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RANDOM WALKS FACING BINOMIAL CATASTROPHES: FROM FIXED TO RANDOM SURVIVAL PROBABILITY

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Abstract

In a Markov chain population model subject to catastrophes, random birth events, promoting growth, are in balance with the effect of binomial catastrophes that cause recurrent mass removal. We study two versions of such population models when the binomial catastrophic events either comes from a fixed or random survival probability. In both

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cases, most of the time, the chain is ergodic and we are left with the description of its invariant equilibrium probability mass function. For such processes, the notion of discrete self-decomposability plays a key role in quantifying the degree of disaster of the equilibrium state.

1. Introduction

In some simple growth process, some population changes its size at random as follows. In the course of its lifetime, the population alternates at random busy and idle periods. In a busy growth period, a random or fixed amount of newborns are produced which are being added to the current population size. In an idle exclusive catastrophic period, the population stands at risk being subject say to external attacks (such as a flood or a drought or a pest outbreak), resulting in the possible death of each of its constitutive members with some fixed mortality probability, independently of each other. Under such a catastrophic event, the population size thus shrinks according to a binomial distribution with survival probability, say u (hence the name binomial catastrophes). It will happen that the current size of the population is reduced to zero at some random time. In a worst disaster scenario for instance, all members can die in a single idle period leading instantaneously to a first disastrous extinction event (the case $u = 0$). From a first extinction event, the organism can then either recover taking advantage of a subsequent busy epoch and starting afresh from zero, or not, being stuck to zero for ever.

Stochastic models subject to binomial catastrophes have a wide application in different fields viz. bioscience, economy, ecology, computer and natural sciences, etc. For instance,

– In forestry, a good management of the biomass depends on the prediction of how steady growth periods alternate with tougher periods, due to the occurrence of cataclysms such as hurricanes or droughts or floods, hitting each tree in a similar and independent way.

– In economy, pomp periods often alternate with periods of scarcity when a crisis equally strikes all the economic agents.

As just described in words, the process under concern turns out to be a Markov chain on the non-negative integers, displaying a subtle balance between generalized birth and death events. Whenever it is ergodic, there are infinitely many local extinctions and the pieces of sample paths separating consecutive passages to zero are called excursions, the height and length of which are of some relevance in the understanding of the population size evolution. Excursions indeed form independent and identically distributed (iid) blocks of such Markov chains.

Discrete-time random population dynamics with catastrophes balanced by random growth has a long history in the literature, starting with [12]. Mathematically, binomial catastrophe models are Markov chains (MCs) which are random walks on the non-negative integers, so differing from standard random walks on the integers in that a one-step move down from some positive integer cannot take one to a negative state, resulting in transition probabilities being state-dependent. Such MCs may thus be viewed as generalized birth and death chains on a countable state-space [13, 21]. The transient and equilibrium behaviour of such stochastic population processes with either disastrous $(u = 0)$ or mild $(u \in (0, 1))$ binomial catastrophes is one of the purposes of this work. We aim at studying the equilibrium distribution of this process and deriving procedures for its approximate computation. Another issue of importance concerns the measures of the risk of extinction, first extinction time, time elapsed between two consecutive extinction times and maximum population size reached in between.

The detailed structure of the manuscript, attempting to realize this program, can be summarized as follows:

• Section 2 is designed to introduce the model in probabilistic terms. We develop three particular important cases:

– Survival probability $u = 1$. In this case, on a catastrophic event, the chain remains in its current state with no depletion of individuals at all. The population size slowly drifts to infinity, a case of transience.

– Survival probability $u = 0$. This is a case of total disasters for which, on a catastrophic event, the population size is instantaneously propelled to state 0, no one surviving the drought.

– The scenario when the adjunction of newborns, on a growth event, is deterministic being reduced to a single element, and $u \neq \{0, 1\}$ so that binomial mortality is not degenerate.

• In Section 3, we discuss the conditions under which the chain with $u \in [0, 1)$ is recurrent (positive or null) or transient. We emphasize that positive recurrence is generic, unless the newborns random variables have unrealistic very heavy tails, with infinite logarithmic moments. This is due to the fact that for a binomial catastrophe model, a catastrophic event produces a huge death toll. When, as in the positive recurrent case, it is non-trivial, we mainly discuss the shape of the invariant probability mass function of the chain. Specifically,

– When $u = 0$ (total disasters), the invariant probability mass function is shown to be the one of a shifted geometric sum of the newborns, so in the compound Poisson class, but not necessarily self-decomposable. We give some sufficient conditions under which it is self-decomposable, so unimodal and then we show that it has mode necessarily at the origin. When the mode is at 0, we speak of a completely disastrous situation, the most probable equilibrium state being zero (corresponding to eventual extinction).

We refer to [16] for the notions of discrete infinite divisibility (compound Poisson), self-decomposability and unimodality which play a key role in this manuscript.

– When $u \in (0, 1)$ (mild disasters), we show that the invariant probability mass function is also the one of a compound Poisson (infinitely divisible) random variable, which is moreover discrete u^k-self-decomposable for all $k \geq 1$. Self-decomposability being α-self-decomposable for all $\alpha \in (0, 1)$ we give conditions under which this strict self-decomposability holds. When strict self-decomposability holds, the invariant probability mass function is unimodal and we give completely disastrous conditions under which the mode is located at 0.

In the positive recurrent case and when $u \in [0, 1)$, we also give new expressions of the distributions of the first return time to zero and the time to a first extinction starting from some positive initial condition.

• In Section 4, we introduce a family of binomial catastrophe models, now with random survival probability U. Think of an extreme drought striking a forest: each tree, independently of its neighbours, will have to face a chance U of survival, and considering U random can be a natural issue to take into account the variability of the trees' reaction in their struggle for life against the drought. When dealing with such a growth model with binomial catastrophes in random environment, the strength of the depletion of individuals in a catastrophic event is also very strong (massive) but more diffuse. To the best of the author's knowledge, the randomness feature of U has not been fully considered in the literature and we go one-step in this direction while considering that U has a $\text{beta}(a, 1)$ distribution, $a > 0$. We show that the condition under which binomial catastrophe models with random U are ergodic is the same as for the model with fixed u (finiteness of the logarithmic moments of the newborns random variables). When ergodic, we give the exact expression of the probability generating function of its invariant measure. It has two factors, one of which is the one of a self-decomposable random variable, the other one being the one obtained for the total disaster equilibrium case with fixed $u = 0$. When strict self-decomposability of the latter factor holds, the invariant probability mass function of the binomial catastrophe models with random U is itself self-decomposable, so unimodal and we give completely disastrous conditions under which its mode is located at 0.

We finally show that, when ergodic, the invariant measure of the binomial catastrophe models with random U can be achieved by a pure-death branching processes with immigration in continuous-time.

2. The Binomial Catastrophe Model with Fixed Survival Probability

We first describe the model and some of its special extreme cases.

2.1. The model

Consider a discrete-time Markov chain (MC) X_n taking values in $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. With $b_n(u)$, $n = 1, 2, \ldots$ an independent identically distributed (iid) sequence of Bernoulli random variables (rv's) with success parameter $u \in (0, 1)$, let

$$u \circ X_n = \sum_{m=1}^{X_n} b_m(u) \tag{1}$$

denote the Bernoulli thinning of X_n [1]. Let β_n, $n = 1, 2, \ldots$ be an iid birth sequence of rv's with values in $\mathbb{N} = \{1, 2, \ldots\}$. The dynamics of the MC under concern here is a balance between birth and death events according as $(p + q = 1)$:

$$X_{n+1} = X_n + \beta_{n+1} \text{ with probability } p$$

$$X_{n+1} = u \circ X_n \text{ with probability } q = 1 - p.$$

[1]In words, the thinning operation acting on a discrete random variable is the natural discrete analog of scaling a continuous variable, i.e., multiplying it by a constant in $[0, 1]$. See [16].

