Journal of Pure and Applied Mathematics: Advances and Applications
Volume 25, Number 1, 2022, Pages 23-48
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jpamaa_7100122242

ON γ^{δ^*}-OPEN AND γ^{δ^*}-DISCONNECTED SETS

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Abstract

In this article, we define an γ^{δ^*}-open sets via γ-operation defined on δ and investigate some of their properties. Moreover, we study the notion of γ^{δ^*}-generalized closed set in topological spaces and present some of their respective features. Ultimately, we introduce and characterize γ^{δ^*}-disconnected and γ^{δ^*}-component sets.

2020 Mathematics Subject Classification: Primary: 54A05, 54A10; Secondary: 54C05.

Keywords and phrases: δ^*-open set, function, γ-operation, δ^*-open set.

Received December 30, 2021; Revised June 23, 2022

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [9] defined the concept of operation on topological spaces and introduced α-closed graphs of an operation. Ogata [11] called the operation α as γ operation and studied the notion of γ-open sets. Ibrahim [8] presented and discussed an operation on the family of all α-open sets, and he introduced the concept of α_γ-open sets. Ibrahim [5] defined the notion of R_γ-open sets by using an operation γ on the family of all regular open sets in topological spaces, and also he introduced P_γ-open sets [2]. Furthermore, Ibrahim [4] investigated the concept of δ^*-open sets. Ibrahim [6] introduced the concept of γgb-closed sets in topological spaces. Soft set is a parameterized general mathematical tool which eals with a collection of approximate descriptions of objects. Each approximate description has two parts, a predicate and an approximate value set. The concept of soft sets was first introduced by Molodtsov [3] in 1999 as a general mathematical tool for dealing with uncertain objects. Ibrahim [7] studied some new classes of soft generalized closed sets by using soft $\tilde{\gamma}$ open set. A subset A of a topological space (Y, δ) is said to be:

(1) α-open [10] if $A \subseteq Int(Cl(Int(A)))$.

(2) preopen [1] if $A \subseteq Int(Cl(A))$.

(3) regular open [12] if $Int(Cl(A)) = A$.

Let (Y, δ) be any topological space and let f be a function from X into Y, then a subset G in δ is called δ^*-open if $f^{-1}(G) = \phi$ or $f^{-1}(G) = X$, that is $\delta^* = \{G \in \delta : f^{-1}(G) = \phi$ or $f^{-1}(G) = X\}$. The family of all δ^*-open sets in Y is denoted by δ^* [4]. An operation γ on a topology δ is a mapping from δ into power set $P(Y)$ of Y such that $V \subseteq V^{\gamma}$ for each $V \in \delta$, where V^{γ} denotes the value of γ at V [9]. A subset A of Y with an operation γ on δ is called γ-open if for each $y \in A$, there exists an open set U such that $y \in U$ and $U^{\gamma} \subseteq A$. The family of all γ-open sets in a topological space (Y, δ) is denoted by δ_{γ} [11].

2. γ^{δ^*}-Open Sets

Definition 2.1. Let (Y, δ) be any topological space and f be a function from X into Y. Then, a subset A of Y with an operation γ on δ is called γ^{δ^*}-open if for each $y \in A$, there exists a δ^*-open set U such that $y \in U$ and $U^{\gamma} \subseteq A$. The family of all γ^{δ^*}-open sets in a topological space (Y, δ) is denoted by $\delta_{\gamma^{\delta^*}}$. Complements of γ^{δ^*}-open sets are called γ^{δ^*}-closed.

Theorem 2.2. *If A* is γ^{δ^*}*-open, then A is* γ*-open.*

Proof. Obvious. □

Remark 2.3. The converse of the above theorem need not be true in general as it is shown below.

Example 2.4. Consider $X = \{a, b\}$ and $Y = \{1, 2, 3, 4, 5\}$ with $\delta = \{\phi, Y, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 3, 4, 5\}\}$. Let $f : X \to Y$ be a function such that

$$f(x) = \begin{cases} 1 & \text{if } x = a, \\ 5 & \text{if } x = b. \end{cases}$$

Then, $\delta^* = \{\phi, Y, \{2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4, 5\}\}$. For each $A \in \delta$ we define an operation γ on δ by

$$A^{\gamma} = \begin{cases} \{2, 3, 4\} \cup \{1\} & \text{if } \{2, 3, 4\} \subseteq A, \\ Y & \text{otherwise}. \end{cases}$$

Then, $\{1, 2, 3, 4\}$ is γ-open but not γ^{δ^*}-open.

Theorem 2.5. *Let (Y, δ) be any topological space and $A \subseteq Y$. Then*:

(1) *If A is γ^{δ^*}-open, then A is δ^*-open.*

(2) *If A is γ^{δ^*}-open, then A is open.*

Proof. Obvious. □

Remark 2.6. The set $\{2, 3, 4\}$ in Example 2.4 is both δ^*-open and open but it is not γ^{δ^*}-open.

Remark 2.7. Let γ be the identity operation on δ and $A \subseteq Y$. Then, A is γ^{δ^*}-open if and only if A is δ^*-open.

Theorem 2.8. *Let (Y, δ) be an indiscrete topological space, then* $\delta_{\gamma^{\delta^*}} = \delta_{\gamma}$.

Theorem 2.9. *Let (Y, δ) be any topological space and f be a constant function, then* $\delta_{\gamma^{\delta^*}} = \delta_{\gamma}$.

Remark 2.10. Let γ be an operation on the family of all preopen sets [2]. Then:

(1) If A is γ^{δ^*}-open, then A is both α_γ-open and P_γ-open.

(2) The concept of γ^{δ^*}-open and R_γ-open are independent in general.

Theorem 2.11. *If A_i is γ^{δ^*}-open for every $i \in I$, then $\cup \{A_i : i \in I\}$ is γ^{δ^*}-open.*

Proof. Let $y \in \cup_{i \in I} A_i$, then $y \in A_i$ for some $i \in I$. Since A_i is γ^{δ^*}-open, then there exists δ^*-open U such that $y \in U \subseteq U^\gamma \subseteq A_i \subseteq \cup_{i \in I} A_i$. Therefore, $\cup_{i \in I} A_i$ is γ^{δ^*}-open. □

Remark 2.12. If A and B are two γ^{δ^*}-open sets in (Y, δ), then the following example shows that $A \cap B$ need not be γ^{δ^*}-open.

Example 2.13. Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\delta = \{\phi, Y, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. Let $f : X \to Y$ be a function such that

$$f(x) = \begin{cases} 2 & \text{if } x = a, \\ 2 & \text{if } x = b, \\ 2 & \text{if } x = c. \end{cases}$$

And for each $A \in \delta$ we define an operation γ on δ by

$$A^\gamma = \begin{cases} Cl(A) & \text{if } 3 \notin A, \\ A & \text{if } 3 \in A. \end{cases}$$

Then, it is obvious that the sets $\{1, 2\}$ and $\{2, 3\}$ are γ^{δ^*}-open, however their intersection $\{2\}$ is not γ^{δ^*}-open.

