$

The conditional probability density function of $Y_l (1 \leq k < l \leq m)$ given $Y_k = y_k$ is determined as

$$g_\beta(y_l | y_k;\, m) = \frac{dG_\beta(y_l | y_k;\, m)}{dy_l} = \frac{1}{B(l-k,\, m-l+1)}$$

$$\times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \exp\left(-\frac{y_l - y_k}{\beta} (m-l+1+j) \right) \frac{1}{\beta}. \tag{36}$$

It follows from (36) that the cumulative probability distribution function of the ancillary statistic

$$W_{lk} = \frac{Y_l - Y_k}{S_k}, \tag{37}$$

is given by

$$G(w_{lk}) = 1 - \frac{1}{B(l-k,\ m-l+1)}$$

$$\times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \frac{[1+(m-l+1+j)w_{lk}]^{-k}}{m-l+1+j}, \tag{38}$$

where

$$S_k = \sum_{i=1}^{k} Y_i + (m-k)Y_k. \tag{39}$$

The probability density function of the ancillary statistic W_{lk} is given by

$$g(w_{lk}) = \frac{dG(w_{lk})}{dw_{lk}} = \frac{k}{B(l-k,\ m-l+1)}$$

$$\times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j [1+(m-l+1+j)w_{lk}]^{-k-1}, \quad w_{lk} \geq 0. \tag{40}$$

3. Constructing Exact Statistical New-Sample Prediction Limits for Anticipated Outcomes under Parametric Uncertainty

Example 1. Let $X_1 \leq \ldots \leq X_k$ be the first $k = 4$ ordered observations of lifetimes from the past sample of size $m = 10$ from the exponential distribution (19) and Y_l be the l-th order statistic in a set of m future ordered observations of lifetimes $Y_1 \leq \ldots \leq Y_m$ also from the distribution (19), where $l = m = 10$. Consider a life test where the above m units, whose lifetimes are distributed according to the same exponential distribution (19), are put on test simultaneously, and where all units are observed until failure. It is necessary to construct prediction limit for the lifetime $Y_l (l = m)$ on the basis of the past k smallest lifetimes $X_1 \leq \ldots \leq X_k$, where $X_1 = 33$, $X_2 = 87$, $X_3 = 125$, and $X_4 = 165$.

The problem considered in this example is to find the lower $(1-\alpha)$-prediction limit $y_l\left(= L_{1-\alpha;\, l}\right)$ for $Y_l,\ l \in \{5, 6, 7, 8, 9, 10\}$, satisfying

$$P_\beta(Y_l > y_l;\ m) = \int_{y_l}^{\infty} g_\beta(y;\ m)dy = \overline{G}_\beta(y_l;\ m) = 1-\alpha, \tag{41}$$

and the upper $(1-\alpha)$-prediction limit $y_l\left(= U_{1-\alpha;\, l}\right)$ for $Y_l,\ l \in \{5, 6, 7, 8, 9, 10\}$, satisfying

$$P_\beta(Y_l \le y_l;\ m) = \int_{y_k}^{y_l} g_\beta(y_l;\ m)dy_l = G_\beta(y_l;\ m) = 1-\alpha, \tag{42}$$

where the parameter β is unknown, $\alpha = 0.05$. The complete sufficient statistic for β is given by

$$S_k = \sum_{i=1}^{k} X_i + (m-k)X_k = 33 + 87 + 125 + 165 + 6 \times 165 = 1400. \tag{43}$$

Solution. Since the parameter β is unknown, we transform (41) as follows:

$$P_\beta\left(\frac{Y_l}{S_k}\frac{S_k}{\beta} > \frac{y_l}{s_k}\frac{s_k}{\beta};\ m\right) = \overline{G}\left(\frac{y_l}{s_k}\frac{s_k}{\beta};\ m\right)$$

$$= \frac{1}{B(l,\ m-l+1)}\sum_{j=0}^{l-1}\binom{l-1}{j}(-1)^j\,\frac{\exp\left(-\frac{y_l}{s_k}\frac{s_k}{\beta}(m-l+1+j)\right)}{m-l+j+1} = 1-\alpha. \tag{44}$$

It follows from (32) and (44) that

$$\Pr(W_l V_k > w_l v_k;\ m) = \overline{G}(w_l v_k;\ m)$$

$$= \frac{1}{B(l,\ m-l+1)}\sum_{j=0}^{l-1}\binom{l-1}{j}(-1)^j\,\frac{\exp(-w_l v_k(m-l+1+j))}{m-l+j+1} = 1-\alpha. \tag{45}$$

To eliminate the unknown parameter β from the problem, (45) is transformed as follows:

$$\Pr(W_l > w_l;\, m) = \int_0^\infty \Pr(W_l V_k > w_l v_k;\, m) f(v_k|k)\, dv_k$$

$$= \int_0^\infty \overline{G}(w_l v_k;\, m) \frac{1}{\Gamma(k)} v_k^{k-1} \exp(-v_k)\, dv_k$$

$$= \frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \frac{[1+(m-l+1+j)w_l]^{-k}}{m-l+1+j}$$

$$= \overline{G}(w_l) = 1 - \alpha. \tag{46}$$

It follows from (32) and (46) that

$$L_{1-\alpha;\, l} = y_l = w_l s_k, \tag{47}$$

where

$$w_l = \arg\min_{w_l} \left(\frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \right.$$

$$\left. \times \frac{[1+(m-l+1+j)w_l]^{-k}}{m-l+1+j} - (1-\alpha) \right)^2, \tag{48}$$

