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BOUNDS FOR THE TORSIONAL RIGIDITY OF CYLINDRICALLY ANISOTROPIC BARS

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Abstract

Bounds for the torsional rigidity of cylindrically anisotropic bars with one plane of elastic symmetry perpendicular to the axis of bar are derived making of use the two minimum theorems of elasticity. All results of the paper are based on the theory of uniform torsion which was developed by Saint-Venant and Prandtl. Illustrative example shows that the one term approximation leads to relative close bounds.

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1. Introduction

While the Saint-Venant torsion of homogeneous Cartesian anisotropic linearly elastic bars has been the subject of several studies from both theoretical and numerical viewpoints [1-7, 14, 15] until then relative few articles and books deal with the task of torsion problem of cylindrically anisotropic bars [2, 3, 4, 6, 8-13]. The object of this paper is the Saint-Venant torsion of homogeneous cylindrical orthotropic solid elliptical cross section. The cylindrical anisotropic materials having a linear elastic stress-strain relations can possess a unique coupling between the radial and circumferential directions. The cylindrical orthotropy is a lower level of the cylindrical anisotropy it has a weaker coupling between the radial and circumferential directions as in the case of cylindrical anisotropy. The bar with elliptical cross section is an important structural component the investigation of its deformation under the torsional load is the subject of several text books of elasticity [1, 5, 6].

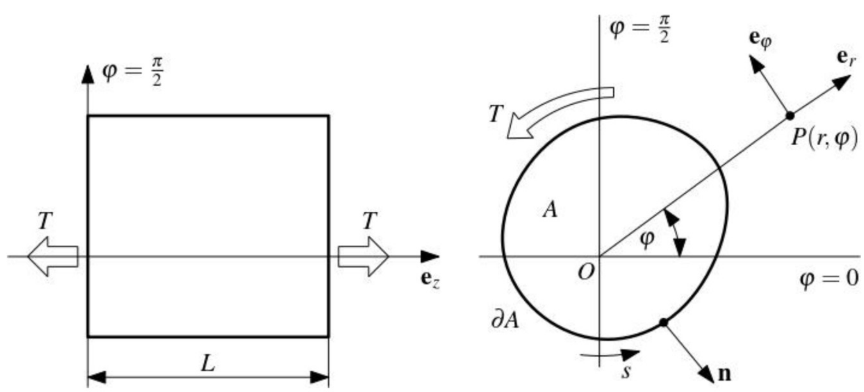

Figure 1. Saint-Venant torsion of cylindrical anisotropic bar.

We consider the Saint-Venant torsion of cylindrical anisotropic linearly elastic homogeneous bar with solid cross section. The polar coordinate system $Or\varphi z$ is positioned at the left end cross section of the bar as shown in Figure 1. The applied torque is T and the rate of twist with respect to axial coordinate z is denoted by ϑ and the length of the bar L. The cross section of the bar is a simply connected bounded two-dimensional domain A and its boundary curve is ∂A. The arc length coordinate defined on ∂A is s and the component of normal vector $\boldsymbol{n}$ in polar coordinate system is n_r, n_φ. The unit vectors of the polar coordinate system $Or\varphi z$ are denoted by $\mathbf{e}_r(\varphi)$, $\mathbf{e}_\varphi(\varphi)$, and $\mathbf{e}_z$ as shown in Figure 1. For cylindrical anisotropic elastic bar the shear moduli are $A_{44}, A_{55}, A_{45} = A_{54}$. The shear flexibility coefficients denoted by $a_{44}, a_{55}, a_{45} = a_{54}$. The connection between the shear moduli and shear flexibility coefficients are as follows [2, 3, 6]:

$$A_{44} = \frac{a_{55}}{a_{44}a_{55} - a_{45}^2}, \quad A_{55} = \frac{a_{44}}{a_{44}a_{55} - a_{45}^2}, \tag{1}$$

$$A_{45} = A_{54} = -\frac{a_{45}}{a_{44}a_{55} - a_{45}^2}. \tag{2}$$

From the non-negativity of strain-energy density, it follows that [2, 3, 4, 6]

$$a_{44} > 0,\ A_{44} > 0, \quad a_{55} > 0,\ A_{55} > 0, \tag{3}$$

$$a_{44}a_{55} - a_{45}^2 > 0, \quad A_{44}A_{55} - A_{45}^2 > 0. \tag{4}$$

2. Governing Equations

The cylindrically anisotropic bar under the uniform torsion follows the next stress-strain relationships [2, 3, 6]