Remark 2.14. From the above example, we notice that the family of all γ^{δ^*}-open subsets of a space Y is a supratopology and need not be a topology in general.

Theorem 2.15. *The set A is δ^*-open in the space Y if and only if for each $y \in A$. there exists a δ^*-open set B such that $y \in B \subseteq A$.*

Definition 2.16. An operation γ on δ is said to be:

(1) δ^*-regular if for every δ^*-open subsets U and V of Y containing $y \in Y$, there exists a δ^*-open set W containing y such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

(2) δ^*-open if for every δ^*-open set U in Y containing $y \in Y$, there exists a γ^{δ^*}-open set V in Y such that $y \in V$ and $V \subseteq U^\gamma$.

Theorem 2.17. *Let γ be a δ^*-regular operation on δ. If A and B are γ^{δ^*}-open sets in Y, then $A \cap B$ is also a γ^{δ^*}-open set and hence, $\delta_{\gamma^{\delta^*}}$ forms a topology on Y.*

Proof. Let $y \in A \cap B$, then $y \in A$ and $y \in B$. Since A and B are γ^{δ^*}-open sets, then there exist δ^*-open sets U_1 and U such that $y \in U_1 \subseteq U_1^\gamma \subseteq A$ and $y \in U \subseteq U^\gamma \subseteq B$. Since γ is δ^*-regular, then there exists a δ^*-open set K such that $y \in K \subseteq K^\gamma \subseteq U_1^\gamma \cap U^\gamma \subseteq A \cap B$. This implies that $A \cap B$ is γ^{δ^*}-open set. □

Theorem 2.18. *Let γ be a monotone operation on δ, then $\delta_{\gamma^{\delta^*}}$ forms a topology on Y.*

Proof. Clearly $\phi, Y \in \delta_{\gamma^{\delta^*}}$ and by Theorem 2.11, the union of any family of γ^{δ^*}-open sets is γ^{δ^*}-open. To complete the proof it is enough to show that the finite intersection of γ^{δ^*}-open sets is γ^{δ^*}-open. Let A and B be two γ^{δ^*}-open sets and $y \in A \cap B$, then $y \in A$ and $y \in B$, so there exist γ^{δ^*}-open sets U and V such that $y \in U \subseteq U^{\gamma} \subseteq A$ and $y \in V \subseteq V^{\gamma} \subseteq B$, since γ is monotone and $U \cap V$ is γ^{δ^*}-open such that $U \cap V \subseteq U$ and $U \cap V \subseteq V$, this implies that $(U \cap V)^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma} \subseteq A \cap B$. Thus, $A \cap B$ is γ^{δ^*}-open. This completes the proof. □

Definition 2.19. Let (Y, δ) be any topological space and $A \subseteq Y$. Then:

(1) A point $y \in Y$ is in $Cl_{\gamma^{\delta^*}}$-closure of A, if $U^{\gamma} \cap A \neq \phi$ for each δ^*-open set U containing y. The $Cl_{\gamma^{\delta^*}}$-closure of A is denoted by $Cl_{\gamma^{\delta^*}}(A)$.

(2) The intersection of all γ^{δ^*}-closed sets containing A is called the γ^{δ^*}-closure of A and denoted by γ^{δ^*}-$Cl(A)$.

(3) A point $y \in A$ is said to be γ^{δ^*}-interior point of A if there exists a δ^*-open set U in Y containing y such that $U^{\gamma} \subseteq A$. We denote the set of all such points by $Int_{\gamma^{\delta^*}}(A)$. Thus, $Int_{\gamma^{\delta^*}}(A) = \{y \in A : y \in U \in \delta^*$ and $U^{\gamma} \subseteq A\} \subseteq A$.

(4) The union of all γ^{δ^*}-open sets contained in A is called the γ^{δ^*}-interior of A and denoted by γ^{δ^*}-$Int(A)$.

(5) A is said to be $\gamma^{\delta*}$-neighbourhood of $y \in Y$, if there exists a $\gamma^{\delta*}$-open set U in Y such that $y \in U \subseteq A$.

The proof of the following theorems are obvious and hence omitted.

Theorem 2.20. *For a point* $y \in Y$, $y \in \gamma^{\delta*}\text{-}Cl(A)$ *if and only if every* $\gamma^{\delta*}$*-open subset of* Y *containing* y *has a non empty intersection with* A.

Theorem 2.21. *Let* (Y, δ) *be any topological space. For any subsets* A *and* B *of* Y, *we have the following properties*:

(1) $A \subseteq \gamma^{\delta*}\text{-}Cl(A)$, $A \subseteq Cl_{\gamma^{\delta*}}(A)$ *and* $\gamma^{\delta*}\text{-}Int(A) \subseteq A$.

(2) $\gamma^{\delta*}\text{-}Cl(A)$ *is* $\gamma^{\delta*}$*-closed and* $\gamma^{\delta*}\text{-}Int(A)$ *is* $\gamma^{\delta*}$*-open.*

(3) *A is* $\gamma^{\delta*}$*-closed if and only if* $A = \gamma^{\delta*}\text{-}Cl(A)$ *and A is* $\gamma^{\delta*}$*-open if and only if* $A = \gamma^{\delta*}\text{-}Int(A)$.

(4) $\gamma^{\delta*}\text{-}Cl(\phi) = \phi$, $\gamma^{\delta*}\text{-}Cl(Y) = Y$, $\gamma^{\delta*}\text{-}Int(\phi) = \phi$, $\gamma^{\delta*}\text{-}Int(Y) = Y$, $Cl_{\gamma^{\delta*}}(\phi) = \phi$ *and* $Cl_{\gamma^{\delta*}}(Y) = Y$.

(5) *If* $A \subseteq B$, *then* $\gamma^{\delta*}\text{-}Cl(A) \subseteq \gamma^{\delta*}\text{-}Cl(B)$, $Cl_{\gamma^{\delta*}}(A) \subseteq Cl_{\gamma^{\delta*}}(B)$, $\gamma^{\delta*}\text{-}Int(A) \subseteq \gamma^{\delta*}\text{-}Int(B)$ *and* $Int_{\gamma^{\delta*}}(A) \subseteq Int_{\gamma^{\delta*}}(B)$.