In a similar way it can be shown that

$$U_{1-\alpha;\, l} = y_l = w_l s_k, \tag{49}$$

where

$$w_l = \arg\min_{w_l} \left(\frac{1}{B(l,\, m-l+1)} \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \right.$$

$$\left. \times \frac{[1+(m-l+1+j)w_l]^{-k}}{m-l+1+j} - \alpha \right)^2. \tag{50}$$

If $l = 5$, it follows from (47) that the exact lower $(1 - \alpha)$-prediction limit $y_l (= L_{1-\alpha;\, l})$ for Y_l is given by

$$L_{1-\alpha;\, l} = y_l = w_l s_k = 0.05206 \times 1400 = 72.88; \tag{51}$$

it follows from (49) that the exact upper $(1 - \alpha)$-prediction limit $y_l (= U_{1-\alpha;\, l})$ for Y_l is given by

$$U_{1-\alpha;\, l} = y_l = w_l s_k = 0.54215 \times 1400 = 759.014. \tag{52}$$

If $l = 10$, it follows from (47) that the exact lower $(1 - \alpha)$-prediction limit $y_l (= L_{1-\alpha;\, l})$ for Y_l is given by

$$L_{1-\alpha;\, l} = y_l = w_l s_k = 0.265 \times 1400 = 370.7; \tag{53}$$

it follows from (49) that the exact upper $(1 - \alpha)$-prediction limit $y_l (= U_{1-\alpha;\, l})$ for Y_l is given by

$$U_{1-\alpha;\, l} = y_l = w_l s_k = 2.418 \times 1400 = 3386. \tag{54}$$

4. Constructing Exact Statistical Within-Sample Prediction Limits for Anticipated Outcomes under Parametric Uncertainty

Example 2. Consider a life test where m units, whose lifetimes are distributed according to the same exponential distribution (43), are put on test simultaneously, and where all units are observed until failure. It is necessary, to construct prediction limit for the lifetime $Y_l (1 \le k < l \le m)$ given $Y_k = y_k$ on the basis of the k smallest lifetimes $Y_1 \le \ldots \le Y_k$. Suppose that $m = 10$ items, whose lifetimes are distributed according to the same exponential distribution, are on test simultaneously, and that the first four items $(k = 4)$ to fail do so at times $Y_1 = 33$, $Y_2 = 87$, $Y_3 = 125$, and $Y_4 = 165$.

The problem considered in this example is to find the lower $(1-\alpha)$-prediction limit $y_l\left(= L_{1-\alpha;\, l}\right)$ for Y_l, $l \in \{5, 6, 7, 8, 9, 10\}$, satisfying

$$P_\beta(Y_l \le y_l | Y_k = y_k;\, m) = \int_{y_l}^{\infty} g_\beta(y|y_k;\, m)dy = \overline{G}_\beta(y_l|y_k;\, m) = 1-\alpha, \quad (55)$$

and the upper $(1-\alpha)$-prediction limit $y_l\left(= U_{1-\alpha;\, l}\right)$ for Y_l, $l \in \{5, 6, 7, 8, 9, 10\}$, satisfying

$$P_\beta(Y_l \le y_l | Y_k = y_k;\, m) = \int_{y_k}^{y_l} g_\beta(y_l|y_k;\, m)dy_l = G_\beta(y_l|y_k;\, m) = 1-\alpha, \quad (56)$$

where the parameter β is unknown, $\alpha = 0.05$. The complete sufficient statistic for β is given by

$$S_k = \sum_{i=1}^{k} Y_i + (m-k)Y_k = 33 + 87 + 125 + 165 + 6 \times 165 = 1400. \quad (57)$$

Solution. Since the parameter β is unknown, we transform (55) as follows:

$$P_\beta\left(\frac{Y_l - Y_k}{S_k}\frac{S_k}{\beta} > \frac{y_l - y_k}{s_k}\frac{s_k}{\beta} \,|\, Y_k = y_k;\, m\right) = \overline{G}\left(\frac{y_l - y_k}{s_k}\frac{s_k}{\beta} \,|\, y_k;\, m\right)$$

$$= \frac{1}{B(l-k,\, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j}$$

$$\times \exp\left(-\frac{y_l - y_k}{\beta}(m-l+1+j)\right) = 1-\alpha. \quad (58)$$

It follows from (37) and (58) that

$$\Pr(W_{lk}V_k > w_{lk}v_k | Y_k = y_k;\ m) = \overline{G}(w_{lk}v_k | y_k;\ m)$$

$$= \frac{1}{B(l-k,\ m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j}$$

$$\times \exp(-w_{lk}v_k(m-l+1+j)) = 1-\alpha. \tag{59}$$

To eliminate the unknown parameter β from the problem, (59) is transformed as follows:

$$\Pr(W_{lk} > w_{lk} | Y_k = y_k;\ m) = \int_0^\infty \Pr(W_{lk}V_k > w_{lk}v_k | y_k;\ m) f(v_k | k) dv_k$$

$$= \int_0^\infty \overline{G}(w_{lk}v_k | y_k;\ m) \frac{1}{\Gamma(k)} v_k^{k-1} \exp(-\nu_k) dv_k$$

$$= \frac{1}{B(l-k,\ m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j$$

$$\times \frac{[1+(m-l+1+j)w_{lk}]^{-k}}{m-l+1+j} = \overline{G}(w_{lk}) = 1-\alpha. \tag{60}$$