This model was considered by [1, 2, 4, 6, 8, 12], and we aim at providing an additional insight. We have $u \circ X_n \overset{d}{\sim} \text{bin}(X_n, u)$ hence the name binomial catastrophe. The binomial effect is appropriate when, on a catastrophic event, the individuals of the current population each die (with a fixed probability $1 - u$) or survive (with probability u) in an independent and even way, resulting in a drastic depletion of individuals at each step. Owing to: $u \circ X_n = X_n - (1 - u) \circ X_n$, the number of stepwise removed individuals is $(1 - u) \circ X_n$ with probability (wp) q. This way of depleting the population size (at shrinkage times) by a fixed random fraction u of its current size is very drastic, especially if X_n happens to be large. Unless u is very close to 1 in which case depletion is modest (the case $u = 1$ is discussed below), it is very unlikely that the size of the upward moves will be large enough to compensate depletion while producing a transient chain drifting at ∞.

With $b_x := \mathbf{P}(\beta = x)$, $x \geq 1$ and $d_{x,y} := \binom{x}{y} u^y (1 - u)^{x-y}$ the binomial probability mass function (pmf), the one-step-transition matrix P of the MC X_n is given by:

$$P(0, 0) = q,\ P(0, y) = pb_y = pb_y,\ y \geq 1$$

$$P(x, y) = q\binom{x}{y} u^y (1 - u)^{x-y} = qd_{x,y},\ x \geq 1 \text{ and } 0 \leq y \leq x$$

$$P(x, y) = pb_{y-x},\ x \geq 1 \text{ and } y > x. \tag{2}$$

Remarks (Special extreme cases).

(i) When $u = 1$, the lower triangular part of P vanishes leading to

$$P(0, 0) = q,\ P(0, y) = pb_y,\ y \geq 1$$

$$P(x, y) = 0,\ x \geq 1 \text{ and } 0 \leq y < x,\ P(x, x) = q,\ x \geq 1$$

$$P(x, y) = pb_{y-x},\ x \geq 1 \text{ and } y > x.$$

The transition matrix P is upper-triangular with diagonal terms. The process X_n is non-decreasing, so it drifts to ∞ with probability 1 (a case of transience).

(ii) If $\beta \overset{d}{\sim} \delta_1$ a move up results in the addition of only one individual, which is the simplest deterministic drift upwards. In this case, the transition matrix P is lower-Hessenberg.

3. Recurrence Against Transience

Using a generating function approach, we start with the transient analysis before switching to the question of equilibrium.

3.1. The transient analysis

Assume $X_0 = x_0 = 0$. Let $\phi_\beta(z) = \mathbf{E}(z^\beta)$ be the pgf of $\beta = \beta_1$, as an absolutely monotone function on $[0, 1]$[2]. Let $\boldsymbol{\pi}_n' = (\pi_n(0), \pi_n(1), \ldots)$ where $\pi_n(x) = \mathbf{P}_0(X_n = x)$ and $'$ denotes the transposition. With $\mathbf{z} = (1, z, z^2, \ldots)'$, a column vector obtained as the transpose $'$ of the row vector $(1, z, z^2, \ldots)$, define

$$\Phi_n(z) = \mathbf{E}_0(z^{X_n}) = \boldsymbol{\pi}_n'\mathbf{z},$$

the pgf of X_n. The time evolution $\pi_{n+1}' = \pi_n' P$ yields

$$\Phi_{n+1}(z) = \boldsymbol{\pi}_{n+1}'\mathbf{z} = \boldsymbol{\pi}_n' P\mathbf{z},$$

[2]A function B is said to be absolutely monotone on $(0, 1)$ if it has all its derivatives $B^{(n)}(z) \geq 0$ for all $z \in (0, 1)$. Pgfs are absolutely monotone and the composition of two pgfs is a pgf. With x integer, we denote by $[z^x]B(z)$ the z^x coefficient in the series expansion of $B(z)$.

leading to the transient dynamics

$$\Phi_{n+1}(z) = p\phi_\beta(z)\Phi_n(z) + q\Phi_n(1-u(1-z)),\ \Phi_0(z) = 1. \tag{3}$$

The fixed point pgf of X_∞, if it exists, solves

$$\Phi_\infty(z) = p\phi_\beta(z)\Phi_\infty(z) + q\Phi_\infty(1-u(1-z)). \tag{4}$$

When $u = 1$ (survival of all), there is no move down possible. The only solution to $\Phi_\infty(z) = p\phi_\beta(z)\Phi_\infty(z) + q\Phi_\infty(z)$ is $\Phi_\infty(z) = 0$, corresponding to $X_\infty \overset{d}{\sim} \delta_\infty$. Indeed, when $u = 1$, combined to $\Phi_\infty(1) = 1$,

$$\Phi_{n+1}(z) = (q + p\phi_\beta(z))\Phi_n(z),\ \Phi_0(z) = 1$$

$$\Phi_n(z)^{1/n} = q + p\phi_\beta(z),$$

showing that, if $1 \le \rho := \phi'_\beta(1) = \mathbf{E}(\beta) < \infty$, $n^{-1}X_n \to q + p\rho \ge 1$ almost surely as $n \to \infty$. The process X_n is transient in that, after a finite number of passages in state 0, it drifts to ∞.

3.2. Existence and shape of an invariant pmf ($u \in [0, 1)$)

We shall distinguish two cases, starting with the extreme total disaster one.

- The case $u = 0$ (total disasters)

When $u = 0$ (total disasters), the transition matrix (2) reads

$$P(0, 0) = q,\ P(0, y) = pb_y,\ y \ge 1$$

$$P(x, y) = 0,\ x \ge 1 \text{ and } 0 < y \le x,\ P(x, 0) = q,\ x \ge 1$$

$$P(x, y) = pb_{y-x},\ x \ge 1 \text{ and } y > x.$$

Some related Markov catastrophe models involving total disasters are described in [19, 7].

For such Markov chains, the time $\tau \geq 1$ elapsed between consecutive catastrophic events is geometric with $\mathbf{P}(\tau = x) = qp^{x-1}$, $x \geq 1$ and the net growth B of the process during this laps of time is

$$B = \sum_{x=1}^{\tau-1} \beta_x, \tag{5}$$

where τ and $(\beta_x;\ x \geq 1)$ are independent. The pgf of B is thus

$$\phi_B(z) = \frac{q}{1 - p\phi_\beta(z)},$$

the one obtained while compounding a shifted-geometric pgf $q/(1 - pz)$ with the pgf $\phi_\beta(z)$ of the β's [3].

When a downward move occurs, it takes instantaneously X_n to zero (a case of total disasters), independently of the value of X_n. This means that, defining $\tau_{x_0,0} = \inf(n \geq 1 : X_n = 0 \mid X_0 = x_0)$, the first extinction time of X_n, $\mathbf{P}(\tau_{x_0,0} = x) = qp^{x-1}$, $x \geq 1$, a geometric distribution with success parameter q, with mean $\mathbf{E}(\tau_{x_0,0}) = 1/q$, independently of $x_0 \geq 0$. Note that $\tau_{0,0} \stackrel{d}{=} \tau$, as the length of any excursion between consecutive visits to 0, also has a geometric distribution with success parameter q and finite mean $1/q$. In addition, the height H of an excursion is clearly distributed like $B = \sum_{x=1}^{\tau_{0,0}-1} \beta_x$ (with the convention $\sum_{x=1}^{0} \beta_x = 0$). Consecutive excursions are the iid pieces of this random walk on the non-negative integers.