(6) $\gamma^{\delta*}\text{-}Cl(A \cup B) \supseteq \gamma^{\delta*}\text{-}Cl(A) \cup \gamma^{\delta*}\text{-}Cl(B)$, $Cl_{\gamma^{\delta*}}(A) \cup Cl_{\gamma^{\delta*}}(B) \subseteq Cl_{\gamma^{\delta*}}(A \cup B)$, $\gamma^{\delta*}\text{-}Int(A \cup B) \supseteq \gamma^{\delta*}\text{-}Int(A) \cup \gamma^{\delta*}\text{-}Int(B)$ *and* $Int_{\gamma^{\delta*}}(A) \cup Int_{\gamma^{\delta*}}(B) \subseteq Int_{\gamma^{\delta*}}(A \cup B)$.

(7) $\gamma^{\delta*}\text{-}Cl(A \cap B) \subseteq \gamma^{\delta*}\text{-}Cl(A) \cap \gamma^{\delta*}\text{-}Cl(B)$, $Cl_{\gamma^{\delta*}}(A \cap B) \subseteq Cl_{\gamma^{\delta*}}(A) \cap Cl_{\gamma^{\delta*}}(B)$ *and* $\gamma^{\delta*}\text{-}Int(A \cap B) \subseteq \gamma^{\delta*}\text{-}Int(A) \cap \gamma^{\delta*}\text{-}Int(B)$.

(8) $Cl_{\gamma_{\delta^*}}(A)$ *is* δ^**-closed and* $Int_{\gamma_{\delta^*}}(A)$ *is* δ^**-open.*

(9) *If* γ *is* δ^**-open, then* $Cl_{\gamma_{\delta^*}}(A) = \gamma^{\delta^*}\text{-}Cl(A)$ *and* $Cl_{\gamma_{\delta^*}}(Cl_{\gamma_{\delta^*}}(A)) = Cl_{\gamma_{\delta^*}}(A)$, *and* $Cl_{\gamma_{\delta^*}}(A)$ *is* γ^{δ^*}*-closed.*

(10) *If* γ *is* δ^**-regular, then* $Int_{\gamma_{\delta^*}}(A) \cap Int_{\gamma_{\delta^*}}(B) = Int_{\gamma_{\delta^*}}(A \cup B)$ *and* $Cl_{\gamma_{\delta^*}}(A \cup B) = Cl_{\gamma_{\delta^*}}(A) \cup Cl_{\gamma_{\delta^*}}(B)$.

(11) *A is* γ^{δ^*}*-open if and only if* $Int_{\gamma_{\delta^*}}(A) = A$ *and A is* γ^{δ^*}*-closed if and only if* $Cl_{\gamma_{\delta^*}}(A) = A$.

(12) *A is* γ^{δ^*}*-open if and only if it is a* γ^{δ^*}*-neighbourhood of each of its points.*

(13) *If* $A \subseteq B$ *and A is a* γ^{δ^*}*-neighbourhood of a point* $y \in Y$, *then B is also* γ^{δ^*}*-neighbourhood of the same point* y.

Theorem 2.22. *Let A be any subset of a topological space* (Y, δ). *Then, the following relation holds*: $A \subseteq \delta^*\text{-}Cl(A) \subseteq Cl_{\gamma_{\delta^*}}(A) \subseteq \gamma^{\delta^*}\text{-}Cl(A)$.

Theorem 2.23. *Let* (Y, δ) *be any topological space and* $A \subseteq Y$. *Then*

(1) $Int_{\gamma_{\delta^*}}(Y \setminus A) = Y \setminus Cl_{\gamma_{\delta^*}}(A)$ *and* $Y \setminus \gamma^{\delta^*}\text{-}Int(A) = \gamma^{\delta^*}\text{-}Cl(Y \setminus A)$.

(2) $Cl_{\gamma_{\delta^*}}(Y \setminus A) = Y \setminus Int_{\gamma_{\delta^*}}(A)$ *and* $Y \setminus \gamma^{\delta^*}\text{-}Cl(A) = \gamma^{\delta^*}\text{-}Int(Y \setminus A)$.

(3) $Int_{\gamma_{\delta^*}}(A) = Y \setminus Cl_{\gamma_{\delta^*}}(Y \setminus A)$ *and* $\gamma^{\delta^*}\text{-}Int(A) = Y \setminus \gamma^{\delta^*}\text{-}Cl(Y \setminus A)$.

(4) $Cl_{\gamma_{\delta^*}}(A) = Y \setminus Int_{\gamma_{\delta^*}}(Y \setminus A)$ *and* $\gamma^{\delta^*}\text{-}Cl(A) = Y \setminus \gamma^{\delta^*}\text{-}Int(Y \setminus A)$.

Theorem 2.24. *Let* γ *be a monotone operation on* δ *and* $A \subseteq Y$. *Then*:

(1) *For every* γ^{δ^*}*-open set* G *of* Y, *we have* $Cl_{\gamma^{\delta^*}}(A) \cap G \subseteq Cl_{\gamma^{\delta^*}}(A \cap G)$.

(2) *For every* γ^{δ^*}*-closed set* F *of* Y, *we have* $Int_{\gamma^{\delta^*}}(A \cup F) \subseteq Int_{\gamma^{\delta^*}}(A) \cup F$.

Proof. (1) Let $y \in Cl_{\gamma^{\delta^*}}(A) \cap G$ and U be a δ^*-open set containing y. Since $y \in Cl_{\gamma^{\delta^*}}(A)$, implies that $U^\gamma \cap A \neq \phi$. Since G is a γ^{δ^*}-open set, there exists a δ^*-open set V in Y containing y such that $V^\gamma \subseteq G$. Thus $(U \cap V)^\gamma \cap A \neq \phi$, this implies that $U^\gamma \cap (A \cap G) \neq \phi$ by monotone and hence $y \in Cl_{\gamma^{\delta^*}}(A \cap G)$. Therefore, $Cl_{\gamma^{\delta^*}}(A) \cap G \subseteq Cl_{\gamma^{\delta^*}}(A \cap G)$.

(2) Follows from (1) and Theorem 2.23 (3). □

Theorem 2.25. *Let* (Y, δ) *be a topological space*, A *be a subset of* Y *and* γ *be* δ^**-open. Then*, $Int_{\gamma^{\delta^*}}(Cl_{\gamma^{\delta^*}}(A)) = \phi$ *if and only if any one of the following conditions hold*:

(1) $Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A)) = Y$.

(2) $A \subseteq Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$.

Proof. (1) $Int_{\gamma^{\delta^*}}(Cl_{\gamma^{\delta^*}}(A)) = \phi$ if and only if $Y \setminus Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A)) = \phi$ by Theorem 2.23 (3) if and only if $Y = Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$.

(2) $A \subseteq Y = Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$ by (1).