$$\tau_{rz} = A_{55}\gamma_{rz} + A_{45}\gamma_{\varphi z}, \tag{5}$$

$$\tau_{\varphi z} = A_{45}\gamma_{rz} + A_{44}\gamma_{\varphi z}, \tag{6}$$

where τ_{rz} and $\tau_{\varphi z}$ are the shearing stresses while γ_{rz} and $\gamma_{\varphi z}$ are the shearing strains. The inverse relations of (5) and (6) are as follows:

$$\gamma_{rz} = a_{55}\tau_{rz} + a_{45}\tau_{\varphi z}, \tag{7}$$

$$\gamma_{\varphi z} = a_{45}\tau_{rz} + a_{44}\tau_{\varphi z}. \tag{8}$$

The shearing strains in terms of torsion function $\omega = \omega(r, \varphi)$ can be represented as [1-4]:

$$\gamma_{rz} = \vartheta \frac{\partial \omega}{\partial r}, \quad \gamma_{\varphi z} = \vartheta\left(\frac{1}{r}\frac{\partial \omega}{\partial \varphi} + r\right). \tag{9}$$

The shearing stresses satisfy the mechanical equilibrium equation [1-5]

$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{\varphi z}}{\partial \varphi} = 0, \quad (r, \varphi) \in A, \tag{10}$$

and the stress boundary condition on the mantle of the bar

$$\tau_{rz}n_r + \tau_{\varphi z}n_\varphi + 0, \quad (r, \varphi) \in \partial A. \tag{11}$$

It is known that Equations (10) and (11) are satisfied if

$$\tau_{rz} = \frac{\partial}{r}\frac{\partial U}{\partial \varphi}, \ \tau_{\varphi z} = -\vartheta \frac{\partial U}{\partial r}, \ (r, \varphi) \in A \cup \partial A, \tag{12}$$

$$U(r, \varphi) = 0, \quad (r, \varphi) \in \partial A, \tag{13}$$

where $U = U(r, \varphi)$ is the Prandtl's stress function of the solid cross section A. The Prandtl's stress function is the solution of the following Dirichlet type boundary value problem [3, 4, 12, 13]:

$$a_{44}\frac{\partial^2 U}{\partial r^2} + a_{44}\frac{1}{r}\frac{\partial U}{\partial r} - 2a_{45}\frac{1}{r}\frac{\partial^2 U}{\partial r \partial \varphi} + a_{55}\frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + 2 = 0, \ (r, \varphi) \in A, \tag{14}$$

$$U(r, \varphi) = 0, \quad (r, \varphi) \in \partial A. \tag{15}$$

The expression of torsional rigidity S in terms of $U = U(r, \varphi)$ can be represented as [3, 4, 12, 13]

$$S = 2\int_A U\mathrm{d}A = \int_A \left[a_{44}\left(\frac{\partial U}{\partial r}\right)^2 - 2a_{45}\frac{1}{r}\frac{\partial U}{\partial r}\frac{\partial U}{\partial \varphi} + a_{55}\frac{1}{r^2}\left(\frac{\partial U}{\partial \varphi}\right)^2 \right] dA. \tag{16}$$

Combination of Equations (5), (6), (9), (10) and (11) gives the torsional boundary value problem in terms of torsion function. A detailed computation leads to the following Neumann's type boundary value problem for $\omega = \omega(r, \varphi)$:

$$A_{55}\frac{\partial^2 \omega}{\partial r^2} + 2A_{45}\frac{1}{r}\frac{\partial^2 \omega}{\partial r \partial \varphi} + A_{44}\frac{1}{r^2}\frac{\partial^2 \omega}{\partial \varphi^2} + 2A_{45} = 0, \quad (r, \varphi) \in A, \tag{17}$$

$$\left(A_{55}\frac{\partial \omega}{\partial r} + A_{45}\left(\frac{1}{r}\frac{\partial \omega}{\partial \varphi} + r\right)\right)n_r + \left(A_{45}\frac{\partial \omega}{\partial r} + A_{44}\left(\frac{1}{r}\frac{\partial \omega}{\partial \varphi} + r\right)\right)n_\varphi = 0,$$

$$(r, \varphi) \in \partial A. \tag{18}$$

The expression of torsional rigidity S in terms of $\omega = \omega(r, \varphi)$ can be represented as

$$S = \int_A \left[A_{55} r \frac{\partial \omega}{\partial r} + A_{45}\left(\frac{\partial \omega}{\partial \varphi} + r^2\right)\right] dA$$

$$= \int_A \left[A_{55}\left(\frac{\partial \omega}{\partial r}\right)^2 + 2A_{45}\frac{\partial \omega}{\partial r}\left(\frac{1}{r}\frac{\partial \omega}{\partial \varphi} + r\right) + A_{44}\left(\frac{1}{r}\frac{\partial \omega}{\partial \varphi} + r\right)^2 \right] dA. \tag{19}$$

