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NOTE ON AN ALGORITHM THAT HIDES AN OTHER ALGORITHM

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Abstract

In this paper, we study and prove some results on an algorithm that hides an other algorithm. We present the main algorithms used in this work and the associated convergence results.

1. Introduction

Algorithms and their convergences play a fundamental role in mathematics applied to all applied sciences.

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In many different fields such as optimization, the performance of a method is often characterized by its rate of convergence. However, accelerating an algorithm requires a lot of knowledge about the problem's structure, and such improvement is done on a case-by-case basis. Many accelerated schemes have been developed in the past few decades and are massively used in practice. Despite their simplicity, such methods are usually based on purely algebraic arguments and often do not have an intuitive explanation.

Recently, heavy work has been done to link accelerated algorithms with other fields of science, such as control theory, differential equations or optimization. However, these explanations often rely on complex arguments, usually using non-conventional tools in their analysis.

In this paper we study and prove some results on an algorithm that hides an other algorithm. We present the main algorithms used in this work and the associated convergence results.

2. Main Results

Theorem 1. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha < \beta$ *and let* f *be an increasing convex function defined on* $I = [\alpha, \beta]$ *such that* $f(\alpha) < 0 < f(\beta)$. *There exists an unique* ζ *in* I *such that* $f(\zeta) = 0$.

We consider the following algorithm:

starting from an arbitrary x_0, $x_0 \in I$ *we make* $n = 0$

(i) *if* $f(x_n) = 0$ *then stop*;

(ii) *else we take* $x_n^* \in \partial f(x_n)$ *and put*

$$x_{n+1} = x_n - \frac{f(x_n)}{x_n^*}$$

return to (i).

Then we have

(a) *if the sequence* $\{x_n\}$ *is not finite, then we have*

$$x_1 > x_2 > \dots > x_n > \dots > \zeta, \quad \forall n;$$

(b) *the sequence* $\{x_n\}$ *converges superlinearly to* ζ, *meaning that we have*

$$\frac{|x_{n+1} - \zeta|}{|x_n - \zeta|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty;$$

(c) *if* $f''(\zeta)$ *exists then the convergence is quadratic.*

Proof. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Let f be an increasing convex function on $I = [\alpha, \beta]$.

$$f : I \ni x \longrightarrow f(x) \text{ such that } f(\alpha) < 0 < f(\beta)$$

$\exists! \, \zeta \in I$ such that $f(\zeta) = 0$. Let the following algorithm:

Algorithm 1:

($\circ$) Initialization: $x_0 \in I$, x_0 arbitrary, for $n = 0$;

(i) if $f(x_n) = 0 \Rightarrow$ stop, else

(ii) take $x_n^* \in \partial f(x_n)$ and we put

$$x_{n+1} = x_n - \frac{f(x_n)}{x_n^*}$$

return to (i).

(a) If $\{x_n\}$ is not finite, then one has

$$x_1 > x_2 > \dots > x_n > x_{n+1} > \dots > \zeta, \quad \forall n.$$

Indeed; let us suppose that $\{x_n\}$ is infinite and that we never stopped. Then we have

$$x_{n+1} = x_n - \frac{f(x_n)}{x_n^*} \Rightarrow x_{n+1} - x_n = -\frac{f(x_n)}{x_n^*},$$

and

$$x_n^* \in \partial f(x_n) \Rightarrow f(x_{n+1}) \geq f(x_n) + x_n^*(x_{n+1} - x_n)$$

$$(x_{n+1} - x_n)x_n^* + f(x_n) = 0 \Rightarrow f(x_{n+1}) \geq 0 = f(\zeta)$$

and since f is an increasing function $\Rightarrow x_{n+1} \geq \zeta$

and therefore

$$f(x_{n+1}) \geq 0 \text{ and } x_{n+1} \geq \zeta.$$

On the other hand

$$x_n^* > 0 \text{ and } f(x_n) > 0 \ \forall n \geq 1 \Rightarrow x_{n+1} - x_n = -\frac{f(x_n)}{x_n^*} < 0 \Rightarrow x_{n+1} < x_n.$$

Therefore, we have

$$x_1 > x_2 > \ldots > x_n > x_{n+1} > \ldots > \zeta, \quad \forall n. \tag{2.1}$$