It follows from (37) and (60) that

$$L_{1-\alpha;\ l} = y_l = y_k + w_{lk}s_k, \tag{61}$$

where

$$w_{lk} = \arg\min_{w_{lk}} \left(\frac{1}{B(l-k,\ m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \right.$$

$$\left. \times \frac{[1+(m-l+1+j)w_{lk}]^{-k}}{m-l+1+j} - (1-\alpha) \right)^2, \tag{62}$$

In a similar way it can be shown that

$$U_{1-\alpha;\, l} = y_l = y_k + w_{lk} s_k, \tag{63}$$

where

$$w_{lk} = \arg\min_{w_{lk}} \left(\frac{1}{B(l-k,\, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \right.$$

$$\left. \times \frac{[1+(m-l+1+j)w_{lk}]^{-k}}{m-l+1+j} - \alpha \right)^2, \tag{64}$$

If $l = 5$, it follows from (61) that the exact lower $(1-\alpha)$-prediction limit $y_l (= L_{1-\alpha;\, l})$ for Y_l is given by

$$L_{1-\alpha;\, l} = y_l = y_k + w_{lk} s_k = 165 + 0.002151 \times 1400 = 168.0114; \tag{65}$$

it follows from (63) that the exact upper $(1-\alpha)$-prediction limit $y_l (= U_{1-\alpha;\, l})$ for Y_l is given by

$$U_{1-\alpha;\, l} = y_l = y_k + w_{lk} s_k = 165 + 0.185791 \times 1400 = 425.1067. \tag{66}$$

If $l = 10$, it follows from (61) that the exact lower $(1-\alpha)$-prediction limit $y_l (= L_{1-\alpha;\, l})$ for Y_l is given by

$$L_{1-\alpha;\, l} = y_l = y_k + w_{lk} s_k = 165 + 0.192433 \times 1400 = 434.4062; \tag{67}$$

it follows from (63) that the exact upper $(1-\alpha)$-prediction limit $y_l (= U_{1-\alpha;\, l})$ for Y_l is given by

$$U_{1-\alpha;\, l} = y_l = y_k + w_{lk} s_k = 165 + 2.098182 \times 1400 = 3102.455. \tag{68}$$

4.1. Estimating the unknown parameter β under parametric uncertainty

It follows from (57) that the maximum likelihood estimate of the unknown parameter β is given by

$$\beta_{ML} = \frac{S_k}{k} = \frac{\sum_{i=1}^{k=4} Y_i + (m-k)Y}{4} = \frac{1400}{4} = 350. \tag{69}$$

Let us assume that the unknown parameter β is equal to β_{ML} Then it follows from (12) that the upper $(1-\alpha)$-prediction limit $y_l (= U_{1-\alpha;\, l})$ for y_l can be found as follows. Minimize

$$\left(\sum_{j=l-k}^{m-k} \binom{m-k}{j} \left[1 - \frac{\overline{F}_\beta(y_l)}{\overline{F}_\beta(y_k)} \right]^j \left[\frac{\overline{F}_\beta(y_l)}{\overline{F}_\beta(y_k)} \right]^{m-k-j} - (1-\alpha) \right)^2$$

$$= \left(\int_0^{1-\frac{\overline{F}_\beta(y_l)}{\overline{F}_\beta(y_k)}} f_{l-k,\, m-l+1}(u)\, du - (1-\alpha) \right)^2 = \left(1 - \frac{\overline{F}_\beta(y_l)}{\overline{F}_\beta(y_k)} - q_{1-\alpha;\,(l-k,\, m-l+1)} \right)^2$$

$$= \left(\left[1 - q_{1-\alpha;\,(l-k,\, m-l+1)} \right] - \exp\left(-\frac{y_l - y_k}{\beta} \right) \right)^2, \tag{70}$$

subject to $y_l \in (0, \infty)$.

If $m = 10,\ l = 10,\ k = 4,\ \alpha = 0.05,\ y_k = 165,\ \beta = \beta_{ML} = 350,$ it follows from (70) that the upper $(1-\alpha)$-prediction limit $U_{1-\alpha;\, l}$ for Y_l is given by

$$U_{1-\alpha;\, l} = y_l = y_k + \beta_{ML} \ln\left(\frac{1}{1 - q_{1-\alpha;\,(l-k,\, m-l+1)}} \right)$$

$$= 165 + 350 \times \ln\left(\frac{1}{1 - 0.991488} \right) = 1833.179. \tag{71}$$

It follows from (68) that the exact upper $(1-\alpha)$-prediction limit $y_l(= U_{1-\alpha;\, l})$ for Y_l is equal to 3102.455. Then it follows from (70) that the adequate value of the unknown parameter β is given by

$$\beta = \frac{y_l - y_k}{\ln\left(\dfrac{1}{1 - q_{1-\alpha;\,(l-k,\, m-l+1)}}\right)} = \frac{3.102.455 - 165}{\ln\left(\dfrac{1}{1 - 0.991488}\right)} = 616.3063. \qquad (72)$$

Relative efficiency. The relative efficiency of β_{ML} as compared with β is given by

$$\text{rel.eff.}_U\{U_{1-\alpha;\, l}(\beta_{ML}),\, U_{1-\alpha;\, l}(\beta)\} = \frac{U_{1-\alpha;\, l}(\beta_{ML})}{U_{1-\alpha;\, l}(\beta)}$$

$$= \frac{1833.179}{3102.455} = 0.59088. \qquad (73)$$

5. Finding Order Quantity to Maximize an Expected Profit in Single-Period Decision-Making Models

The single-period decision-making models are phrased as the Newsboy Problem, where a newsboy determines the optimal order quantity to maximize the expected profit. For the newsboy who sells newspapers, the demand is uncertain, and the newsboy must decide how many newspapers to buy from his supplier. If he buys too many newspapers he is left with unsold newspapers that have no value at the end of the day; if he buys too few newspapers he has lost the opportunity of making a higher profit.