The total strain energy of the bar is as follows:

$$W = \frac{1}{2}\vartheta^2 LS = \frac{1}{2}\frac{T^2}{S}L. \tag{20}$$

3. Lower Bound for the Torsional Rigidity

In this case, the minimum principle of the complementary energy of linear elasticity leads to the following inequality [16-18]:

$$\prod\nolimits_C [U(r, \varphi)] \geq \prod\nolimits_C \left[\tilde{U}(r, \varphi)\right], \tag{21}$$

where $\tilde{U} = \tilde{U}(r, \varphi)$ is a statically adissible stress function which satisfies the homogeneous boundary condition

$$\tilde{U}(r, \varphi) = 0, \quad (r, \varphi) \in \partial A, \tag{22}$$

and its second order partial derivatives with respect to r and φ are continuous functions in $\overline{A} = A \cup \partial A$ and $\prod_C \left[\tilde{U}(r, \varphi)\right]$ is defined as

$$\prod\nolimits_C \left[\tilde{U}(r, \varphi)\right] = 4\int_A \tilde{U} dA - \int_A \left[a_{44}\left(\frac{\partial \tilde{U}}{\partial r}\right)^2 - 2a_{45}\,\frac{1}{r}\,\frac{\partial \tilde{U}}{\partial \varphi}\,\frac{\partial \tilde{U}}{\partial r} + a_{55}\,\frac{1}{r^2}\left(\frac{\partial \tilde{U}}{\partial \varphi}\right)^2\right] dA. \tag{23}$$

It must be noted that

$$S = \prod\nolimits_C [U(r, \varphi)]. \tag{24}$$

It is obvious if $\tilde{U}(r, \varphi)$ statically admissible stress function then $\lambda\tilde{U}(r, \varphi)$ is also a statically admissible function for the arbitrary value of λ, that is the following inequality is valid:

$$S > \lambda\int_A \tilde{U} dA - \lambda^2 \int_A \left[a_{44}\left(\frac{\partial \tilde{U}}{\partial r}\right)^2 - 2a_{45}\,\frac{1}{r}\,\frac{\partial \tilde{U}}{\partial \varphi}\,\frac{\partial \tilde{U}}{\partial r} + a_{55}\,\frac{1}{r^2}\,\frac{\partial^2 \tilde{U}}{\partial \varphi^2}\right] dA, -\infty < \lambda < \infty. \tag{25}$$

By a simple calculation can be derived the inequality relations formulated in Equation (26)

$$\lambda\int_A \tilde{U} \mathrm{d}A - \lambda^2 \int_A \left[a_{44}\left(\frac{\partial \tilde{U}}{\partial r}\right)^2 - 2a_{45}\,\frac{1}{r}\,\frac{\partial \tilde{U}}{\partial \varphi}\,\frac{\partial \tilde{U}}{\partial r} + a_{55}\,\frac{1}{r^2}\left(\frac{\partial \tilde{U}}{\partial \varphi}\right)^2\right] dA$$

$$\leq \frac{\left(\int_A 2\tilde{U} dA\right)^2}{\int_A \left[a_{44}\left(\frac{\partial \tilde{U}}{\partial r}\right)^2 - 2a_{45}\,\frac{1}{r}\,\frac{\partial \tilde{U}}{\partial \varphi}\,\frac{\partial \tilde{U}}{\partial r} + a_{55}\,\frac{1}{r^2}\left(\frac{\partial \tilde{U}}{\partial \varphi}\right)^2\right] dA}, \; -\infty < \lambda < \infty. \tag{26}$$

The correctness of the following lower bound formula follows from inequalities (16) and (25)

$$S \geq \frac{\left(\int_A 2\tilde{U} dA\right)^2}{\int_A \left[a_{44}\left(\frac{\partial \tilde{U}}{\partial r}\right)^2 - 2a_{45}\frac{1}{r}\frac{\partial \tilde{U}}{\partial \varphi}\frac{\partial \tilde{U}}{\partial r} + a_{55}\frac{1}{r^2}\left(\frac{\partial \tilde{U}}{\partial \varphi}\right)^2\right] dA}. \tag{27}$$