Conversely, $A \subseteq Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$, implies that $Cl_{\gamma^{\delta^*}}(A) \subseteq Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$ by Theorem 2.21 (9). Since $Y = Cl_{\gamma^{\delta^*}}(A) \cup (Y \setminus Cl_{\gamma^{\delta^*}}(A))$, implies that $Y \subseteq Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A)) \cup (Y \setminus Cl_{\gamma^{\delta^*}}(A)) = Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$. Thus, $Y = Cl_{\gamma^{\delta^*}}(Y \setminus Cl_{\gamma^{\delta^*}}(A))$. □

Theorem 2.26. *Let (Y, δ) be a topological space, A be a subset of Y and the operation γ be both δ^*-regular and δ^*-open. If $Int_{\gamma^{\delta^*}}(Cl_{\gamma^{\delta^*}}(A)) = \phi$, then every non empty γ^{δ^*}-open set U contains a non empty γ^{δ^*}-open set B disjoint with A.*

Proof. Given $Int_{\gamma^{\delta^*}}(Cl_{\gamma^{\delta^*}}(A)) = \phi$. This implies that $Cl_{\gamma^{\delta^*}}(A)$ does not contain any non empty γ^{δ^*}-open set. Hence for any non empty γ^{δ^*}-open set U, $U \cap (Y \setminus Cl_{\gamma^{\delta^*}}(A)) \neq \phi$. Thus by Theorem 2.21 (9) and Theorem 2.17, $B = U \cap (Y \setminus Cl_{\gamma^{\delta^*}}(A)) = U \setminus Cl_{\gamma^{\delta^*}}(A)$ is a non empty γ^{δ^*}-open set contained in U and disjoint with A. □

Definition 2.27. Let A be a subset of a topological space (Y, δ). The γ^{δ^*}-kernel of A, denoted by γ^{δ^*}-ker(A) is defined to be the set γ^{δ^*}-ker$(A) = \cap \{V : A \subseteq V, V \in \delta_{\gamma^{\delta^*}}\}$.

Proposition 2.28. *Let (Y, δ) be a topological space and $x \in Y$. Then, $y \in \gamma^{\delta^*}$-ker$(\{x\})$ if and only if $x \in \gamma^{\delta^*}$-$Cl(\{y\})$.*

Proof. Suppose that $y \notin \gamma^{\delta^*}$-ker$(\{x\})$. Then there exists an γ^{δ^*}-open set V containing x such that $y \notin V$. Therefore by Theorem 2.20, we have $x \notin \gamma^{\delta^*}$-$Cl(\{y\})$. The proof of the converse case can be done similarly. □

Proposition 2.29. *Let* (Y, δ) *be a topological space and* A *be a subset of* Y. *Then,* $\gamma^{\delta^*}\text{-ker}(A) = \{x \in Y : \gamma^{\delta^*}\text{-}Cl(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in \gamma^{\delta^*}\text{-ker}(A)$ and suppose $\gamma^{\delta^*}\text{-}Cl(\{x\}) \cap A \neq \phi$. Hence $x \notin Y \setminus \gamma^{\delta^*}\text{-}Cl(\{x\})$, which is a γ^{δ^*}-open set containing A. This is impossible, since $x \in \gamma^{\delta^*}\text{-ker}(A)$. Consequently, $\gamma^{\delta^*}\text{-}Cl(\{x\}) \cap A \neq \phi$. Next, let $x \in Y$ such that $\gamma^{\delta^*}\text{-}Cl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \gamma^{\delta^*}\text{-ker}(A)$. Then, there exists a γ^{δ^*}-open set V containing A and $x \notin V$. Let $y \in \gamma^{\delta^*}\text{-}Cl(\{x\}) \cap A$. Hence, V is a γ^{δ^*}-open set containing y which does not contain x. By this contradiction $x \in \gamma^{\delta^*}\text{-ker}(A)$ and the claim. □

Proposition 2.30. *The following properties hold for the subsets A and B of a topological space* (Y, δ):

(1) $A \subseteq \gamma^{\delta^*}\text{-ker}(A)$.

(2) $A \subseteq B$ *implies that* $\gamma^{\delta^*}\text{-ker}(A) \subseteq \gamma^{\delta^*}\text{-ker}(B)$.

(3) *If* A is γ^{δ^*}*-open, then* $A = \gamma^{\delta^*}\text{-ker}(A)$.

(4) $\gamma^{\delta^*}\text{-ker}(\gamma^{\delta^*}\text{-ker}(A)) = \gamma^{\delta^*}\text{-ker}(A)$.

Proof. (1), (2) and (3): Are immediate consequences of $\gamma^{\delta^*}\text{-ker}(A) = \cap \{U \in \delta_{\gamma^{\delta^*}} : A \subseteq U\}$.

(4) First observe that by (1) and (2), we have $\gamma^{\delta^*}\text{-ker}(A) \subseteq \gamma^{\delta^*}\text{-ker}(\gamma^{\delta^*}\text{-ker}(A))$. If $x \notin \gamma^{\delta^*}\text{-ker}(A)$, then there exists $U \in \delta_{\gamma^{\delta^*}}$ such that $A \subseteq U$ and $x \notin U$. Hence $\gamma^{\delta^*}\text{-ker}(A) \subseteq U$, and so we have $x \notin \gamma^{\delta^*}\text{-ker}(\gamma^{\delta^*}\text{-ker}(A))$. Thus $\gamma^{\delta^*}\text{-ker}(\gamma^{\delta^*}\text{-ker}(A)) = \gamma^{\delta^*}\text{-ker}(A)$. □

Proposition 2.31. *The following statements are equivalent for any points x and y in a topological space* (Y, δ):

(1) γ^{δ^*}-ker$(\{x\}) \neq \gamma^{\delta^*}$-ker$(\{y\})$.

(2) γ^{δ^*}-$Cl(\{x\}) \neq \gamma^{\delta^*}$-$Cl(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Suppose that γ^{δ^*}-ker$(\{x\}) \neq \gamma^{\delta^*}$-ker$(\{y\})$, then there exists a point z in Y such that $z \in \gamma^{\delta^*}$-ker$(\{x\})$ and $z \notin \gamma^{\delta^*}$-ker$(\{y\})$. From $z \in \gamma^{\delta^*}$-ker$(\{x\})$, it follows that $\{x\} \cap \gamma^{\delta^*}$-$Cl(\{z\}) \neq \phi$ which implies $x \in \gamma^{\delta^*}$-$Cl(\{z\})$. By $z \notin \gamma^{\delta^*}$-ker$(\{y\})$, we have $\{y\} \cap \gamma^{\delta^*}$-$Cl(\{z\}) = \phi$. Since $x \in \gamma^{\delta^*}$-$Cl(\{z\})$, γ^{δ^*}-$Cl(\{x\}) \subseteq \gamma^{\delta^*}$-$Cl(\{z\})$ and $\{y\} \cap \gamma^{\delta^*}$-$Cl(\{x\}) = \phi$. Therefore, it follows that γ^{δ^*}-$Cl(\{x\}) \neq \gamma^{\delta^*}$-$Cl(\{y\})$. Now γ^{δ^*}-ker$(\{x\})$ $\neq \gamma^{\delta^*}$-ker$(\{y\})$ implies that γ^{δ^*}-$Cl(\{x\}) \neq \gamma^{\delta^*}$-$Cl(\{y\})$.