The newsboy problem is a well-known single-item and single-period inventory problem in which the following is given:

Y : quantity demanded (random variable),

$f_\beta(y)$: the exponential probability density function of Y (when the parameter β is known) (1),

$f_{s_k}(y)$: the statistical pivot-based estimate of $f_\beta(y)$ (when the parameter β is not known) (20),

u: the order quantity (decision variable) to satisfy the demand Y,

$Q(Y|u)$: the profit (random variable) which depends on the demand Y and the order quantity u,

c_u : unit cost price,

c_y : unit selling price $(c_y > c_u)$,

c_u^-: unit salvage cost $(c_u > c_u^-)$,

c_y^-: unit shortage penalty.

When a quantity u is ordered and Y is the demand (random variable), the profit $Q(Y|u)$ (random variable) is determined by

$$Q(Y|u) = \begin{cases} c_y Y + c_u^-(u - Y) - c_u u, & Y \le u \\ c_y u - c_y^-(Y - u) - c_u u, & Y \ge u \end{cases}$$

$$= \begin{cases} (c_y - c_u^-)Y - (c_u - c_u^-)u, & Y \le u \\ (c_y + c_y^- - c_u)u - c_y^- Y, & Y \ge u, \end{cases} \tag{74}$$

where $Q(Y|u)$ increases for $Y < u$ and decreases for $Y > u$. To demonstrate how to solve a typical newsboy problem with continuous demand (random variable), we formulate a problem as follows:

Maximize the expected profit

$$E_\beta\{Q(Y|u)\} = \int_0^u [(c_y - c_u^-)Y - (c_u - c_u^-)u] f_\beta(y)dy$$

$$+ \int_u^\infty [(c_y + c_y^- - c_u)u - c_y^- Y] f_\beta(y)dy$$

$$= (c_y - c_u^-)\int_0^u y f_\beta(y)dy - (c_u - c_u^-)u\int_0^u f_\beta(y)dy$$

$$+ (c_y + c_y^- - c_u)u\int_u^\infty f_\beta(y)dy - c_y^-\int_u^\infty y f_\beta(y)dy$$

$$= (c_y - c_u^-)\int_0^u y f_\beta(y)dy + c_y^-\int_0^u y f_\beta(y)dy - c_y^-\int_0^u y f_\beta(y)dy$$

$$- c_y^-\int_u^\infty y f_\beta(y)dy - (c_u - c_u^-)u\int_0^u f_\beta(y)dy - (c_u - c_u^-)u\int_u^\infty f_\beta(y)dy$$

$$+ (c_u - c_u^-)u\int_u^\infty f_\beta(y)dy + (c_y + c_y^- - c_u)u\int_u^\infty f_\beta(y)dy$$

$$= (c_y + c_y^- - c_u^-)\int_0^u y f_\beta(y)dy - c_y^- E_\beta\{Y\} - (c_u - c_u^-)u$$

$$+ (c_y + c_y^- - c_u^-)u\int_u^\infty f_\beta(y)dy$$

$$= (c_y + c_y^- - c_u^-)\int_0^u y f_\beta(y)dy - c_y^- E_\beta\{Y\} - (c_u - c_u^-)u$$

$$+ (c_y + c_y^- - c_u^-)u\int_u^\infty f_\beta(y)dy$$

$$= (c_y + c_y^- - c_u^-)\left[- u\overline{F}_\beta(u) + \int_0^u \overline{F}_\beta(y)dy\right] - c_y^- E_\beta\{Y\} - (c_u - c_u^-)u$$

$$+ (c_y + c_y^- - c_u^-)u\overline{F}_\beta(u)$$

$$= (c_y + c_y^- - c_u^-)\int_0^u \overline{F}_\beta(y)dy - c_y^- E_\beta\{Y\} - (c_u - c_u^-)u \tag{75}$$

subject to

$$u > 0. \tag{76}$$

Solution.

$$\frac{dE_\beta\{Q(Y|u)\}}{du} = (c_y + c_y^- - c_u^-)\,\overline{F}_\beta(u) - (c_u - c_u^-) = 0. \tag{77}$$

It follows from (77) that

$$\overline{F}_\beta(u) = \exp(-u/\beta) = \frac{c_u - c_u^-}{c_y + c_y^- - c_u^-}. \tag{78}$$

It follows from (78) that the optimal order quantity is given by

$$u^* = \beta \ln \frac{c_y + c_y^- - c_u^-}{c_u - c_u^-}. \tag{79}$$

It follows from (75) and (79) that the maximum expected profit, which can be realized, is given by

$$E_\beta\{Q(Y|u^*)\} = (c_y + c_y^- - c_u^-)\int_0^{u^*} \overline{F}_\beta(y)dy - c_y^- E_\beta\{Y\} - (c_u - c_u^-)u^*. \tag{80}$$