4. Upper Bound for the Torsional Rigidity

According to the minimum principle of potential energy of the linear theory of elasticity, we have [16-18]

$$\prod\nolimits_L [\tilde{\omega}(r, \varphi)] \geq \prod\nolimits_L [\omega(r, \varphi)], \tag{28}$$

where $\tilde{\omega} = \tilde{\omega}(r, \varphi)$ is a kinematically admissible torsion function and

$$\prod\nolimits_L [\tilde{\omega}(r, \varphi)] = \int_A \left[A_{55}\left(\frac{\partial \tilde{\omega}}{\partial r}\right)^2 + 2A_{45}\frac{\partial \tilde{\omega}}{\partial r}\left(\frac{1}{r}\frac{\partial \tilde{\omega}}{\partial \varphi} + r\right) + A_{44}\left(\frac{1}{r}\frac{\partial \tilde{\omega}}{\partial \varphi} + r\right)^2\right] dA. \tag{29}$$

In Equation (28), $\tilde{\omega} = \tilde{\omega}(r, \omega)$ is an arbitrary function whose second order partial derivatives with respect to r and φ are continuous functions in $\bar{A} = A \cup \partial A.$ It should be mentioned that

$$S = \prod\nolimits_L [\omega(r, \varphi)]. \tag{30}$$

It is obvious if $\tilde{\omega} = \tilde{\omega}(r, \varphi)$ kinematically admissible torsion function then $\lambda\tilde{\omega}(r, \varphi)$ is also a kinematically admissible torsion function for arbitrary value of $\lambda,$ that the upper bound given by in equality (31) is valid for S with arbitrary λ

$$S \leq \int_A \left[A_{55}\left(\lambda \frac{\partial \tilde{\omega}}{\partial r}\right) + 2A_{45}\lambda \frac{\partial \tilde{\omega}}{\partial r}\left(\frac{\lambda}{r}\frac{\partial \omega}{\partial \varphi} + r\right) + A_{44}\left(\frac{\lambda}{r}\frac{\partial \tilde{\omega}}{\partial \varphi} + r\right)^2 \right] dA, \ -\infty \leq \lambda \leq \infty. \tag{31}$$

The following inequality can be proven by a simple extreme value calculation:

$$\int_A \left[A_{55}\left(\lambda \frac{\partial \tilde{\omega}}{\partial r}\right)^2 + 2A_{45}\lambda \frac{\partial \tilde{\omega}}{\partial r}\left(\frac{\lambda}{r}\frac{\partial \omega}{\partial \varphi} + r\right) + A_{44}\left(\frac{\lambda}{r}\frac{\partial \tilde{\omega}}{\partial \varphi} + r\right)^2 \right] dA$$

$$\geq \int_A A_{44} r^2 dA - \frac{\left(\int_A \left[A_{45} r \frac{\partial \tilde{\omega}}{\partial r} + A_{44}\frac{\partial \tilde{\omega}}{\partial \varphi} \right] dA \right)^2}{\int_A \left[A_{55}\left(\frac{\partial \tilde{\omega}}{\partial r}\right) + 2A_{45}\frac{1}{r}\frac{\partial \tilde{\omega}}{\partial \varphi}\frac{\partial \tilde{\omega}}{\partial r} + A_{44}\frac{1}{r^2}\left(\frac{\partial \tilde{\omega}}{\partial \varphi}\right)^2 \right] dA}, \ -\infty < \lambda < \infty. \tag{32}$$

According to inequality relation (32) the form of the sharpest upper bound in terms of $\tilde{\omega} = \tilde{\omega}(r, \varphi)$ is as follows:

$$S \leq \int_A A_{44} r^2 dA - \frac{\left(\int_A \left[A_{45} r \frac{\partial \tilde{\omega}}{\partial r} + A_{44}\frac{\partial \tilde{\omega}}{\partial \varphi} \right] dA \right)^2}{\int_A \left[A_{55}\left(\frac{\partial \tilde{\omega}}{\partial r}\right)^2 + 2A_{45}\frac{1}{r}\frac{\partial \tilde{\omega}}{\partial \varphi}\frac{\partial \tilde{\omega}}{\partial r} + A_{44}\frac{1}{r^2}\left(\frac{\partial \tilde{\omega}}{\partial \varphi}\right)^2 \right] dA}$$

for

$$\int_A (\tilde{\omega}(r, \varphi))^2 dA \neq 0, \tag{33}$$

and

$$S \leq \int_A A_{44} r^2 dA \ \text{ for } \int_A (\tilde{\omega}(r, \varphi))^2 dA = 0. \tag{34}$$