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[3]A geometric(q) rv with success probability q takes values in $N = \{1, 2, \ldots\}$. A shifted geometric(q) rv with success probability q takes values in $N_0 = \{0, 1, 2, \ldots\}$. It is obtained while shifting the former one by one unit.

Combined to $\Phi_\infty(1) = 1$, (4) yields

$$\Phi_\infty(z) = \phi_B(z), \text{ equivalently } X_\infty \overset{d}{=} B, \tag{6}$$

as an admissible pgf solution, whatever the distribution of β. We just obtained:

Proposition (total disaster)**.** *Combined to* $\Phi_\infty(1) = 1$, (6) *is an admissible pgf solution. When* $u = 0$, X_n *is ergodic* (*positive recurrent*) *and the law of* X_∞ *is a compound shifted-geometric of the* β' *s*, *whatever the distribution of* β.

Statistical properties of the rv $X_\infty \overset{d}{=} B$ of the total disaster model

(1) The probabilities $\pi(x) = [z^x]\Phi_\infty(z) = \mathbf{P}(B = x)$, $x \geq 1$ are in principle explicitly given by the Faa di Bruno formula for compositions of two pgf's, (see [3], p. 146). It involves the ordinary Bell polynomials

$$B_{x,y}(b_1, b_2, \ldots, b_{x-y+1}) = \frac{y!}{x!} \sum^* \prod_{z \geq 1} \frac{b_z^{c_z}}{c_z!},$$

where the star-sum runs over the non-negative integers c_z obeying $\sum_{z\geq 1} zc_z = x$ and $\sum_{z\geq 1} c_z = y$ (upon partitioning the integer x into y summands). In more detail $\pi(0) = q$ and

$$\pi(x) = q\sum_{y=1}^{x} p^y B_{x,y}(b_1, b_2, \ldots, b_{x-y+1}),\ x \geq 1.$$

We also observe that

$$\frac{1 - \phi_B(z)}{1 - z} = \frac{p(1 - \phi_\beta(z))}{(1 - z)(1 - p\phi_\beta(z))} = \frac{p}{q}\frac{1 - \phi_\beta(z)}{1 - z}\phi_B(z), \tag{7}$$

so that (with $\mathbf{P}(B = 0) = q$) the following recurrences on $\pi(x) = [z^x]\Phi_\infty(z) = \mathbf{P}(B = x),\ x \geq 1,$ hold

$$\mathbf{P}(B > x) = \frac{p}{q}\sum_{y=0}^{x} \mathbf{P}(\beta > x - y)\mathbf{P}(B = y),$$

$$\mathbf{P}(B = x) = p\sum_{y=0}^{x-1} \mathbf{P}(\beta = x - y)\mathbf{P}(B = y),\ x \geq 1.$$

The latter convolution-like relation can be useful to generate recursively $\pi(x) = [z^x]\Phi_\infty(z) = \mathbf{P}(B = x),\ x \geq 1$ on a laptop, thereby by-passing its complex combinatorial representation in terms of Bell polynomials.

(2) With $p \in (0, 1)$ and $b(p)$ a Bernoulli rv with success parameter $p,$ define the thinning operator $p * X = \sum_{n=1}^{b(p)} X_n$ (where $X_n \overset{d}{=} X$ and with the convention $\sum_{n=1}^{0} X_n = 0$). Compare with (1). We have $\mathbf{E}(z^{p*X}) = q + p\mathbf{E}(z^X)$. Supposing we search for a solution X to the equation in distribution (assuming $X' \overset{d}{=} X$ and $p * X'$ and β independent)

$$X \overset{d}{=} p * X' + \beta,$$

we get

$$\mathbf{E}(z^X) = \Phi(z) = [q + p\Phi(z)]\phi_\beta(z),$$

$$\Phi(z) = \frac{q\phi_\beta(z)}{1 - p\phi_\beta(z)},$$

$\Phi(z)$ is the pgf of a compound geometric rv (see Theorem 4.2 of [16]).

Proposition (total disaster)**.** *The limiting rv* X_∞, *as a compound shifted geometric rv, is the solution of the fixed point equation*

$$X_\infty \overset{d}{=} p * (X'_\infty + \beta). \tag{8}$$

Proof. Searching now for a solution X_∞ to the equation in distribution (8) we get

$$\Phi_\infty(z) = q + p\Phi_\infty(z)\phi_\beta(z)$$

$$\Phi_\infty(z) = \frac{q}{1 - p\phi_\beta(z)}.$$

(3) The rv $X_\infty \overset{d}{=} B$ is at least infinitely divisible (ID), else compound Poisson, because $\phi_B(z) = \exp - r(1 - \psi(z))$ where $r > 0$ and $\psi(z)$ is a pgf with $\psi(0) = 0$. Indeed, with $q = e^{-r}$,

$$\psi(z) = \frac{-\log(1 - p\phi_\beta(z))}{-\log q} \tag{9}$$

is a pgf (the one of a Fisher-log-series rv, [5]).

(4) We now address the self-decomposability of $X_\infty \overset{d}{=} B$ (with $\phi_B(z) = \frac{q}{1 - p\phi_\beta(z)}$) question: would X_∞ be self-decomposable, (assuming $X'_\infty \overset{d}{=} X_\infty$ and $u \circ X'_\infty$ and Y_u independent), it should solve

$$X_\infty \overset{d}{=} u \circ X'_\infty + Y_u,$$

for all $u \in (0, 1)$. See [16] and [18] for an account on discrete SD rv's, as a remarkable subclass of compound Poisson ones.

Theorem. *With* $b_x = [z^x]\phi_\beta(z)$, $x \geq 1$, $B \overset{d}{=} X_\infty$ *is ID and furthermore self-decomposable* (*SD*) *if*

$$\frac{b_{x+1}}{b_x} \leq \frac{x - pb_1}{x+1} \text{ for any } x \geq 1. \tag{10}$$

Proof. If B is SD then (see [14], Lemma 2.13)

$$\phi_B(z) = e^{-r\int_z^1 \frac{1-h(z')}{1-z'}dz'},$$

for some $r > 0$ and some pgf $h(z)$ obeying $h(0) = 0$. We are led to check if

$$\psi'(z) = \frac{p\phi'_\beta(z)}{1 - p\phi_\beta(z)} = r\frac{1-h(z)}{1-z},$$

for some pgf h and $r = pb_1$, where

$$h(z) = 1 - \frac{1}{b_1}(1-z)\frac{\phi'_\beta(z)}{1 - p\phi_\beta(z)} = \frac{1}{b_1}\frac{b_1(1 - p\phi_\beta(z)) - (1-z)\phi'_\beta(z)}{1 - p\phi_\beta(z)}$$

$$= \phi_B(z)\phi(z), \text{ where } \phi(z) := \frac{1}{qb_1}[b_1(1 - p\phi_\beta(z)) - (1-z)\phi'_\beta(z)].$$

A sufficient condition for h to be a pgf is thus that $\phi(z)$ be a pgf itself. In particular, if $\phi(1) = 1$ (which holds) and if

$$[z^x]\phi(z) \geq 0 \text{ for all } x \geq 1.$$

The proof ends while observing

$$\phi(z) = \frac{1}{qb_1}\sum_{x\geq 1} z^x[(x - pb_1)b_x - (x+1)b_{x+1}]. \qquad \square$$

Let us show on four examples that these conditions can be met.

Examples. (1) Suppose $\phi_\beta(z) = z$. Then $0 = \frac{b_{x+1}}{b_x} \leq \frac{x-p}{x+1}$ for all $x \geq 1$. The simple shifted-geometric rv B is SD.