If the parameter β is unknown, then, using (74) and (27), we obtain

$$\frac{dE_{s_k}\{Q(Y|u)\}}{du} = (c_y + c_y^- - c_u^-)\,\overline{F}_{s_k}(u) - (c_u - c_u^-) = 0. \tag{81}$$

It follows from (81) and (27) that

$$\overline{F}_{s_k}(u) = \left(1 + \frac{u}{s_k}\right)^{-k} = \frac{c_u - c_u^-}{c_y + c_y^- - c_u^-}. \tag{82}$$

It follows from (81) that the optimal order quantity (based on S_k (27)) under parametric uncertainty is given by

$$u^{\bullet} = s_k \left[\left(\frac{c_y + c_y^- - c_u^-}{c_u - c_u^-} \right)^{1/k} - 1 \right]. \tag{83}$$

It follows from (75) and (80) that the maximum expected profit (based on S_k (27)) under parametric uncertainty, which can be realized, is given by

$$E_{s_k}\{Q(Y|u^{\bullet})\} = (c_y + c_y^- - c_u^-) \int_0^{u^{\bullet}} \overline{F}_{s_k}(y)dy - c_y^- E_{s_k}\{Y\} - (c_u - c_u^-)u^{\bullet}. \tag{84}$$

5.1. Generalized single-period decision-making model

The model described above can be generalized as follows. Consider n different items (say, newspapers) that are bought and sold. Here we have to determine the optimal order quantity for each of n different items in order to maximize the expected total profit. It is assumed that the volume level of the total order quantity is limited by the available monetary resource (c).

To demonstrate how to solve a generalized newsboy problem with n continuous demands (random variables), we formulate a problem as follows.

Maximize the expected total profit

$$\sum_{i=1}^{n} E_{\beta_i}\{Q(Y_i|u_i)\} = \sum_{i=1}^{n} \left((c_{y_i} + c_{y_i}^- - c_{u_i}^-) \int_0^{u_i} \overline{F}_{\beta_i}(y_i)dy_i \right.$$

$$\left. - c_{y_i}^- E_{\beta_i}\{Y_i\} - (c_{u_i} - c_{u_i}^-)u_i \right) \tag{85}$$

subject to

$$\sum_{i=1}^{n} c_{u_i} u_i = c, \quad u_i \geq 0, \quad i = 1, \ldots, n. \tag{86}$$

Solution.

Using the Lagrange multiplier method, the Lagrange function is determined as

$$L(u_1, \ldots, u_n, \lambda) = \sum_{i=1}^{n} E_{\beta_i}\{Q(Y_i|u_i)\} - \lambda\left(\sum_{i=1}^{n} c_{u_i} u_i - c\right), \tag{87}$$

where a new variable (λ) called a Lagrange multiplier is introduced. Then

$$\frac{\partial L}{\partial u_i} = (c_{y_i} + c_{y_i}^{-} - c_{u_i}^{-})\,\overline{F}_{\beta_i}(u_i) - (c_{u_i} - c_{u_i}^{-}) - \lambda c_{u_i}$$

$$= (c_{y_i} + c_{y_i}^{-} - c_{u_i}^{-})\,\overline{F}_{\beta_i}(u_i) - [(\lambda + 1)c_{u_i} - c_{u_i}^{-}] = 0, \quad i = 1, \ldots, n, \tag{88}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} c_{u_i} u_i - c = 0. \tag{89}$$

It follows from (88) that

$$u_i = \beta_i \ln \frac{c_{y_i} + c_{y_i}^{-} - c_{u_i}^{-}}{(\lambda + 1)c_{u_i} - c_{u_i}^{-}}, \; i = 1, \ldots, n, \tag{90}$$

Substituting (90) into (89), we have that

$$\lambda^* = \underset{\lambda \in (-\infty, \infty)}{\arg}\left(\sum_{i=1}^{n} c_{u_i}\beta_i \ln \frac{c_{y_i} + c_{y_i}^{-} - c_{u_i}^{-}}{(\lambda + 1)c_{u_i} - c_{u_i}^{-}} = c\right),$$

or

$$\lambda^* = \underset{\lambda \in (-\infty, \infty)}{\arg\min}\left(\sum_{i=1}^{n} c_{u_i}\beta_i \ln \frac{c_{y_i} + c_{y_i}^{-} - c_{u_i}^{-}}{(\lambda + 1)c_{u_i} - c_{u_i}^{-}} - c\right)^2. \tag{91}$$

Substituting (91) into (90), we determine the optimal order quantity for each from n different items to maximize the expected total profit,

$$u_i^* = \beta_i \ln \frac{c_{y_i} + c_{y_i}^- - c_{u_i}^-}{(\lambda^* + 1)c_{u_i} - c_{u_i}^-}, \; i = 1, \ldots, n. \tag{92}$$

It follows from (85) and (92) that the maximum expected total profit, which can be realized, is given by

$$\sum_{i=1}^{n} E_{\beta_i}\{Q(Y_i|u_i^*)\} = \sum_{i=1}^{n} \left((c_{y_i} + c_{y_i}^- - c_{u_i}^-) \int_0^{u_i^*} \overline{F}_{\beta_i}(y_i) dy_i \right.$$

$$\left. - c_{y_i}^- E_{\beta_i}\{Y_i\} - (c_{u_i} - c_{u_i}^-) u_i^* \right). \tag{93}$$