5. Example: Cylindrical Anisotropic Elliptical Bar

Figure 2 shows the cross section of the cylindrical anisotropic elastic bar. The equation of the boundary curve ∂A in polar coordinates r and φ is

$$R(\varphi) = \frac{ab}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}, \ 0 \leq \varphi \leq 2\pi. \tag{35}$$

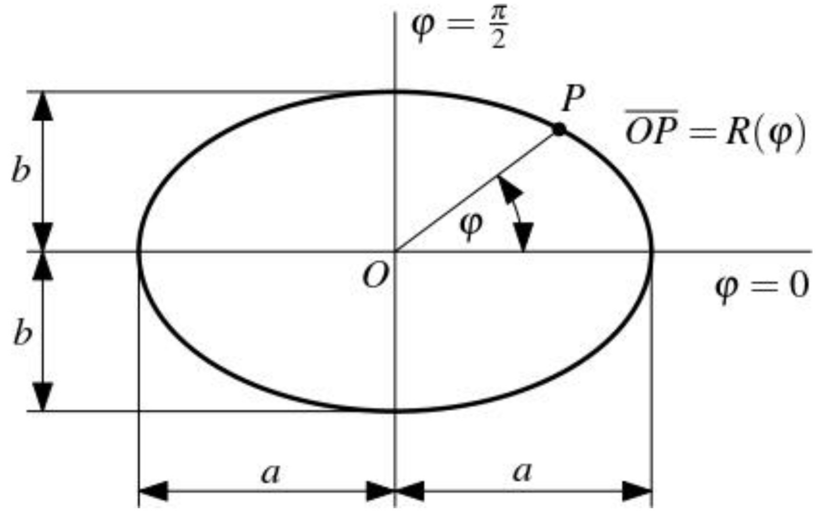

Figure 2. Cylindrical anisotropic elliptical cross section.

Assumed form of the statically admissible function $\tilde{U} = \tilde{U}(r, \varphi)$ is as follows:

$$\tilde{U}(r, \varphi) = a^2 b^2 - r^2 \left(a^2 \sin \varphi + b^2 \cos^2 \varphi\right), \quad (r, \varphi) \in A \cup \partial A. \tag{36}$$

Substitution of $\tilde{U} = \tilde{U}(r, \varphi)$ given by Equation (36) into lower bound formula (27) gives

$$S \geq S_L = \frac{a^3 b^3 \pi}{a_{55}\left(a^2 + b^2\right) + 2ab(a_{44} - a_{55})}. \tag{37}$$

In the lower bound formula (37) for $a_{44} = a_{55}$ the sign of the equality is valid, in this case the considered bar is isotropic.

By the use of kinematically admissible torsion function in upper bound formula yields the result

$$S \le S_U = \frac{A_{44} a^3 b^3 \left(A_{55} a^2 + 2 A_{44} ab + A_{55} b^2 \right) \pi}{A_{44} \left(a^4 + b^4 \right) + 4 A_{55} a^2 b^2 + 2 A_{44} ab \left(a^2 + ab + b^2 \right)}. \tag{38}$$

In the upper bound formula (38) for $A_{44} = A_{55}$ the sign of the equality is valid that is we have [1, 5]

$$S = A_{44} \frac{a^3 b^3}{a^2 + b^2} \pi. \tag{39}$$

For $a = 0.06\text{m}$, $b = 0.12\text{m}$, $a_{44} = 9.5 \times 10^{-9} \frac{\text{m}^2}{N}$, $a_{55} = 9 \times 10^{-9} \frac{\text{m}^2}{N}$, $a_{45} = -\ 6.5 \times 10^{-9} \frac{\text{m}^2}{N}$ the bounding formulae (37), (38) give the following numerical result:

$$2128.891022\text{Nm}^2 \le S \le 2150.064306\text{Nm}^2. \tag{40}$$

6. Conclusions

In the present paper, the elastic torsion of cylindrical bar with solid cross section made of cylindrical anisotropic material which displays planes of elastic symmetry perpendicular to the axis of bar is considered. Lower and upper bounds are derived to the torsional rigidity of the bar. To formulate the upper and lower bounds of torsional rigidity two minimum theorems of linear elasticity are used. The application of the obtained formulae of torsional rigidity is illustrated in the example of elliptical cross section.

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