(b) $\{x_n\}$ converges superlinearly to ζ, meaning that we have

$$\frac{|x_{n+1} - \zeta|}{|x_n - \zeta|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty?$$

• Let us show the sequence $\{x_n\}$ converges to ζ, else there exists

$$\xi > \zeta \text{ with } \{x_n\} \longrightarrow \xi.$$

Or, $f(\xi) > 0 = f(\zeta)$ (because f is an increasing function)

$x_{n+1} - x_n = -\dfrac{f(x_n)}{x_n^*} \Rightarrow |x_{n+1} - x_n| = \dfrac{f(x_n)}{x_n^*} \geq \dfrac{f(\xi)}{x_1^*} > 0$ since $\{x_n^*\}$ is a decreasing sequence according to the expression (2.1), it is not possible, then $\{x_n\} \longrightarrow \zeta$.

• Superlinear convergence

$$x_{n+1} - \zeta = x_n - \frac{f(x_n)}{x_n^*} - \zeta = x_n - \zeta + \frac{f(\zeta) - f(x_n)}{x_n^*} \text{ since } f(\zeta) = 0.$$

Or, since $\{x_n\} \longrightarrow \zeta$ and $x_n > \zeta$ according to (2.1), we have

$$\lim_{(n \longrightarrow +\infty)} \frac{f(\zeta) - f(x_n)}{\zeta - x_n} = f'_+(\zeta),$$

and

$$x_{n+1} - \zeta = x_n - \zeta + \frac{f(\zeta) - f(x_n)}{x_n^*} \Rightarrow x_{n+1} - \zeta = (x_n - \zeta)(1 - \frac{f(\zeta) - f(x_n)}{(\zeta - x_n)x_n^*})$$

$$\Rightarrow |x_{n+1} - \zeta| = |x_n - \zeta|\,(1 - \frac{f'_+(\zeta)}{x_n^*}).$$

If $n \longrightarrow +\infty$, $x_n \longrightarrow \zeta$ and $x_n^* \longrightarrow f'_+(\zeta)$ by the continuity of sub-differential therefore

$$\frac{|x_{n+1} - \zeta|}{|x_n - \zeta|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

(c) If $f''(\zeta)$ exists the convergence is quadratic ?

Indeed, if $f''(\zeta)$ exists, then f' exists in a neighbourhood of ζ, namely, for n large enough, we have

$$f'(x_n) = x_n^*.$$

Moreover

$$f'(x_n) = f'(\zeta) + (x_n - \zeta)f''(\zeta) + (x_n - \zeta)\varepsilon(x_n - \zeta).$$

That is, by applying

$$f(x_n) = f(\zeta) + (x_n - \zeta)f'(\zeta) + (x_n - \zeta)\varepsilon(x_n - \zeta),$$

with

$$\varepsilon(x_n - \zeta) \longrightarrow 0,\ (x_n - \zeta \ \longrightarrow 0),\ f'(\zeta) = f'_+(\zeta).$$

Then

$$|x_{n+1} - \zeta| = |x_n - \zeta|\,(1 - \frac{f'(x_n)}{f'(\zeta)}) = \frac{|x_n - \zeta|^2}{f'(\zeta)}\,(f''(\zeta) + \varepsilon(x_n - \zeta)),$$

hence the quadratic convergence. □

Theorem 2. *Consider the following problem of optimization*:

$$\alpha := Inf(\frac{f(x)}{g(x)}/x \in C), \tag{$\mathcal{P}$}$$

where C *is a compact subset of* $\mathbb{R}^n$, f *and* g *are defined and continuous on* C *and* g *strictly positive on* C. *For any* $\theta \in \mathbb{R}$, *we associate the following problem*:

$$F(\theta) = \underset{x \in C}{Inf}(f(x) - \theta g(x)). \tag{$\mathcal{P}_\theta$}$$

Let M *and* $M(\theta)$, *respectively be the set of optimal solutions for the problems* $(\mathcal{P})$ *and* $(\mathcal{P}_\theta)$. *Then we have*

(a) *the function* F *is strictly decreasing and concave on* $\mathbb{R}$, *that* $F(\alpha) = 0$ *and* $M(\theta) = M$;

(b) *if* $x \in M(\theta)$ *we have*

$$F(\mu) \leqq F(\theta) + (\theta - \mu)g(x), \quad \forall \mu \in \mathbb{R};$$

(c) *consider the following algorithm*:

Algorithm 2. (i) *We start from an arbitrary* $x_0 \in C$ *and let* $n = 0$;

(ii) *we put*

$$\theta_n = \frac{f(x_n)}{g(x_n)}.$$

If $F(\theta_n) = 0$ *then stop*; *else take* $x_{n+1} \in M(\theta_n)$, *we put* $n = n + 1$ *and we return to* (ii).