(2) Suppose $\phi_\beta(z) = b_1 z + b_2 z^2$ with $b_2 = 1 - b_1$. We need to check conditions under which $\frac{b_2}{b_1} \leq \frac{1 - pb_1}{2}$. This condition is met if and only if the polynomial $pb_1^2 - 3b_1 + 2 \leq 0$ which holds if and only if $b_1 \geq b_1^*$, where $b_1^* \in (0, 1)$ is the zero of this polynomial in $(0, 1)$.

(3) Suppose $\phi_\beta(z) = \overline{\alpha} z/(1 - \alpha z)$, $\alpha \in (0, 1)$, the pgf of a geometric$(\overline{\alpha})$ rv, with $b_x = \overline{\alpha}\alpha^{x-1}$. The condition reads: $\alpha \leq \frac{x - p\overline{\alpha}}{x+1}$. It is fulfilled if $\alpha \leq \frac{x-p}{x+q}$ for all $x \geq 1$ which is $\alpha \leq q/(1+q) < 1$ (or $\overline{\alpha} = b_1 \geq b_1^* = 1/(1+q)$).

(4) Suppose $\phi_\beta(z) = z \exp - \theta(1-z)$, $\theta > 0$, the pgf of a shifted Poisson(θ) rv, with $b_x = \theta^{x-1} e^{-\theta}/(x-1)!$, $x \geq 1$. The SD condition holds for all $x \geq 1$ if and only if: $\theta \leq (1 - pb_1)/2 < 1$. Under this condition, $\frac{b_{x+1}}{b_x} = \frac{\theta}{x} < 1$ (b_x is a decreasing sequence), and β has its mode at $x = 1$.

(5) Sibuya, [15]. Suppose $\phi_\beta(z) = 1 - (1-z)^\alpha$, $\alpha \in (0, 1)$, with $b_x = \alpha [\overline{\alpha}]_{x-1}/x!$, $x \geq 1$ (where $[\overline{\alpha}]_x = \overline{\alpha}(\overline{\alpha}+1)\ldots(\overline{\alpha}+x-1)$, $x \geq 1$ are the rising factorials of $\overline{\alpha}$ and $[\overline{\alpha}]_0 := 1$). The condition reads: $\frac{b_{x+1}}{b_x} = \frac{x-\alpha}{x+1} \leq \frac{x - p\alpha}{x+1}$ which is always fulfilled. The shifted-geometric rv with Sibuya (with heavy tail index α) distributed compounding rv is always SD. Note $\frac{b_{x+1}}{b_x} < 1$ (b_x is a decreasing sequence), so that β has its mode at $x = 1$.

When the rv $X_\infty \overset{d}{=} B$ is SD, it is unimodal, with mode at the origin if $\mathbf{P}(B = 1) < \mathbf{P}(B = 0)$, or with two modes at $\{0, 1\}$ if $\frac{\mathbf{P}(B = 1)}{\mathbf{P}(B = 0)} = 1$ (see [18], Theorem 4.20).

Proposition (total disaster)**.** *Under the condition that* $X_\infty \overset{d}{=} B$ *is SD and so unimodal,* X_∞ *has always mode at the origin.*

Proof. The condition to check here is that, with $\mathbf{P}(B = 0) = \phi_B(0) = q$ and $\mathbf{P}(B = 1) = \phi'_B(0) = pq\phi'_\beta(0) = pqb_1$, $\mathbf{P}(B = 1)/\mathbf{P}(B = 0) = pb_1 < 1$, which is always satisfied.

Definition. A random walk with binomial catastrophe X_n is said to be **completely disastrous** if the limit law of X_∞ is SD, unimodal, with mode at the origin.

The above random walk with total disaster is completely disastrous when B is SD: The most probable equilibrium state is then 0 (corresponding to eventual extinction).

• **The case** $\boldsymbol{u \in (0, 1)}$**.** From (4) and (6), the limit law pgf $\Phi_\infty(z)$, if it exists, solves the functional equation

$$\Phi_\infty(z) = \phi_B(z)\Phi_\infty(1 - u(1 - z)), \tag{11}$$

so that, formally

$$\Phi_\infty(z) = \prod_{n \geq 0} \phi_B(1 - u^n(1 - z)), \tag{12}$$

as an infinite product pgf.

Proposition. *The invariant measure exists for all* $u \in (0, 1)$ *if and only if* $\mathbf{E}(\log_+ B) < \infty$ *or equivalently* $\mathbf{E}(\log \beta) < \infty$.

Proof (Theorem 2 in [12]). By a comparison argument, we need to check the conditions under which $\pi(0) = \Phi_\infty(0)$ converges to a positive number. We get

$$\Phi_\infty(0) = \prod_{n\geq 0} \phi_B(1-u^n) > 0 \Leftrightarrow \sum_{n\geq 0}(1-\phi_B(1-u^n)) < \infty$$

$$\Leftrightarrow \int_0^1 \frac{1-\phi_B(z)}{1-z}\,dz < \infty \Leftrightarrow \sum_{x\geq 1} \log x\mathbf{P}(B = x) = \mathbf{E}(\log_+ B) < \infty,$$

meaning that B has a finite logarithmic first moment. It follows from (7) that $\int_0^1 \frac{1-\phi_B(z)}{1-z}\,dz < \infty \Leftrightarrow \int_0^1 \frac{1-\phi_\beta(z)}{1-z}\,dz < \infty.$

This condition is extremely weak and, for most β's therefore (but the ones with infinite logarithmic moments), the process X_n is positive recurrent, in particular if β has finite mean.

When β has finite first and second order moments, so do B and X_∞ which exist. Indeed:

If $\phi'_\beta(1) = \mathbf{E}\beta = \rho < \infty$, (with $\mathbf{E}B = \phi'_B(1) = (p\rho)/q$)

$$\Phi'_\infty(1) = q\left(\frac{p\rho}{q^2} + \frac{u}{q}\Phi'_\infty(1)\right) \Rightarrow \Phi'_\infty(1) = \mathbf{E}(X_\infty) =: \mu = \frac{\mathbf{E}(B)}{1-u} < \infty.$$

If $\phi''_\beta(1) < \infty$ or equivalently if $\phi''_B(1) < \infty$, X_∞ has finite variance found to be:

$$\sigma^2(X_\infty) = \frac{1}{1-u^2}\left(\phi''_B(1) - \phi'_B(1)^2 + (1+u)\phi'_B(1)\right)$$

$$= \frac{1}{1-u^2}\left(\sigma^2(B) + u\mathbf{E}(B)\right).$$

Example. With $\theta, C > 0$, suppose that $\mathbf{P}(\beta > x) \sim_{x \uparrow \infty} C(\log x)^{-\theta}$ translating that β has very heavy logarithmic tails. Then $\mathbf{E}\beta^q = \infty$ for all $q > 0$ and β has no moments of arbitrary positive order. For such a (logarithmic tail) model of β, one can check that X_n remains positive recurrent if $\theta > 1$ and starts being transient only if $\theta < 1$. The case $\theta = 1$ is a critical null-recurrent situation. Being strongly attracted to 0, the binomial catastrophe model exhibits a recurrence/transience transition but only for such very heavy-tailed choices of β.

Recall that:

When positive recurrent, the chain visits state 0 infinitely often and the expected return time $\tau_{0,0}$ to 0 has finite mean given, by Kac's theorem, [11], as $\mathbf{E}\tau_{0,0} = 1/\pi(0)$.

When null recurrent, the chain visits state 0 infinitely often but the expected return time to 0 has infinite mean.

When transient, the chain visits state 0 a finite number of times before drifting to ∞ for ever after an infinite number of steps (no finite time explosion is possible for discrete-time Markov chains).