(2) $\Rightarrow$ (1): Suppose that γ^{δ^*}-$Cl(\{x\}) \neq \gamma^{\delta^*}$-$Cl(\{y\})$. Then there exists a point z in Y such that $z \in \gamma^{\delta^*}$-$Cl(\{x\})$ and $z \notin \gamma^{\delta^*}$-$Cl(\{y\})$. Then, there exists an γ^{δ^*}-open set containing z and therefore x but not y, namely, $y \notin \gamma^{\delta^*}$-ker$(\{x\})$ and thus γ^{δ^*}-ker$(\{x\}) \neq \gamma^{\delta^*}$-ker$(\{y\})$. □

Proposition 2.32. *Let* (Y, δ) *be a topological space. Then,* $\cap \{\gamma^{\delta^*}$-Cl $(\{x\}): x \in Y\} = \phi$ *if and only if* γ^{δ^*}-ker$(\{x\}) \neq Y$ *for every* $x \in Y$.

Proof. Necessity: Suppose that $\cap \{\gamma^{\delta^*}$-$Cl(\{x\}) : x \in Y\} = \phi$. Assume that there is a point y in Y such that γ^{δ^*}-ker$(\{y\}) = Y$. Let x be any point of Y. Then $x \in V$ for every γ^{δ^*}-open set V containing y and hence $y \in \gamma^{\delta^*}$-$Cl(\{x\})$ for any $x \in Y$. This implies that $y \in \cap \{\gamma^{\delta^*}$-$Cl(\{x\}) : x \in Y\}$. But this is a contradiction.

Sufficiency: Assume that $\gamma^{\delta^*}\text{-ker}(\{x\}) \neq Y$ for every $x \in Y$. If there exists a point y in Y such that $y \in \cap \{\gamma^{\delta^*}\text{-}Cl(\{x\}) : x \in Y\}$, then every γ^{δ^*}-open set containing y must contain every point of Y. This implies that the space Y is the unique γ^{δ^*}-open set containing y. Hence $\gamma^{\delta^*}\text{-ker}(\{y\}) = Y$ which is a contradiction. Therefore, $\cap \{\gamma^{\delta^*}\text{-}Cl(\{x\}) : x \in Y\} = \phi$. □

Definition 2.33. A subset A of a topological space (Y, δ) is said to be γ^{δ^*}-generalized closed (briefly, γ^{δ^*}-g.closed) if $Cl_{\gamma^{\delta^*}}(A) \subseteq U$ whenever $A \subseteq U$ and U is γ^{δ^*}-open.

Remark 2.34. It is clear that every γ^{δ^*}-closed set is γ^{δ^*}-g.closed, but the converse is not true in general as it is shown in the following example.

Example 2.35. Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\delta = \{\phi, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$. Let $f : X \to Y$ be a function such that

$$f(x) = \begin{cases} 3 & \text{if } x = a, \\ 1 & \text{if } x = b, \\ 3 & \text{if } x = c. \end{cases}$$

And for each $A \in \delta$ we define an operation γ on δ by

$$A^{\gamma} = \begin{cases} A & \text{if } A = \{2\} \text{ or } \{1, 3\}, \\ Y & \text{otherwise.} \end{cases}$$

Let $A = \{1\}$, since the only γ^{δ^*}-open supersets of A are $\{1, 3\}$ and Y, then $Cl_{\gamma^{\delta^*}}(A) = \{1, 3\}$ and so A is γ^{δ^*}-g.closed. But it is easy to see that A is not γ^{δ^*}-closed

Theorem 2.36. *If A is γ^{δ^*}-open and γ^{δ^*}-g.closed, then A is γ^{δ^*}-closed.*

Proof. Suppose that A is γ^{δ^*}-open and γ^{δ^*}-g.closed. Since $A \subseteq A$, we have $Cl_{\gamma^{\delta^*}}(A) \subseteq A$, also $A \subseteq Cl_{\gamma^{\delta^*}}(A)$, therefore $Cl_{\gamma^{\delta^*}}(A) = A$. That is, A is γ^{δ^*}-closed. □

Theorem 2.37. *Let A a subset of a topological space (Y, δ). Then the following statements are equivalent*:

(1) A is γ^{δ^*}-*g.closed.*

(2) γ^{δ^*}-$Cl(\{x\}) \cap A \neq \phi$ *for every* $x \in Cl_{\gamma^{\delta^*}}(A)$.

(3) $Cl_{\gamma^{\delta^*}}(A) \subseteq \gamma^{\delta^*}$-ker$(A)$.

Proof. (1) $\Rightarrow$ (2): Let A be a γ^{δ^*}-g.closed set of (Y, δ). Suppose that there exists a point $x \in Cl_{\gamma^{\delta^*}}(A)$ such that γ^{δ^*}-$Cl(\{x\}) \cap A = \phi$. By Theorem 2.21 (2), γ^{δ^*}-$Cl(\{x\})$ is γ^{δ^*}-closed. Put $U = Y \setminus \gamma^{\delta^*}$-$Cl(\{x\})$. Then, we have that $A \subseteq U$, $x \notin U$ and U is a γ^{δ^*}-open set of (Y, δ). Since A is a γ^{δ^*}-g.closed set, $Cl_{\gamma^{\delta^*}}(A) \subseteq U$. Thus, we have $x \notin Cl_{\gamma^{\delta^*}}(A)$. This is a contradiction.

(2) $\Rightarrow$ (3): Follows from Proposition 2.29.

(3) $\Rightarrow$ (1): Let U be any γ^{δ^*}-open set such that $A \subseteq U$. Let x be a point such that $x \in Cl_{\gamma^{\delta^*}}(A)$. By (3), $x \in \gamma^{\delta^*}$-ker(A). Namely, we have $x \in U$, because $A \subseteq U$ and $U \in \delta_{\gamma^{\delta^*}}$. □

Theorem 2.38. *Let* (Y, δ) *be a topological space. If a subset* A *of* Y *is* γ^{δ^*}*-g.closed, then* $Cl_{\gamma^{\delta^*}}(A) \setminus A$ *does not contain any non-empty* γ^{δ^*}*-closed set.*

Proof. Suppose that there exists a non-empty γ^{δ^*}-closed set F such that $F \subseteq Cl_{\gamma^{\delta^*}}(A) \setminus A$. Then we have $A \subseteq Y \setminus F$ and $Y \setminus F$ is γ^{δ^*}-open. It follows from the assumption that $Cl_{\gamma^{\delta^*}}(A) \subseteq X \setminus F$ and so $F \subseteq (Cl_{\gamma^{\delta^*}}(A) \setminus A) \cap (Y \setminus Cl_{\gamma^{\delta^*}}(A))$. Therefore, we have $F = \phi$. □

Remark 2.39. In the above theorem, if γ is a δ^*-open operation, then the converse of the above Theorem is true.