Similarly, it can be considered the *Generalized Single-Period Decision-Making Model* under parametric uncertainty. In this case we have the following:

$$u_i = s_{k_i} \left[\left(\frac{c_{y_i} + c_{y_i}^- - c_{u_i}^-}{(\lambda + 1)c_{u_i} - c_{u_i}^-} \right)^{1/k_i} - 1 \right]. \tag{94}$$

Substituting (94) into (89), we have that

$$\lambda^* = \underset{\lambda \in (-\infty, \infty)}{\arg} \left(\sum_{i=1}^{n} c_{u_i} s_{k_i} \left[\left(\frac{c_{y_i} + c_{y_i}^- - c_{u_i}^-}{(\lambda + 1)c_{u_i} - c_{u_i}^-} \right)^{1/k_i} - 1 \right] = c \right)$$

or

$$\lambda^* = \underset{\lambda \in (-\infty, \infty)}{\arg} \min \left(\sum_{i=1}^{n} c_{u_i} s_{k_i} \left[\left(\frac{c_{y_i} + c_{y_i}^- - c_{u_i}^-}{(\lambda + 1)c_{u_i} - c_{u_i}^-} \right)^{1/k_i} - 1 \right] - c \right)^2. \tag{95}$$

Substituting (95) into (94), we determine the optimal order quantity for each from n different items to maximize the expected total profit under parametric uncertainty,

$$u_i^* = s_{k_i}\left[\left(\frac{c_{y_i} + c_{y_i}^- - c_{u_i}^-}{(\lambda^* + 1)c_{u_i} - c_{u_i}^-}\right)^{1/k_i} - 1\right], \; i = 1, \ldots, n. \tag{96}$$

It follows from (85) and (96) that the maximum expected total profit, which can be realized under parametric uncertainty, is given by

$$\sum_{i=1}^{n} E_{s_{k_i}}\{Q(Y_i|u_i^*)\} = \sum_{i=1}^{n}\left((c_{y_i} + c_{y_i}^- - c_{u_i}^-)\int_0^{u_i^*} \overline{F}_{s_{k_i}}(y_i)dy_i \right.$$

$$\left. - c_{y_i}^- E_{s_{k_i}}\{Y_i\} - (c_{u_i} - c_{u_i}^-)u_i^*\right). \tag{97}$$

6. Two-Parameter Exponential Distribution

Let $Z = (Z_1 \le \ldots \le Z_r)$ be the first r ordered observations (order statistics) in a sample of size m from the two-parameter exponential distribution with the probability density function

$$f_\omega(x) = \vartheta^{-1}\exp\left(-\frac{x-\delta}{\vartheta}\right), \quad \vartheta > 0, \; x > 0, \tag{98}$$

and the cumulative probability distribution function

$$F_\omega(x) = 1 - \exp\left(-\frac{x-\delta}{\vartheta}\right), \quad \overline{F}_\omega(x) = 1 - F_\omega(x) = \exp\left(-\frac{x-\delta}{\vartheta}\right), \tag{99}$$

where $\omega = (\delta, \vartheta)$, is the shift parameter and ϑ is the scale parameter. It is assumed that these parameters are unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and Z_k is a random variable. In this case, the likelihood function is given by

$$L(\delta,\ \vartheta) = \prod_{i=1}^{r} f_{\omega}(z_i)(\overline{F}_{\omega}(z_r))^{m-r}$$

$$= \frac{1}{\vartheta^r} \exp\left(- \left[\sum_{i=1}^{r} (z_i - \delta) + (m-r)(z_r - \delta) \right] \Big/ \vartheta \right)$$

$$= \frac{1}{\vartheta^r} \exp\left(- \left[\sum_{i=1}^{r} (z_i - z_1 + z_1 - \delta) + (m-r)(z_r - z_1 + z_1 - \delta) \right] \Big/ \vartheta \right)$$

$$= \frac{1}{\vartheta^{r-1}} \exp\left(- \left[\sum_{i=1}^{r} (z_i - z_1) + (m-r)(z_r - z_1) \right] \Big/ \vartheta \right) \times \frac{1}{\vartheta} \exp\left(- \frac{m(z_1 - \delta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta^{r-1}} \exp\left(- \frac{s_r}{\vartheta} \right) \times \frac{1}{\vartheta} \exp\left(- \frac{m(s_1 - \delta)}{\vartheta} \right), \tag{100}$$

where

$$S = \left(S_1 = Z_1,\ S_r = \sum_{i=1}^{r} (Z_i - Z_1) + (m-r)(Z_r - Z_1) \right), \tag{101}$$

is the complete sufficient statistic for ω. The probability density function of $S = (S_1,\ S_r)$ is given by

$$f_{\omega}(s_1,\ s_r) = \frac{\dfrac{1}{\vartheta^{r-1}} \exp\left(- \dfrac{s_r}{\vartheta} \right) \times \dfrac{1}{\vartheta} \exp\left(- \dfrac{m(s_1 - \delta)}{\vartheta} \right)}{\dfrac{1}{s_r^{r-2}} \displaystyle\int_0^{\infty} \dfrac{s_r^{r-2}}{\vartheta^{r-1}} \exp\left(- \dfrac{s_r}{\vartheta} \right) ds_r \times \dfrac{1}{m} \int_0^{\infty} \dfrac{m}{\vartheta} \exp\left(- \dfrac{m(s_1 - \delta)}{\vartheta} \right) ds_1}$$