If the algorithm stops at step n then $\theta_n = \alpha$ *and* $x_n \in M$, *that in the opposite case the sequence* $\{\theta_n\}$ *tends superlinearly towards* α *decreasing and that any adherence value of the sequence* $\{x_n\}$ *belongs to* M.

Proof. Let

$$\alpha := Inf(\frac{f(x)}{g(x)} / \quad x \in C), \qquad (\mathcal{P})$$

C is a compact subset of $\mathbb{R}^n$, f and g are defined and continuous on C, $g > 0$ on C, $\forall\ \theta \in \mathbb{R}$, we associate the problem

$$F(\theta) = \underset{x \in C}{Inf}(f(x) - \theta g(x)). \qquad (\mathcal{P}_\theta)$$

M : the set of optimal solutions of $(\mathcal{P})$;

$M(\theta)$: the set of optimal solutions of $(\mathcal{P}_\theta)$.

(a)-(b) F strictly decreasing. Let $\xi \in M(\theta)$, then,

$$F(\theta) = f(\xi) - \theta g(\xi) = f(\xi) - \mu g(\xi) + \mu g(\xi) - \theta g(\xi)$$

$$\Rightarrow F(\theta) = f(\xi) - \mu g(\xi) + (\mu - \theta) g(\xi)$$

or,

$$f(\xi) - \mu g(\xi) \geq \underset{x \in C}{Inf}(f(x) - \mu g(x)) = F(\mu).$$

$$\Rightarrow F(\theta) = f(\xi) - \mu g(\xi) + (\mu - \theta) g(\xi) \geq F(\mu) + (\mu - \theta) g(\xi),$$

then $\forall \xi \in M(\theta)$, we have

$$F(\theta) \geq F(\mu) + (\mu - \theta) g(\xi).$$

Or,

$$g(\xi) > 0,$$

hence

$$F(\mu) \leq F(\theta) + (\theta - \mu) g(\xi), \quad \forall \xi \in M(\theta), \quad \forall \mu \in \mathbb{R}.$$

This is question (b), it also implies F strictly decreasing.

Let us show that F is concave over $\mathbb{R}$?

Show that F is a concave function over $\mathbb{R}$, that $F(\alpha) = 0$ and that $M(\theta) = M$?

Indeed, $\theta \longrightarrow f(x) - \theta g(x)$ is concave $\forall x \in C$, hence

$$F(\theta) = \underset{x \in C}{Inf} (f(x) - \theta g(x))$$

is concave.

Let us show that $F(\alpha) = 0$?

Indeed;

$$\alpha = \underset{x \in C}{Inf} \frac{f(x)}{g(x)} \Rightarrow \alpha \leq \frac{f(x)}{g(x)}, \quad \forall x \in C.$$

$$\Rightarrow f(x) - \alpha g(x) \geq 0 \quad \forall x \in C \Rightarrow F(\alpha) \geq 0.$$

Let $\zeta \in M$, so,

$$\alpha = \frac{f(\zeta)}{g(\zeta)} \Rightarrow f(\zeta) - \alpha g(\zeta) = 0 \Rightarrow F(\alpha) \leq 0.$$

Therefore,

$$F(\alpha) = 0 \text{ and } M \subset M(\alpha).$$

Let us show that $M(\alpha) \subset M$?

Indeed; if $\xi \in M(\alpha)$ then $F(\alpha) = 0,$ namely,

$$f(\xi) = \alpha g(\xi) \Rightarrow \alpha = \frac{f(\xi)}{g(\xi)} \Rightarrow \xi \in M,$$

therefore $M(\alpha) \subset M$ and hence $M(\alpha) = M.$

(b) Already done.