Proof. Let U be an γ^{δ^*}-open set such that $A \subseteq U$. Since γ is a δ^*-open operation, it follows from Theorem 2.21 (9), that $Cl_{\gamma^{\delta^*}}(A)$ is γ^{δ^*}-closed in (Y, δ). Thus by Definition 2.1 and Theorem 2.11, we have $Cl_{\gamma^{\delta^*}}(A) \cap (Y \setminus U) = F$ is γ^{δ^*}-closed in (Y, δ). Since $Y \setminus U \subseteq Y \setminus A$, $F \subseteq Cl_{\gamma^{\delta^*}}(A) \setminus A$. Using the assumptions of the converse of Theorem 2.38 above, $F = \phi$ and hence $Cl_{\gamma^{\delta^*}}(A) \subseteq U$. □

Theorem 2.40. *Let* (Y, δ) *be a topological space. Then for each* $x \in Y$, $\{x\}$ *is* γ^{δ^*}*-closed or* $Y \setminus \{x\}$ *is* γ^{δ^*}*-g.closed in* (Y, δ).

Proof. Suppose that $\{x\}$ is not γ^{δ^*}-closed, then $Y \setminus \{x\}$ is not γ^{δ^*}-open. Let U be any γ^{δ^*}-open set such that $Y \setminus \{x\} \subseteq U$. Then $U = Y$. Hence, $Cl_{\gamma^{\delta^*}}(Y \setminus \{x\}) \subseteq U$. Thereore, $Y \setminus \{x\}$ is a γ^{δ^*}-g.closed set. □

Proposition 2.41. *A subset A of Y is γ^{δ^*}-g.closed if and only if $F \subseteq \gamma^{\delta^*}$-$Int(A)$ whenever $F \subseteq A$ and F is γ^{δ^*}-closed in Y.*

Proof. Let A be γ^{δ^*}-g.open and $F \subseteq A$ where F is γ^{δ^*}-closed. Since $Y \setminus A$ is γ^{δ^*}-g.closed and $Y \setminus F$ is a γ^{δ^*}-open set containing $Y \setminus A$ implies γ^{δ^*}-$Cl(Y \setminus A) \subseteq Y \setminus F$. By Theorem 2.23 (1), $Y \setminus \gamma^{\delta^*}$-$Int(A) \subseteq Y \setminus F$. That is $F \subseteq \gamma^{\delta^*}$-$Int(A)$.

Conversely, suppose that F is γ^{δ^*}-closed and $F \subseteq A$ implies $F \subseteq \gamma^{\delta^*}$-$Int(A)$. Let $Y \setminus A \subseteq U$, where U is γ^{δ^*}-open. Then $Y \setminus U \subseteq A$ where $Y \setminus U$ is γ^{δ^*}-closed. By hypothesis $Y \setminus U \subseteq \gamma^{\delta^*}$-$Int(A)$. That is $Y \setminus \gamma^{\delta^*}$-$Int(A) \subseteq U$. By Theorem 2.23 (1), γ^{δ^*}-$Cl(Y \setminus A) \subseteq U$. This implies $Y \setminus A$ is γ^{δ^*}-g.closed and A is γ^{δ^*}-g.open. □

Definition 2.42. Two subsets A and B of a topological space (Y, δ) are called γ^{δ^*}-separated if $(\gamma^{\delta^*}\text{-}Cl(A) \cap B) \cup (A \cap \gamma^{\delta^*}\text{-}Cl(B)) = \phi$.

Each two γ^{δ^*}-separated sets are always disjoint, since $A \cap B \subseteq A \cap \gamma^{\delta^*}\text{-}Cl(B) = \phi$. The converse may not be true in general, as it is shown in the following example.

Example 2.43. Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\delta = \{\phi, Y, \{1\}, \{2\}, \{1, 2\}\}$. Let $f : X \to Y$ be a function such that

$$f(x) = \begin{cases} 2 & \text{if } x = a, \\ 2 & \text{if } x = b, \\ 2 & \text{if } x = c. \end{cases}$$

And for each $A \in \delta$, we define an operation γ on δ by

$$A^{\gamma} = \begin{cases} A & \text{if } 2 \in A, \\ Y & \text{if } 2 \notin A. \end{cases}$$

Then, the sets $\{1\}$ and $\{2\}$ are disjoint subsets of Y, but not γ^{δ^*}-separated.

Theorem 2.44. *If A and B are any two nonempty subsets in a space Y, then the following statements are true*:

(1) *If A and B are γ^{δ^*}-separated, $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are also γ^{δ^*}-separated.*

(2) *If $A \cap B = \phi$ such that each of A and B are both γ^{δ^*}-closed $(\gamma^{\delta^*}$-open$)$, then A and B are γ^{δ^*}-separated.*

(3) *If each of A and B is γ^{δ^*}-closed $(\gamma^{\delta^*}$-open$)$ and if $H = A \cap (Y \setminus B)$ and $G = B \cap (Y \setminus B)$, then H and G are γ^{δ^*}-separated.*

Proof. (1) Since $A_1 \subseteq A$, then $\gamma^{\delta^*}\text{-}Cl(A_1) \subseteq \gamma^{\delta^*}\text{-}Cl(A)$. Then, $B \cap \gamma^{\delta^*}\text{-}Cl(A) = \phi$ implies $B_1 \cap \gamma^{\delta^*}\text{-}Cl(A) = \phi$ and $B_1 \cap \gamma^{\delta^*}\text{-}Cl(A_1) = \phi$. Similarly $A_1 \cap \gamma^{\delta^*}\text{-}Cl(B_1) = \phi$. Hence, A_1 and B_1 are γ^{δ^*}-separated.

(2) Since $A = \gamma^{\delta^*}\text{-}Cl(A)$, $B = \gamma^{\delta^*}\text{-}Cl(B)$ and $A \cap B = \phi$, then $\gamma^{\delta^*}\text{-}Cl(A) \cap B = \phi$ and $\gamma^{\delta^*}\text{-}Cl(B) \cap A = \phi$. Hence, A and B are γ^{δ^*}-separated. If A and B are γ^{δ^*}-open, then their complements are γ^{δ^*}-closed. Hence, $\gamma^{\delta^*}\text{-}Cl(A) \subseteq Y \setminus B$ and $\gamma^{\delta^*}\text{-}Cl(B) \subseteq Y \setminus A$. Therefore, A and B are γ^{δ^*}-separated.