$$= \frac{\dfrac{1}{\vartheta^{r-1}} \exp\left(- \dfrac{s_r}{\vartheta} \right) \times \dfrac{1}{\vartheta} \exp\left(- \dfrac{m(s_1 - \delta)}{\vartheta} \right)}{\dfrac{\Gamma(r-1)}{s_r^{r-2}} \times \dfrac{1}{m}}$$

$$= \frac{1}{\Gamma(r-1)\vartheta^{r-1}} s_r^{r-2} \exp\left(- \frac{s_r}{\vartheta} \right) \times \frac{m}{\vartheta} \exp\left(- \frac{m(s_1 - \delta)}{\vartheta} \right) = f_{\vartheta}(s_r) f_{\omega}(s_1), \tag{102}$$

where

$$f_{\omega}(s_1) = \frac{m}{\vartheta} \exp\left(-\frac{m(s_1 - \delta)}{\vartheta}\right), \quad s_1 \geq \delta, \tag{103}$$

$$f_{\vartheta}(s_r) = \frac{1}{\Gamma(r-1)\vartheta^{r-1}} s_r^{r-2} \exp\left(-\frac{S_r}{\vartheta}\right), \quad s_r \geq 0. \tag{104}$$

$$V_1 = \frac{S_1 - \delta}{\vartheta}, \tag{105}$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1) = m \exp(-mv_1), \quad v_1 \geq 0, \tag{106}$$

$$V_r = \frac{S_r}{\vartheta}, \tag{107}$$

is the pivotal quantity, the probability density function of which is given by

$$f_r(v_r) = \frac{1}{\Gamma(r-1)} v_r^{r-2} \exp(-v_r), \; v_r \geq 0. \tag{108}$$

6.1. Constructing one-sided γ-content tolerance limit with a confidence level β

Theorem 4. *Let* $Z_1 \leq \ldots \leq Z_r$ *be the first r ordered observations from the preliminary sample of size m from a two-parameter exponential distribution defined by the density function* (98). *Then the lower one-sided* γ*-content tolerance limit with a confidence level* β, $L_k \equiv L_k(S)$ (*on the k-th order statistic* Y_k *from a set of n future ordered observations* $Y_1 \leq \ldots \leq Y_n$ *also from the distribution* (98)), *which satisfies*

$$E\{\Pr(P_{\omega}(Y_k > L_k|n) \geq \gamma)\} = \beta, \tag{109}$$

is given by

$$L_k = \begin{cases} S_1 + \dfrac{S_r}{m}\left[1 - \left(\dfrac{\Delta_{1-\gamma}^m}{1-\beta}\right)^{\frac{1}{r-1}}\right], & if \quad m \geq \dfrac{\ln(1-\beta)}{\ln \Delta_{1-\gamma}}, \\ S_1 - \dfrac{S_r}{m}\left[\left(\dfrac{\Delta_{1-\gamma}^m}{1-\beta}\right)^{\frac{1}{r-1}} - 1\right], & if \quad m < \dfrac{\ln(1-\beta)}{\ln \Delta_{1-\gamma}}, \end{cases} \tag{110}$$

where

$$\Delta_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} \ \ (Beta(k, n-k+1), 1-\gamma \ quantile). \tag{111}$$

Proof. It follows from (1), (99) and (109) that

$$\Pr(P_\omega(Y_k > L_k|n) \geq \gamma) = \Pr\left(1 - \int_0^{F_\omega(L_k)} f_{k,n-k+1}(u)du \geq \gamma\right)$$

$$= \Pr\left(\int_0^{F_\omega(L_k)} f_{k,n-k+1}(u)du \leq 1-\gamma\right)$$

$$= \Pr\big(F_\omega(L_k) \leq q_{k,n-k+1;1-\gamma}\big) = \Pr\left(1 - \exp\left(-\frac{L_k - \delta}{\vartheta}\right) \leq q_{k,n-k+1;1-\gamma}\right)$$

$$= \Pr\left(\exp\left(-\frac{L_k - \delta}{\vartheta}\right) \geq 1 - q_{k,n-k+1;1-\gamma}\right)$$

$$= \Pr\left(\frac{L_k - S_1}{S_r}\frac{S_r}{\vartheta} + \frac{S_1 - \delta}{\vartheta} \leq -\ln(1 - q_{k,n-k+1;1-\gamma})\right)$$

$$= \Pr\left(\frac{S_1 - \delta}{\vartheta} \leq -\frac{L_k - S_1}{S_r}\frac{S_r}{\vartheta} - \ln(1 - q_{k,n-k+1;1-\gamma})\right)$$

$$= \Pr\big(V_1 \leq -\eta_{L_k} V_r - \ln \Delta_{1-\gamma}\big) = \int_0^{-\eta_{L_k} V_r - \ln \Delta_{1-\gamma}} f_1(v_1)dv_1, \tag{112}$$

where

$$V_1 = \frac{S_1 - \delta}{\vartheta}, \ \eta_{L_k} = \frac{L_k - S_1}{S_r}, \ V_r = \frac{S_r}{\vartheta}, \quad \Delta_{1-\gamma} = 1 - q_{k,n-k+1;1-\gamma}. \quad (113)$$