Corollary. *If the process X_n is null recurrent or transient, no non-trivial $(\neq \mathbf{0}')$ invariant measure exists.*

Proof. This is because, $\Phi_\infty(z)$ being an absolutely monotone function on $[0, 1]$ if it exists,

$$\pi(0) = \Phi_\infty(0) = 0 \Rightarrow \Phi_\infty(1-u) = 0 \Rightarrow \pi(x) = 0 \text{ for all } x \geq 1.$$

Sampling X_n at times when thinning occurs and time change:

Let $\tau = \inf(n \geq 1 : B_n(p) = 0)$, with $\mathbf{P}(\tau = k) = p^{k-1}q,\ k \geq 1$, $\mathbf{E}(z^{\tau-1}) = \frac{q}{1 - pz}$. The rv τ is the time elapsed between two consecutive catastrophic events. So long as there is no thinning of X_n (a catastrophic event), the process grows of $B = \sum_{x=1}^{\tau-1} \beta_x$ individuals. Consider a time-changed process $\overline{X}_k$ of X_n whereby one time unit is the time elapsed between consecutive catastrophic events. During this laps of time, the original process X_n grew of B individuals, before shrinking to a random amount of its current size at subsequent catastrophe times. We are thus led to consider the time-changed integral-valued Ornstein-Uhlenbeck process with fixed initial condition $\overline{X}_0 = x_0 \geq 0$:

$$\overline{X}_{k+1} = u \circ \overline{X}_k + B_{k+1},\ k \geq 0, \tag{13}$$

with $B_k,\ k = 1, 2, \ldots$ an iid sequence of compound shifted geometric rv's.

In this form, $\overline{X}_k$ is a pure-death subcritical branching process with immigration, B_{k+1} being the number of immigrants at generation $k+1$, independent of $\overline{X}_k$. With $\overline{\Phi}_k(z) = \mathbf{E}(z^{\overline{X}_k})$, we have

$$\overline{\Phi}_{k+1}(z) = \phi_B(z)\overline{\Phi}_k(1 - u(1 - z)),\ \overline{\Phi}_0(z) = z^{x_0}, \tag{14}$$

corresponding to

$$\overline{X}_k = \sum_{l=0}^{k-1} u^l \circ B_{k-l} + u^k \circ x_0$$

$$\overset{d}{=} \sum_{l=0}^{k-1} u^l \circ B_{l+1} + u^k \circ x_0.$$

The limit pgf $\overline{\Phi}_\infty(z)$ (if it exists) also solves (11), so $\overline{\Phi}_\infty(z) = \Phi_\infty(z)$. Thus (12) holds, corresponding, if $u^k \circ X_0 \to 0$ as $k \to \infty$, to

$$X_\infty \stackrel{d}{=} \overline{X}_\infty \stackrel{d}{=} \sum_{l \geq 0} u^l \circ B_{l+1},$$

an iterated version of $\overline{X}_\infty \stackrel{d}{=} u \circ \overline{X}'_\infty + B$, where $\overline{X}'_\infty \stackrel{d}{=} \overline{X}_\infty$ and $u \circ \overline{X}'_\infty$ independent of B.

The time-changed process $\overline{X}_k$ has the same limit law as the original binomial catastrophe model X_n, which exists if and only if $\mathbf{E} \log_+ B < \infty$ holds. Note $\overline{X}_k$ is the sum of the independent rv's $u^l \circ B_{l+1}$, each obtained while scaling the iid B_{l+1}, $l \geq 0$.

Proposition. *When the law of* X_∞ *exists* ($\mathbf{E} \log \beta < \infty$), *it is infinitely divisible* (*compound Poisson*).

Proof. The rv B being compound-Poisson from (9), as a result of (12), X_∞ is obtained as a weak limit of the sum of a sequence of independent compound-Poisson random variables, each with pgf, $n \geq 0$:

$$\begin{aligned}\phi_B(1 - u^n(1 - z)) &= \exp - r(1 - \psi(1 - u^n(1 - z))) \\ &= \exp - r_n(1 - \widetilde{\psi}_n(z)),\end{aligned}$$

where $r_n = r(1 - \psi(1 - u^n))$ and $\widetilde{\psi}_n(z) = (\psi(1 - u^n(1 - z)) - \psi(1 - u^n))/(1 - \psi(1 - u^n))$ obeying $\widetilde{\psi}_n(0) = 0$.

So it is compound-Poisson. See Proposition 4.1 p. 26 in [18].

Proposition. *When the law of* X_∞ ($\mathbf{E} \log \beta < \infty$), *it is discrete u-self-decomposable* (*SD*) *and then* u^k*-SD for all* $k \geq 1$.

Proof. This follows from [17].

Full SD distributions are those which are u-SD for all $u \in (0, 1)$.

Corollary. *When the law of* X_∞ *exists* $(\mathbf{E} \log \beta < \infty)$, if B *is SD, so is* X_∞.

Proof. If B is SD, from (12), X_∞ is obtained as a weak limit of the sum of a sequence of independent SD random variables, each with pgf, $n \geq 0$:

$$\phi_B(1 - u^n(1 - z)) = \exp - r\int_z^1 \frac{1 - h(1 - u^n(1 - z'))}{1 - z'} dz'$$

$$= \exp - r_n \int_z^1 \frac{1 - \widetilde{h}_n(z')}{1 - z'} dz',$$

where $r_n = r(1 - h(1 - u^n))$ and $\widetilde{h}_n(z) = (h(1 - u^n(1 - z)) - h(1 - u^n))/(1 - h(1 - u^n))$ obeying $\widetilde{h}_n(0) = 0$.

So it is SD; see Proposition 4.4, p. 264 in [18]. We know the conditions on the law of β under which B is SD.

When the rv X_∞ is SD, it is unimodal, with mode at the origin if $\pi(1) < \pi(0)$, or with two modes at $\{0, 1\}$ if $\frac{\pi(1)}{\pi(0)} = 1$ (see [18], Theorem 4.20, p. 274).

With $\mathbf{P}(B = 1) = \phi_B'(0) = pq\mathbf{P}(\beta = 1)$, we have

$$\pi(0) = \Phi_\infty(0) = q\Phi_\infty(1 - u),$$

$$\pi(1) = \Phi_\infty'(0) = \mathbf{P}(B = 1)\Phi_\infty(1 - u) + qu\Phi_\infty'(1 - u)$$

$$= \frac{\mathbf{P}(B = 1)}{q}\pi(0) + qu\Phi_\infty'(1 - u) > p\mathbf{P}(\beta = 1)\pi(0).$$

A condition for unimodality at 0 is thus

$$(\log \Phi_\infty)' (1-u) < \frac{1 - p\mathbf{P}(\beta = 1)}{u}. \tag{15}$$

Note also

$$\begin{aligned}\pi(1) = \Phi_\infty'(0) &= \sum_{m\geq 0} u^m \phi_B'(1-u^m) \prod_{n\neq m} \phi_B(1-u^n) \\ &= \pi(0) \sum_{m\geq 0} u^m (\log \phi_B)' (1-u^m) \\ &= \pi(0) \sum_{m\geq 0} u^m \frac{p\phi_\beta'(1-u^m)}{1-p\phi_\beta(1-u^m)}\end{aligned}$$

with $\pi(1)/\pi(0) < 1$ giving a closed-form condition for unimodality at 0. For instance, if $\phi_\beta(z) = z$, $\pi(1) < \pi(0)$ if and only if

$$\sum_{m\geq 0} \frac{pu^m}{q + pu^m} < 1.$$

We have proved:

Proposition. *Under the condition that* B *is SD*, X_∞ *is SD and so unimodal*; X_n *is completely disastrous with a mode at the origin if and only*

$$\sum_{m\geq 0} u^m \frac{p\phi_\beta'(1-u^m)}{1-p\phi_\beta(1-u^m)} < 1.$$

First return time to 0 of both $\overline{X}_k$ and X_n. We end up this section by supplying a new expression of the distribution of the first return time to 0 of X_n when $u \in (0, 1)$.