(3) If A and B are γ^{δ^*}-open, then $Y \setminus A$ and $Y \setminus B$ are γ^{δ^*}-closed. Since $H \subseteq Y \setminus B$, $\gamma^{\delta^*}\text{-}Cl(H) \subseteq \gamma^{\delta^*}\text{-}Cl(Y \setminus B) = Y \setminus B$ and so $\gamma^{\delta^*}\text{-}Cl(H) \cap B = \phi$. Thus $G \cap \gamma^{\delta^*}\text{-}Cl(H) = \phi$. Similarly, $H \cap \gamma^{\delta^*}\text{-}Cl(G) = \phi$. Hence H and G are γ^{δ^*}-separated. If A and B are γ^{δ^*}- closed, then $\gamma^{\delta^*}\text{-}Cl(H) \subseteq A$ and $\gamma^{\delta^*}\text{-}Cl(G) \subseteq B$. Thus, H and G are γ^{δ^*}-separated. □

Theorem 2.45. *The sets A and B in a space Y are γ^{δ^*}-separated. if and only if there exist U and V in $\delta_{\gamma^{\delta^*}}$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$ and $B \cap U = \phi$.*

Proof. Let A and B be γ^{δ^*}-separated sets. Set $V = Y \setminus \gamma^{\delta^*}\text{-}Cl(A)$ and $U = Y \setminus \gamma^{\delta^*}\text{-}Cl(B)$. Then $U, V \in \delta_{\gamma^{\delta^*}}$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. On the other hand, let $U, V \in \delta_{\gamma^{\delta^*}}$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. Since $Y \setminus V$ and $Y \setminus U$ are γ^{δ^*}-closed, then $\gamma^{\delta^*}\text{-}Cl(A) \subseteq Y \setminus V \subseteq Y \setminus B$ and $\gamma^{\delta^*}\text{-}Cl(B) \subseteq Y \setminus U \subseteq Y \setminus A$. Thus $\gamma^{\delta^*}\text{-}Cl(A) \cap B = \phi$ and $\gamma^{\delta^*}\text{-}Cl(B) \cap A = \phi$. □

Theorem 2.46. *In any topological space (Y, δ), the following statements are equivalent*:

(1) *ϕ and Y are the only γ^{δ^*}-open and γ^{δ^*}-closed sets in Y.*

(2) *Y is not the union of two disjoint nonempty γ^{δ^*}-open sets.*

(3) *Y is not the union of two disjoint nonempty γ^{δ^*}-closed sets.*

(4) *Y is not the union of two nonempty γ^{δ^*}-separated sets.*

Proof. (1) $\Rightarrow$ (2): Suppose (2) is false and that $Y = A \cup B$, where A, B are disjoint nonempty $\gamma^{\delta*}$-open sets. Since $Y \setminus A = B$ is $\gamma^{\delta*}$-open and nonempty, we have that B is a nonempty proper $\gamma^{\delta*}$-open and $\gamma^{\delta*}$-closed set in Y, which shows that (1) is false.

(2) $\Leftrightarrow$ (3): This is clear.

(3) $\Rightarrow$ (4): If (4) is false, then $Y = A \cup B$, where A, B are nonempty and $\gamma^{\delta*}$-separated. Since $\gamma^{\delta*}\text{-}Cl(B) \cap A = \phi$, we conclude that $\gamma^{\delta*}\text{-}Cl(B) \subseteq B$, so B is $\gamma^{\delta*}$-closed. Similarly, A must be $\gamma^{\delta*}$-closed. Therefore, (3) is false.

(4) $\Rightarrow$ (1): Suppose (1) is false and that A is a nonempty proper $\gamma^{\delta*}$-open and $\gamma^{\delta*}$-closed subset of Y. Then, $B = Y \setminus A$ is nonempty, $\gamma^{\delta*}$-open and $\gamma^{\delta*}$-closed, so A and B are $\gamma^{\delta*}$-separated and $Y = A \cup B$, so (4) is false. □

Definition 2.47. A subset C of a space Y is said to be $\gamma^{\delta*}$-disconnected if there are nonempty $\gamma^{\delta*}$-separated subsets A and B of Y such that $C = A \cup B$, otherwise C is called $\gamma^{\delta*}$-connected. If C is $\gamma^{\delta*}$-disconnected, such a pair of sets A, B will be called a $\gamma^{\delta*}$-disconnection of C.

Example 2.48. Consider $X = \{1, 2\}$ and $Y = \{a, b, c\}$ with $\delta = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Let $f : X \to Y$ be an identity function such that

$$f(x) = \begin{cases} c & \text{if } x = 1, \\ b & \text{if } x = 2. \end{cases}$$

And for each $A \in \delta$ we define an operation γ on δ by

$$A^{\gamma} = \begin{cases} A & \text{if } c \in A, \\ Cl(A) & \text{if } c \notin A. \end{cases}$$

Then, Y is γ^{δ^*}-disconnected because there exist a pair $\{a\}, \{b, c\}$ subsets of Y such that $\{a\} \cup \{b, c\} = Y$, and $(\gamma^{\delta^*}\text{-}Cl(\{a\}) \cap \{b, c\}) \cup (\{a\} \cap \gamma^{\delta^*}\text{-}Cl(\{b, c\})) = (\{a\} \cap \{b, c\}) \cup (\{a\} \cap \{b, c\}) = \phi$.

Remark 2.49. Every indiscrete topological space (Y, δ) is γ^{δ^*}-connected.

Remark 2.50. A space Y is γ^{δ^*}-connected if any (therefore all) of the conditions (1)-(4) in Theorem 2.46 hold.

Remark 2.51. According to the Definition 2.47 and Remark 2.50, a space Y is γ^{δ^*}-disconnected if we can write $Y = A \cup B$, where the following (equivalent) statements are true:

(1) A and B are disjoint, nonempty and γ^{δ^*}-open.

(2) A and B are disjoint, nonempty and γ^{δ^*}-closed.

(3) A and B are nonempty and γ^{δ^*}-separated.

Theorem 2.52. *A space Y is γ^{δ^*}-disconnected if and only if there exists a nonempty proper subset A of Y which is both γ^{δ^*}-open. and γ^{δ^*}-closed. in Y.*

Proof. Follows from Remark 2.51. □

Lemma 2.53. *Suppose M, N are γ^{δ^*}-separated subsets of Y. If $C \subseteq M \cup N$ and C is γ^{δ^*}-connected, then $C \subseteq M$ or $C \subseteq N$.*

Proof. Since $C \cap M \subseteq M$ and $C \cap N \subseteq N$, then $C \cap M$ and $C \cap N$ are $\gamma^{\delta*}$-separated and $C = C \cap (M \cup N) = (C \cap M) \cup (C \cap N)$. But C is $\gamma^{\delta*}$-connected so $(C \cap M)$ and $(C \cap N)$ can not form a $\gamma^{\delta*}$-disconnection of C. Therefore, either $C \cap M = \phi$, so $C \subseteq N$ or $C \cap N = \phi$, so $C \subseteq M$.

