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A SIMPLE PROOF THAT MINIMAL NORMAL SUBGROUPS OF FINITE GROUPS ARE DIRECT PRODUCTS OF ISOMORPHIC SIMPLE GROUPS

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Abstract

We give a short self-contained proof of the important classical result that a minimal normal subgroup of a finite group is an internal direct product of isomorphic simple groups (e.g., Theorem 8.6.1 of M. Hall's *The Theory of Groups*, The MacMillan Co., 1966; Corollary 5.27 of J. J. Rotman's *An Introduction to the Theory of Groups*, Springer Verlag, 1995; Theorem 4.3A (iii) of J. D. Dixon & B. Mortimer's *Permutation Groups*, Springer Verlag, 1996).

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1. Introduction

A normal subgroup $M \neq \mathbf{1}$ of a group G is a *minimal normal subgroup* of G if M contains no other non-trivial normal subgroup of G. That is, $\mathbf{1} \neq K \lhd G$ with $K \leq M$ implies $K = M$. It is clear that minimal normal subgroups of non-trivial finite groups exist. We will use this well-known

Lemma. *If A and B are normal subgroups of some ambient group H and $A \cap B = \mathbf{1}$, then $ab = ba$ for all $a \in A$, $b \in B$. Moreover, $G = \langle A, B \rangle$ is the internal direct product of A and B, and is normal in H.*

Proof. $(b^{-1}a^{-1}b)a \in A$ since $b^{-1}a^{-1}b \in A$, and similarly $b^{-1}(a^{-1}ba) \in B$ since $a^{-1}ba \in B$. So $b^{-1}a^{-1}ba = 1$, i.e., a and b commute. In any product in $G = \langle A, B \rangle$, we can move all a's to the left and all b's to the right, so $G = AB$. To multiply in AB, we have $(ab)(a'b') = a(ba')b' = aa'bb'\,(a, a' \in A, b, b' \in B)$. Each element of G can be uniquely written in the form ab : for if $ab = a'b'$ then it follows $a^{-1}a' = bb'^{-1} \in A \cap B = \mathbf{1}$, whence $a = a'$ and $b = b'$. Thus the map $(a, b) \mapsto ab$ is a surjective and injective morphism, showing $A \times B \cong AB = G$. Also for $h \in H$, $ab \in AB$, we have $h^{-1}abh = (h^{-1}ah)(h^{-1}bh) \in AB$, so G is normal in H. □

Theorem. *A minimal normal subgroup M of a finite group G is the internal direct product of isomorphic simple groups that are conjugate in G.*

Proof. Let S be a minimal normal subgroup of M. Since conjugation by $g \in G$ is an automorphism of G, $g^{-1}Sg$ is minimal normal in $M = g^{-1}Mg$. Let $g_0^{-1}Sg_0, \ldots, g_n^{-1}Sg_n$ be the (pairwise) distinct conjugates of S in $G\,(g_0 = 1, g_j \in G, 0 \leq j \leq n, n \geq 0)$. Since $g_i^{-1}Sg_i \cap g_j^{-1}Sg_j$ is normal in M, by minimal normality of $g_i^{-1}Sg_i$ in M, this intersection is $\mathbf{1}$ for $i \neq j$.

Now define groups M_i for $0 \leq i \leq n$ by $M_0 = S$, and $M_{i+1} = \langle M_i, g_{i+1}^{-1}Sg_{i+1} \rangle$ if $g_{i+1}^{-1}Sg_{i+1} \cap M_i = \mathbf{1}$, and $M_{i+1} = M_i$ otherwise. We show by induction up to n, that *each* $M_i \lhd M$, $g_i^{-1}Sg_i$ *is contained in* M_i, *and* M_i *is the internal direct product of distinct conjugates of* S. This is trivial for $i = 0$. Assume the induction hypothesis for i. Consider the case $g_{i+1}^{-1}Sg_{i+1} \cap M_i = \mathbf{1}$: Since $M_i \lhd M$ (by induction hypothesis) and $g_{i+1}^{-1}Sg_{i+1} \lhd M$, by the Lemma M_{i+1} is the internal direct product of M_i and $g_{i+1}^{-1}Sg_{i+1}$, hence of distinct conjugates of S, and also normal in M. On the other hand, in the case $\mathbf{1} \neq g_{i+1}^{-1}Sg_{i+1} \cap M_i$, the intersection is normal in M, so we have $g_{i+1}^{-1}Sg_{i+1} \leq M_i$ since $g_{i+1}^{-1}Sg_{i+1}$ is minimal normal in M. So, in either case, $g_{i+1}^{-1}Sg_{i+1}$ is always contained in M_{i+1}. Thus, by induction, M_n is the internal direct product of conjugates of S. Moreover, since M_n contains M_i, which contains $g_i^{-1}Sg_i$, $M_n = \langle g_i^{-1}Sg_i : 0 \leq i \leq n \rangle = \langle g^{-1}Sg : g \in G \rangle$, the normal closure of S in G. By minimal normality of M in G, it follows that $M_n = M$.

Finally, S is simple: Suppose $K \neq \mathbf{1}$ is normal in S. Since K is contained in the single factor S in the direct product M and is normal in that factor, K is normal in M. By minimal normality of S in M, we have $K = S$, showing S is simple. Thus M is isomorphic to the internal direct product of conjugates of S in G. □

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