Let $\overline{\tau}_{0,0} = \inf(k \geq 1 : \overline{X}_k = 0 | \overline{X}_0 = 0)$. We have $\mathbf{P}(\overline{\tau}_{0,0} = 1) = q$, and for each $l \geq 2$,

$$\mathbf{P}(\overline{\tau}_{0,0} = l) = \mathbf{P}_{\overline{X}_0 = 0}(\overline{X}_1 > 0, \dots, \overline{X}_{l-1} > 0, \overline{X}_l = 0)$$

$$= \mathbf{P}_{\overline{X}_0 = 0}(\overline{X}_1 > 0) \prod_{k=2}^{l-1} \mathbf{P}(\overline{X}_k > 0 | \overline{X}_{k-1} > 0) \mathbf{P}(\overline{X}_l = 0 | \overline{X}_{l-1} > 0), \tag{16}$$

which is computable as follows.

First we have $\mathbf{P}_{\overline{X}_0 = 0}(\overline{X}_1 > 0) = p$. Second, with $\overline{\Phi}_k(z)$ given from (13), obeying $\overline{\Phi}_k(0) = q\overline{\Phi}_{k-1}(1-u)$ and due to and $\mathbf{P}(u \circ \overline{X}_{k-1} = 0) = \mathbf{E}\left[(1-u)^{\overline{X}_{k-1}}\right] = \overline{\Phi}_{k-1}(1-u)$,

$$\mathbf{P}(\overline{X}_k = 0 | \overline{X}_{k-1} > 0) = \frac{\mathbf{P}(\overline{X}_k = 0, \overline{X}_{k-1} > 0)}{\mathbf{P}(\overline{X}_{k-1} > 0)}$$

$$= \frac{\mathbf{P}[u \circ \overline{X}_{k-1} + B_k = 0, \overline{X}_{k-1} > 0]}{1 - \overline{\Phi}_{k-1}(0)}$$

$$= \frac{q}{1 - \overline{\Phi}_{k-1}(0)} (\overline{\Phi}_{k-1}(1-u) - \overline{\Phi}_{k-1}(0)) = \frac{\overline{\Phi}_k(0) - q\overline{\Phi}_{k-1}(0)}{1 - \overline{\Phi}_{k-1}(0)}$$

giving third

$$\mathbf{P}(\overline{X}_k = 0 | \overline{X}_{k-1} > 0) = \frac{1 - (p\overline{\Phi}_{k-1}(0) + \overline{\Phi}_k(0))}{1 - \overline{\Phi}_{k-1}(0)}.$$

From (14), using $\overline{\Phi}_k(z) = \prod_{k'=1}^{k} \phi_B(1 - u^{k'-1}(1-z))$ evaluated at $z = 0$, the three identities yield a closed-form expression of $\mathbf{P}(\overline{\tau}_{0,0} = l)$ in (16).

Let $\overline{B}_l$, $l \geq 1$, be a random sequence of iid rv's each obtained from B_l when $\beta \overset{d}{\sim} \delta_1$, so with $\mathbf{E}\left(z^{\overline{B}}\right) = q/(1 - pz)$ as a shifted geometric rv. Exploiting the fact that $\overline{X}_k$ is X_n sampled at catastrophic times, the time elapsed between consecutive catastrophes of X_n being $\overline{B}_l + 1$, we obtain.

Proposition. *The first return time to* 0 *of* X_n, *say* $\tau_{0,0} = inf(n \geq 1 : X_n = 0 | X_0 = 0)$, *is given as a random sum of independent geometric rv's*

$$\tau_{0,0} = \sum_{l=1}^{\overline{\tau}_{0,0}} (\overline{B}_l + 1),$$

where the law of $\overline{\tau}_{0,0}$ *is from* (16) *and the* $\overline{B}_l$'s *are independent from* $\overline{\tau}_{0,0}$.

Remark. The latter reasoning is also useful to compute the law of the **first time to extinction** of both $\overline{X}_k$ and X_n.

With $x_0 > 0$, let $\overline{\tau}_{x_0,0} = \inf(k \geq 1 : \overline{X}_k = 0 | \overline{X}_0 = x_0)$ be the time to first local extinction of $\overline{X}_k$. We now have $\mathbf{P}(\overline{\tau}_{x_0,0} = 1) = \mathbf{P}_{x_0}(\overline{X}_1 = 0) = q(1-u)^{x_0}$, and for each $l \geq 2$,

$$\mathbf{P}(\overline{\tau}_{x_0,0} = l) = \mathbf{P}_{x_0}(\overline{X}_1 > 0, \ldots, \overline{X}_{l-1} > 0, \overline{X}_l = 0)$$

$$= \mathbf{P}_{x_0}(\overline{X}_1 > 0) \prod_{k=2}^{l-1} \mathbf{P}(\overline{X}_k > 0 | \overline{X}_{k-1} > 0) \mathbf{P}(\overline{X}_l = 0 | \overline{X}_{l-1} > 0), \tag{17}$$

which is computable in a similar way as before except for the first term in the latter product (which is known) and while considering $\overline{\Phi}_k(z) = (1 - u^k (1-z))^{x_0} \cdot \prod_{k'=1}^{k} \phi_B\left(1 - u^{k'-1}(1-z)\right)$ evaluated at 0 instead for the remaining terms, taking into account $\overline{X}_0 = x_0$ from (14).

With $\tau_{x_0,0} = \inf(n \geq 1 : X_n = 0 \mid X_0 = x_0)$ the time to first local extinction of X_n, $\tau_{x_0,0} = \sum_{l=1}^{\overline{\tau}_{x_0,0}} (\overline{B}_l + 1)$.

4. The Binomial Catastrophe Model with Random Beta(α, 1) Survival Probability

We now want to deal with a growth model with binomial catastrophes having random survival probability. Whenever a catastrophic event occurs, each individual present in the current population, independently of the others, is subject to survival (death) with now random probability U (respectively, $\overline{U}$). This situation occurs when a catastrophic event strikes simultaneously and independently all currently alive members of some population. Think of an extreme drought striking a forest: Each tree, independently of its neighbours, will have to face a chance U of survival, and considering U random can be a natural issue to take into account the variability of the trees in their struggle against the drought (binomial catastrophes in random environment). To the best of the authors' knowledge, the randomness feature of U has not been fully considered in the literature and we go one step in this direction.

4.1. Preliminaries

For any fixed x integer and U a $[0, 1]$-valued random variable (rv), first consider the rv

$$U \circ x = \sum_{i=1}^{x} B_i(U),$$

where $(B_i(U), i \geq 1)$ is an iid sequence of Bernoulli rv's with $P(B_i(U) = 1) = U$, random ($U \circ x$ is the Bernoulli thinning of x, see [16]). Clearly $U \circ 0 = 0$ and

$$\mathbf{E}(z^{U \circ x}) = \mathbf{E}[(1 - U(1 - z))^x], \text{ equivalently}$$

$$\mathbf{P}(U \circ x = y) = \binom{x}{y} \mathbf{E}\left[U^y (1 - U)^{x-y}\right], \ 0 \leq y \leq x$$

and the support of $U \circ x$ is $\{0, \dots, x\}$. With $U \overset{d}{\sim}$ fixing the probability distribution of U, examples are:

$$\mathbf{P}(U \circ x = y) = \binom{x}{y} u^y (1 - u)^{x-y} \text{ if } U \overset{d}{\sim} \delta_u, \ u \in (0, 1),$$

$$\mathbf{P}(U \circ x = y) = \delta_{y,0} \text{ if } U \overset{d}{\sim} \delta_0,$$

$$\mathbf{P}(U \circ x = y) = \delta_{y,x} \text{ if } U \overset{d}{\sim} \delta_1,$$

$$\mathbf{P}(U \circ x = y) = \frac{1}{x+1} \text{ if } U \text{ is uniform,}$$

$$\mathbf{P}(U \circ x = y) = \binom{x}{y} \frac{B(a + y, b + x - y)}{B(a, b)} \text{ if } U \overset{d}{\sim} \mathrm{Beta}(a, b), \ a, b > 0, \tag{18}$$

where $B(a, b)$ is the beta function. For the first binomial example with U concentrated in a point u of $(0, 1)$, both mean and variance of $U \circ x$ are of order x for large x (xu and $xu(1 - u)$, respectively). For the second and third examples with U concentrated on the extreme points of $[0, 1]$, $U \circ x$ is concentrated on the extreme points of its support, 0 and x, respectively. For the last two examples with U truly random and

$\text{Beta}(a, b)$ distributed, the mean is of order x and the variance of order x^2 for large x ($x\,\mathbf{E}(U)$ and $x^2\sigma^2(U) + x(\mathbf{E}(U) - \mathbf{E}(U^2))$ respectively). The Bernoulli thinning operator allows for the definition of the size of the population immediately after a catastrophic event.