(c) Consider the following algorithm:

(i) we start from an arbitrary $x_0 \in C$ with $n = 0,$

(ii) put

$$\theta_n = \frac{f(x_n)}{g(x_n)}.$$

If $F(\theta_n) = 0 \Rightarrow$ stop; else, take $x_{n+1} \in M(\theta_n),$ we put $n = n+1$ and we return to (ii).

Let us show that if the algorithm stops at step n then $\theta_n = \alpha$ and $x_n \in M,$ in the contrary case the sequence $\{\theta_n\}$ tends superlinearly to α decreasing ?

Indeed;

$$F(\mu) \leq F(\theta) + (\theta - \mu)g(x), \quad \forall \mu \in \mathbb{R}.$$

$$\Rightarrow - F(\mu) \geq - F(\theta) + (\mu - \theta)g(x), \quad \forall \mu \in \mathbb{R}.$$

$$\Rightarrow g(x) \in \partial(-F(\theta)).$$

according to (a), F is concave and strictly decreasing therefore the function $H = - F$ is convex and strictly increasing.

Find α is to show that $H(\alpha) = 0,$ that is, $-F(\alpha) = 0.$

Algorithm 1 (Theorem 1) is like building:

$$\theta_{n+1} = \theta_n - \frac{H(\theta_n)}{x_n^*}, \text{ where } x_n^* \in \partial H(\theta_n).$$

Right here

$$x_n^* = g(x_{n+1}), \quad H(\theta_n) = -f(x_{n+1}) + \theta_n g(x_{n+1}).$$

So, we must take

$$\theta_{n+1} = \theta_n + \frac{f(x_{n+1}) - \theta_n g(x_{n+1})}{g(x_{n+1})} = \frac{f(x_{n+1})}{g(x_{n+1})}.$$

Therefore

$$\theta_{n+1} = \frac{f(x_{n+1})}{g(x_{n+1})}.$$

This is what Algorithm 2 (Theorem 2) does.

So we have convergence of $\theta_n \longrightarrow \alpha$ superlinearly.

Any adherence value of the sequence $\{x_n\}$ belongs to M ?

Indeed; let ξ be the adherence value of the sequence $\{x_n\}$. So,

$$\theta_n = \frac{f(x_n)}{g(x_n)} \longrightarrow \alpha \Rightarrow \frac{f(\xi)}{g(\xi)} = \alpha$$

therefore $\xi \in M$. □

Theorem 3. *Consider the optimization problem defined in Theorem* 2 (*Algorithm* 2) *with the following additional assumptions*:

f is convex and positive on C, g *is concave*, C *is convex.*

Then the auxiliary problems $(\mathcal{P}_{\theta_n})$ *intervening in the Algorithm* 2 *are convex programming.*

Proof. Let

$$\alpha = Inf(\frac{f(x)}{g(x)}, \quad x \in C). \tag{$\mathcal{P}$}$$

Let us given the following assumptions:

f is convex and positive on C, g is concave, C is convex.

Then, g concave $\Rightarrow \frac{1}{g}$ is convex, therefore $\frac{f}{g}$ is convex and $(\mathcal{P})$ is a convex optimization problem.

If f is convex > 0, g concave > 0, so $\alpha > 0 \Rightarrow \theta_n > 0$. Therefore

$$x \longrightarrow f(x) - \theta_n g(x)$$

is convex. Then we have a convex programming problem for $(\mathcal{P}_{\theta_n})$. □

We obtain the following special case:

Proposition 1. *In the particular case where f and g are affine and C is a convex polyhedron, the dual problem of $(\mathcal{P}_\theta)$ is the dual in linear programming.*

Proof. We have

$$F(\theta) = \underset{x \in C}{Inf}(f(x) - \theta g(x)). \tag{$\mathcal{P}_\theta$}$$

f, g are affine $\Leftrightarrow f(x) =< a, x > +\alpha$, $g(x) =< b, x > +\beta$, then

$$F(\theta) = \underset{x}{Inf}(< a, x > +\alpha - \theta(< b, x > +\beta)/Cx \leq c)$$

$$= \alpha - \theta\beta + \underset{x}{Inf}(< a - \theta b, x > /Cx \leq c).$$

We take the dual in linear programming ([2], [3]). □

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