It follows from (109) and (112) that

$$E\{\Pr(P_\omega(Y_k > L_k|n) \geq \gamma)\} = E\left\{\int_0^{-\eta_{L_k}V_r - \ln \Delta_{1-\gamma}} f_1(v_1)dv_1\right\}$$

$$= E\left\{\int_0^{-\eta_{L_k}V_r - \ln \Delta_{1-\gamma}} m \exp(-mv_1)dv_1\right\}$$

$$= E\{1 - \exp(-m[-\eta_{L_k}V_r - \ln \Delta_{1-\gamma}])\}$$

$$= E\{1 - \exp(m\eta_{L_k}V_r)\exp(\ln \Delta_{1-\gamma}^m)\} = E\{1 - \Delta_{1-\gamma}^m \exp(m\eta_{L_k}V_r)\}$$

$$= \int_0^\infty \left(1 - \Delta_{1-\gamma}^m \exp(m\eta_{L_k}V_r)\right) f_r(v_r)dv_r$$

$$= \int_0^\infty \left(1 - \Delta_{1-\gamma}^m \exp(m\eta_{L_k}V_r)\right) \frac{1}{\Gamma(r-1)} v_r^{r-2} \exp(-v_r)dv_r$$

$$= 1 - \frac{\Delta_{1-\gamma}^m}{\left[1 - m\eta_{L_k}\right]^{r-1}} = \beta. \quad (114)$$

It follows from (114) that

$$\eta_{L_k} = \frac{1}{m}\left(1 - \left[\frac{\Delta_{1-\gamma}^m}{1-\beta}\right]^{\frac{1}{r-1}}\right). \quad (115)$$

There are two possible cases here:

$$1-\left[\frac{\Delta_{1-\gamma}^{m}}{1-\beta}\right]^{\frac{1}{r-1}} \le 0\left(if\ \ m \ge \frac{\ln(1-\beta)}{\ln \Delta_{1-\gamma}}\right),$$

or

$$1-\left[\frac{\Delta_{1-\gamma}^{m}}{1-\beta}\right]^{\frac{1}{r-1}} > 0\left(if\ \ m < \frac{\ln(1-\beta)}{\ln \Delta_{1-\gamma}}\right). \tag{116}$$

Then (110) it follows from (116), (115) and (113). This ends the proof of Theorem 4.

Corollary 4.1. *Let* $Z_1 \le \ldots \le Z_r$ *be the first r ordered observations from the preliminary sample of size m from a two-parameter exponential distribution defined by the probability density function* (98). *Then the upper one-sided* γ*-content tolerance limit with a confidence level* β, $U_k \equiv U_k(S)$ *(on the k-th order statistic* Y_k *from a set of n future ordered observations* $Y_1 \le \ldots \le Y_n$ *also from the distribution* (98), *which satisfies*

$$E\{\Pr(P_\omega(Y_k \le U_k|n) \ge \gamma)\} = \beta, \tag{117}$$

is given by

$$U_k = \begin{cases} S_1 + \dfrac{S_r}{m}\left[1-\left(\dfrac{\Delta_\gamma^m}{\beta}\right)^{\frac{1}{r-1}}\right], & if \quad m \ge \dfrac{\ln \beta}{\ln \Delta_\gamma}, \\ S_1 - \dfrac{S_r}{m}\left[\left(\dfrac{\Delta_\gamma^m}{\beta}\right)^{\frac{1}{r-1}} - 1\right], & if \quad m < \dfrac{\ln \beta}{\ln \Delta_\gamma}, \end{cases} \tag{118}$$

where

$$\Delta_\gamma = 1 - q_{(k,n-k+1),\gamma}(Beta(k, n-k+1),\ \gamma\ quantile). \tag{119}$$

6.2. Numerical practical example

Let us assume that $k = 1$, $r = m = n = 15$, $\gamma = \beta = 0.95$,

$$S = \left(S_1 = Z_1 = 9,\ S_r = \sum_{i=1}^{r=m} (Z_i - Z_1) + (m - r)(Z_r - Z_1) = 192.2508 \right). \tag{120}$$

Then the lower one-sided γ-content tolerance limit with a confidence level β, $L_{k=1} \equiv L_{k=1}(S)$ can be obtained from (110). Since

$$m = 15 < \frac{\ln(1-\beta)}{\ln \Delta_{1-\gamma}} = \frac{\ln(1-\beta)}{\ln\left(1 - q_{(k,\,n-k+1),\,1-\gamma}\right)} = 876, \tag{121}$$

where the quantile of $Beta(k,\ n-k+1),\ 1-\gamma$ is given by

$$q_{(k,\,n-k+1),\,1-\gamma} = 0.003414. \tag{122}$$

It follows from (121) and (110) that

$$L_1(S) = S_1 - \frac{S_r}{m}\left[\left(\frac{\Delta_{1-\gamma}^{m}}{1-\beta}\right)^{\frac{1}{r-1}} - 1\right] = 9 - 3 = 6. \tag{123}$$

Statistical inference. From (123), it follows that there is a 95% certainty that failures will not occur in the proportion $\gamma = 0.95$ or more of a set of n selected items before the end of the lower one-sided γ-content tolerance limit $L_1(S) = 6$ monthly intervals.

7. Conclusion

The new intelligent computational method proposed in this article is a conceptually simple, efficient and useful method for constructing exact statistical prediction (or tolerance) limits and optimal (or improved) statistical decision rules under the parametric uncertainty of applied stochastic models. This technique is based on the constructive use of the

principle of invariance in mathematical statistics. We have illustrated a technique for the exponential distribution. Applications to other log-location-scale distributions can follow directly.

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