4.2. The binomial model with random survival probability

Let $(\beta_n; n \geq 1)$ be a sequence of iid rv's taking values in $\mathbb{N} = \{1, 2, 3, \ldots\}$. Consider now the discrete time-homogeneous Markov chain $(X_n; n \geq 0)$ with state-space $\mathbb{N}_0$ and non-homogeneous spatial transition probabilities characterized by:

- Given $X_n \geq 1$,

$$X_{n+1} = \begin{cases} X_n + \beta_{n+1} \text{ with probability } p \\ U \circ X_n \text{ with probability } q \end{cases} \tag{19}$$

the support of $U \circ x$ being $\{0, \ldots, x\}$. Note $X_n = 1 \Rightarrow X_{n+1} = 0$ with probability q and $U \circ X_n = X_n - (\overline{U} \circ X_n)$, where $\overline{U} := 1 - U$. Note also that given $X_n = x$, there is a probability $\mathbf{P}(U \circ x = x) = \mathbf{E}(U^x)$ that a catastrophic event produces no death toll, all individuals surviving independently to it.

- Given $X_n = 0$, the increment of X_n is β_{n+1} with probability p and 0 with probability q so that X_n is reflected at 0.

The stochastic transition matrix of the new model (19) is $P = [P(x, y)]$, where

$$P(x, y) = q\mathbf{P}(U \circ x = y) \text{ if } 0 \leq y \leq x,$$

$$P(x, y) = p\mathbf{P}(\beta = y - x) \text{ if } y > x,$$

and the shape of the invariant measure was discussed in [8] when $U \overset{d}{\sim} \delta_u$, for some fixed $u \in (0, 1)$.

Special extreme cases are: When U is uniform, $P(x, y) = q/(x+1)$ if $0 \le y \le x$ (the uniform model of [12]) whereas when $U \sim \delta_u$, $P(x, y) = q\binom{x}{y} u^y (1-u)^{x-y}$ if $0 \le y < x$ (the binomial model of [12]). In the uniform case, a catastrophic event takes $X_n = x$ to any state $y \in \{0, \dots, x\}$ with uniform probability $1/(x+1)$.

If $\beta \overset{d}{\sim} \delta_1$ and $U \overset{d}{\sim} \delta_0$, $U \circ X_n = 0$, a catastrophic event takes instantaneously $X_n = x \ge 1$ to state 0, a **total disaster** event, [19, 7]. In that extreme case, the transition matrix P is primitive, with its first column consisting of the q's and an upper diagonal filled with the p's.

In the sequel, we shall choose $P(U \in dv) = av^{a-1}dv$, $a > 0$ so that U has a skewed $\text{beta}(a, 1)$ distribution, with mean $a/(a+1)$. In order to compare with $U \overset{d}{\sim} \delta_u$ as in the fixed survival probability setup, we choose a so that $\mathbf{E}(U) = a/(a+1) = u$ is fixed in $(0, 1)$, (else $a = u/(1-u)$). We also have $\sigma^2(U) = a/(a+2) - [a/(a+1)]^2 = u/(2-u) - u^2$, with a maximum at $u^* = 2(\sqrt{2}-1)/(2\sqrt{2}-1) < 1/2$.

Note:

– $a > 1 \Leftrightarrow u > 1/2$: small values of the survival probability U have little chance to occur.

– $a < 1 \Leftrightarrow u < 1/2$: small values of U are enhanced (favouring the total disaster case).

– $a = 1 \Leftrightarrow u = 1/2$, on a catastrophic event, half the current population size will be decimated and U is uniform.

For such a choice of U, with Γ the Euler gamma function, the transition matrix P of the chain is

$$P(x,\ y) = qa\frac{\Gamma(x+1)}{\Gamma(x+a+1)}\frac{\Gamma(y+a)}{\Gamma(y+1)} \text{ if } 0 \leq y \leq x,$$

$$P(x,\ y) = pb_{y-x} \text{ if } y > x.$$

Theorem. *Let* $U \overset{d}{\sim} beta(a, 1)$. *If and only if* $\mathbf{E}\log_+ B < \infty$ (*or* $\mathbf{E}\log\beta < \infty$), *does* X_∞ *have a proper distribution characterized by its factorized pgf*

$$\Phi_\infty(z) = \phi_B(z)e^{-a\int_z^1 \frac{1-\phi_B(z')}{1-z'}dz'},$$

the second factor of which being SD.

Proof. The limit law pgf $\Phi_\infty(z)$, if it exists, solves the functional equation

$$\Phi_\infty(z) = p\phi_\beta(z)\Phi_\infty(z) + q\mathbf{E}\Phi_\infty(1-U(1-z)). \tag{20}$$

Then

$$\begin{aligned}(1-p\phi_\beta(z))\Phi_\infty(z) &= aq\int_0^1 v^{a-1}\Phi_\infty(1-v(1-z))dv \\ &= \frac{aq}{(1-z)^a}\int_z^1(1-z')^{a-1}\Phi_\infty(z')dz' \\ &= (1-z)^{-a}\left[\Phi_\infty(0) - aq\int_0^z(1-z')^{a-1}\Phi_\infty(z')dz'\right].\end{aligned}$$

Taking the derivative with respect to z, we are led to a linear ordinary differential equation for $\psi(z) = \int_0^z(1-z')^{a-1}\Phi_\infty(z')dz'$ which can be solved to give

$$\psi(z) = \Phi_\infty(0)e^{-aqA(z)}\int_0^z e^{aqA(z')}dA(z') = \frac{\Phi_\infty(0)}{aq}\left(1-e^{-aqA(z')}\right),$$

where $A(z) = \int_0^z a(z')dz'$ and $a(z) = 1/\left[(1-z)\left(1-p\phi_\beta(z)\right)\right]$. Then

$$\Phi_\infty(z) = \Phi_\infty(0)\,(1-z)^{-(a-1)}a(z)e^{-aqA(z)}$$

$$= \frac{\Phi_\infty(0)}{q}\,\phi_B(z)e^{-a\left(\int_0^z [qa(z')-1/(1-z')]dz'\right)}.$$

Observing

$$1/(1-z') - qa(z') = (1-\phi_B(z'))/(1-z'),$$

we get

$$\Phi_\infty(z) = \frac{\Phi_\infty(0)}{q}\,\phi_B(z)e^{a\int_0^z \frac{1-\phi_B(z')}{1-z'}dz'}.$$

Imposing $\Phi_\infty(1) = 1$ yields $\Phi_\infty(0) = qe^{-a\int_0^1 \frac{1-\phi_B(z')}{1-z'}dz'}$ and we finally have

$$\Phi_\infty(z) = \phi_B(z)e^{-a\int_z^1 \frac{1-\phi_B(z')}{1-z'}dz'}$$

This defines a pgf if and only if $\int_0^1 \frac{1-\phi_B(z')}{1-z'}dz' < \infty$ which is $\mathbf{E}\log_+ B < \infty$. The condition for positive recurrence of this chain with random survival probability is the same as for the model with fixed survival probability. Note that as $a \to 0$ (or $u \to 0$), $U \overset{a.s.}{\to} 0$ and $\Phi_\infty(z) \to \phi_B(z)$ and we are back to the total disorder model. The factor $\Phi_\infty^{(2)}(z) :=$ $e^{-a\int_z^1 \frac{1-\phi_B(z')}{1-z'}dz'}$ is the pgf of a discrete SD distribution.