□

Theorem 2.54. *Suppose C and $C_i (i \in I)$ are $\gamma^{\delta*}$-connected subsets of Y and that for each i, C_i and C are not $\gamma^{\delta*}$-separated. Then, $S = C \cup C_i$ is $\gamma^{\delta*}$-connected.*

Proof. Suppose that $S = M \cup N$, where M and N are $\gamma^{\delta*}$-separated. By Lemma 2.53, either $C \subseteq M$ or $C \subseteq N$. Without loss of generality, assume $C \subseteq M$. By the same reasoning we conclude that for each i, either $C_i \subseteq M$ or $C_i \subseteq N$. But if some $C_i \subseteq N$, then C and C_i would be $\gamma^{\delta*}$-separated. Hence every $C_i \subseteq M$. Therefore, $N = \phi$ and the pair M, N is not a $\gamma^{\delta*}$-disconnection of S. □

Corollary 2.55. *Suppose that for each $i \in I$, C_i is a $\gamma^{\delta*}$-connected. subset of Y and that for all $i \neq j$, $C_i \cap C_j \neq \phi$. Then, $\cup \{C_i : i \in I\}$ is $\gamma^{\delta*}$-connected.*

Proof. If $I = \phi$, then $\cup \{C_i : i \in I\} = \phi$ is $\gamma^{\delta*}$-connected If $I \neq \phi$, pick $i_0 \in I$ and let C_{i_0} be the central set C in Theorem 2.54. For all $i \in I$, $C_i \cap C_{i_0} \neq \phi$, so C_i and C_{i_0} are not $\gamma^{\delta*}$-separated. By Theorem 2.54, $\cup \{C_i : i \in I\}$ is $\gamma^{\delta*}$-connected. □

Corollary 2.56. *Suppose that for all* $x, y \in Y$, *there exists* γ^{δ^*}*-connected set* $C_{xy} \subseteq Y$ *with* $x, y \in C_{xy}$. *Then,* Y *is* γ^{δ^*}*-connected.*

Proof. Certainly $Y = \phi$ is γ^{δ^*}-connected. If $Y \neq \phi$, choose $a \in Y$. By hypothesis there is, for each $y \in Y$, a γ^{δ^*}-connected set C_{ay} containing both a and y. By Corollary 2.55, $Y = \cup \{C_{ay} : y \in X\}$ is γ^{δ^*}-connected. □

Corollary 2.57. *Suppose* C *is a* γ^{δ^*}*-connected subset of* Y *and* $A \subseteq Y$. *If* $C \subseteq A \subseteq \gamma^{\delta^*}\text{-}Cl(C)$, *then* A *is* γ^{δ^*}*-connected.*

Proof. For each $a \in A$, $\{a\}$ and C are not γ^{δ^*}-separated. By Theorem 2.54, $A = C \cup \bigcup\{\{a\} : a \in A\}$ is γ^{δ^*}-connected. □

Remark 2.58. In particular, the γ^{δ^*}-closure of a γ^{δ^*}-connected set is γ^{δ^*}-connected.

Definition 2.59. A set C is called a maximal γ^{δ^*}-connected set if it is γ^{δ^*}-connected and if $C \subseteq D \subseteq Y$ where D is γ^{δ^*}-connected, then $C = D$. A maximal γ^{δ^*}-connected subset C of a space Y is called a γ^{δ^*}-component of Y. If Y is itself γ^{δ^*}-connected, then Y is the only γ^{δ^*}-component of Y.

Theorem 2.60. *For each* $x \in Y$, *there is exactly one* γ^{δ^*}*-component of* Y *containing* x.

Proof. For any $x \in Y$, let $C_Y = \bigcup\{A : x \in A \subseteq Y$ and A is γ^{δ^*}-connected$\}$. Then, $\{x\} \in C_x$, since C_x is a union of γ^{δ^*}-connected sets each containing x, C_x is γ^{δ^*}-connected by Corollary 2.55. If

$C_x \subseteq D$ and D is γ^{δ^*}-connected, then D was one of the sets A in the collection whose union defines C_x, so $D \subseteq C_x$ and therefore $C_x = D$. Therefore, C_x is γ^{δ^*}-component of Y that contains x. □

Corollary 2.61. *A space Y is the union of its γ^{δ^*}-components.*

Proof. Follows from Theorem 2.60. □

Corollary 2.62. *Two γ^{δ^*}-components are either disjoint or coincide.*

Proof. Let C_x and C_y be γ^{δ^*}-components and $C_x \neq C_y$. If $p \in C_x \cap C_y$, then by Corollary 2.55, $C_x \cup C_y$ would be a γ^{δ^*}-connected set strictly larger than C_x. Therefore, $C_x \cap C_y = \phi$. □

Theorem 2.63. Each γ^{δ^*}-connected subset of Y is contained in exactly one γ^{δ^*}-component of Y.

Proof. Let A be a γ^{δ^*}-connected subset of Y which is not in exactly one γ^{δ^*}-component of Y. Suppose that C_1 and C_2 are γ^{δ^*}-components of Y such that $A \subseteq C_1$ and $A \subseteq C_2$. Since $C_1 \cap C_2 \neq \phi$ and by Corollary 2.55, $C_1 \cup C_2$ is aother γ^{δ^*}-connected set which contains C_1 as well as C_2, a contradiction to the fact that C_1 and C_2 are γ^{δ^*}-components. This proves that A is contained in exactly one γ^{δ^*}-component of Y. □

Theorem 2.64. *A nonempty γ^{δ^*}-connected subset of Y which is both γ^{δ^*}-open and γ^{δ^*}-closed is γ^{δ^*}-component.*

Proof. Suppose that A is γ^{δ^*}-connected subset of Y which is both γ^{δ^*}-open and γ^{δ^*}-closed. By Theorem 2.63, A is contained in exactly one γ^{δ^*}-component C of Y. If A is a proper subset of C, then $C = (C \cap A) \cup (C \cap (Y \setminus A))$ and $(C \cap A), (C \cap (Y \setminus A))$ is a γ^{δ^*}-disconnection of C, which is acontradiction. Thus, $A = C$. □

Theorem 2.65. *Every γ^{δ^*}-component of Y is γ^{δ^*}-closed.*

Proof. Suppose that C is a γ^{δ^*}-component of Y. Then, by Remark 2.58, γ^{δ^*}-$Cl(C)$ is γ^{δ^*}-connected containing γ^{δ^*}-component C of Y. This implies that $C = \gamma^{\delta^*}$-$Cl(C)$ and hence C is γ^{δ^*}-closed. □

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DOI: https://doi.org/10.1090/S0002-9947-1937-1501905-7

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