Remark. To be exact, the latter factor $\Phi_{\infty}^{(2)}$ is the pgf of a discrete SD distribution if it is of the form

$$e^{-r\int_{z}^{1}\frac{1-\psi_B(z')}{1-z'}dz'},$$

for some $r > 0$ and some pgf $\psi_B(z)$ obeying $\psi_B(0) = 0$. But $\Phi_{\infty}^{(2)}$ is amenable to the latter form while letting

$$\psi_B(z') = \frac{\phi_B(z') - \phi_B(0)}{1 - \phi_B(0)} \text{ and } r = a(1 - \phi_B(0)) = ap.$$

The pgf ψ_B is the one, ϕ_B, of B conditioned to be positive (successful immigration events). In the factor $\Phi_{\infty}^{(2)}(z) := e^{-a\int_{z}^{1}\frac{1-\phi_B(z')}{1-z'}dz'}$, there is a chance $q = 1 - p$ that immigration events will fail, leading to no input of immigrants.

Corollary. *If B is SD, so is X_{∞} which is then unimodal, with mode at the origin if and only if $p(a + b_1) < 1$, equivalently $u < u_c := \frac{1 - pb_1}{1 + p(1 - b_1)}$. In that case, X_n is completely disastrous.*

Proof. If $\phi_B(z)$ is the pgf of a discrete SD distribution, then so is $\Phi_{\infty}(z)$ as a product of two pgf's of discrete SD distributions, see Proposition 4.3 in [18]. Then,

$$\pi(0) = \mathbf{P}(X_{\infty} = 0) = qe^{-a\int_{0}^{1}\frac{1-\phi_B(z')}{1-z'}dz'},$$

$$\pi(1) = \mathbf{P}(X_{\infty} = 1) = pq(a + b_1)e^{-a\int_{0}^{1}\frac{1-\phi_B(z')}{1-z'}dz'},$$

$$\pi(1)/\pi(0) = p(a + b_1).$$

When $p(a+b_1)<1$, or equivalently when

$$u < u_c := \frac{1-pb_1}{1+p(1-b_1)},$$

the random walk with $\mathrm{beta}(a,1)$-distributed survival probability (with mean u), is completely disastrous, X_∞ having mode at 0.

Let us finally observe that:

If $\mathbf{E}(B)<\infty$,

$$\Phi'_\infty(1) = \mathbf{E}(X_\infty) = \mathbf{E}(B)(1+a) = \mathbf{E}(B)/(1-u) < \infty.$$

If $\mathbf{E}(B^2)<\infty$,

$$\Phi''_\infty(1) = \mathbf{E}[X_\infty(X_\infty-1)] = \phi''_B(1)(1+a) + \phi'_B(1)^2(2a+a^2)$$

$$\sigma^2(X_\infty) = \mathbf{E}(B^2)(1+a) - \mathbf{E}(B)^2 = \frac{1}{1-u}\left(\sigma^2(B) + u\mathbf{E}(B)^2\right) < \infty.$$

4.3. Self-decomposable rv's and pure-death branching processes with immigration in continuous-time

There is an alternative construction of a regenerative process in continuous-time which produces discrete-SD distributions in the long-time run, [20]. The following result then holds:

Theorem. *Let* Y_t, *with* $Y_0=0$, *be a continuous-time pure-death branching processes with immigration. Let* $X_t = X_0 + Y_t$ *where* X_0 *and* Y_t *are independent. Suppose that* $X_0 \overset{d}{=} B$ *and that the pgf of the number of immigrants is* $h(z) = \psi_B(z)$. *Then* X_t *has the same limit law as the discrete-time binomial catastrophe process with random* $\mathrm{beta}(a,1)$ *survival probability if the incoming rate of immigrants is* $r = ap$.

Proof. Consider a continuous-time homogeneous compound Poisson process $P_r(t)$, $t \geq 0$, $P_r(0) = 0$, so with pgf

$$\mathbf{E}_0\left(z^{P_r(t)}\right) = \exp\{-rt(1-h(z))\}, \tag{21}$$

where $h(z) = \mathbf{E}z^M$ (with $h(0) = 0$) is the pgf of the number of the immigrants M, arriving in groups at the jump times of $P_r(t)$ having rate $r > 0$. Let now

$$\phi_t(z) = 1 - e^{-t}(1-z), \tag{22}$$

be the pgf of a pure-death Greenwood branching process started with one particle at $t = 0$; see [9]. This expression of $\phi_t(z)$ is easily seen to be the solution to $\dot{\phi}_t(z) = f(\phi_t(z)) = 1 - \phi_t(z)$, $\phi_0(z) = z$, as is usual for a pure-death continuous-time Bellman-Harris branching processes with affine branching mechanism $f(z) = r_d(1-z)$ and fixing, without loss of generality, the death rate to be $r_d = 1$; see [10]. The distribution function of the lifetime of each initial particle is $1 - e^{-t}$. Let Y_t, with initial condition $Y_0 = 0$, be a random process counting the current size of some population for which a random number of individuals M (determined by $h(z)$) immigrate at the jump times of $P_r(t)$, each of which being independently and immediately subject to the latter pure death Greenwood process. Let $X_t = X_0 + Y_t$ as in the statement of the theorem, with X_0 representing a random initial reservoir of eternal individuals not subject to ageing and death. With $\Phi_0(z) = \mathbf{E}\left(z^{X_0}\right)$, we then get

$$\Phi_t(z) := \mathbf{E}\left(z^{X_t}\right) = \Phi_0(z)\,\mathbf{E}\left(z^{Y_t}\right) = \Phi_0(z)\exp - r\int_0^t (1 - h(\phi_s(z)))ds, \tag{23}$$

with

$$\Phi_t(0) = \mathbf{P}(X_t = 0) = \Phi_0(0)\exp - r\int_0^t (1 - h(1 - e^{-s}))ds$$

$$= \Phi_0(0)\exp - r\int_0^{1-e^{-t}} \frac{1 - h(u)}{1 - u}\, du,$$

the probability that the population is extinct at t. As $t \to \infty$,

$$\Phi_t(z) \to \Phi_\infty(z) = \Phi_0(z)\mathbf{E}\left(z^{Y_\infty}\right) = \Phi_0(z)e^{-r\int_0^\infty (1-h(1-e^{-s}(1-z)))ds}$$

$$= \Phi_0(z)e^{-r\int_z^1 \frac{1-h(u)}{1-u}du}. \tag{24}$$

So, $Y := Y_\infty$, as the limiting population size of this pure-death branching process with immigration, is an SD rv, [20]. In such models typically, a subcritical branching population is regenerated by the incoming of immigrants at random Poissonian times. This is the continuous-time version of the discrete-time branching process with immigration X_k introduced in (13).

The proof ends while identifying the triple (Φ_0, h, r) to (ϕ_B, ψ_B, ap). The true rate r of the underlying compound Poisson process $P_r(t)$ has the special factorized form ap when there is a probability $q = 1 - p$ that a Poisson immigration event carrying M immigrants fails.

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