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ON GENERALIZED-FUNCTION SOLUTIONS OF A FIRST ORDER LINEAR SINGULAR DIFFERENTIAL EQUATION IN THE SPACE K' VIA FOURIER TRANSFORM

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Abstract

The main purpose of this work is to obtain and describe the solutions of the non-homogeneous first order linear singular differential equation of the following type:

$$Ax^{p}y'(x)+B_{x}{}^{q}y(x)=\delta^{(s)}(x),$$

with A, B two real numbers, s and $p \in \mathbb{N}, q \in Z_{+}$, in the space of generalized functions K'.

We look for the solution of this equation in the form of $y(x)=\sum_{k=0}^{N}C_{k}\delta^{(k)}(x)$ with the unknown coefficients C_k to be determined.

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We also obtain and analyze the classical solutions of the homogeneous equation.

1. Introduction

The importance of differential equations as scientific tools to construct models of reality is well-known. However, sometimes solving even very simple first order linear differential equation in some specific cases may be not easy. Many scientific papers recently are devoted to considerable interest in problems concerning the existence of solutions to differential and functional differential equations (FDE) in various known spaces of generalized functions. As we notice, a lot of serious areas in theoretical and mathematical physics, theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis use widely the methods of the distributions theory. This work is devoted to the question of the investigation of the solvability in the space of generalized functions, linear differential equations of the first order with singularity and Dirac delta function (or its derivative of some order) in the second hand side. Namely, we consider the equation of the following type:

$$Ax^{p}y'(x) + Bx^{q}y(x) = \delta^{(s)}(x), \tag{1}$$

where A, B are real numbers and,

$$p \in \mathbb{N},\ q,\ s \in \mathbb{N} \cup \{0\}. \tag{2}$$

Among others we can note similar investigations recently done by various authors as Liangprom and Nonlaopon [10], Jhanthanam [11]. We also remark, in particular following Abdourahman, see papers [12, 13-14] which contain references to previous works that it is well-known that normal linear homogeneous systems of ODE with infinitely smooth coefficients have no generalized-function solutions other than the classical solutions.

Equation (1) is studied in the space of generalized functions K'. The main characteristic of such equation is the fact that beside classical solutions, the homogeneous equation (1) may admit solutions centered at zero and more, their number in the case when $p = q - 1$ may be superior than the natural number $\min(q, p-1) - 1$ for one and its appearance already is connected not only with p and q, but also with coefficients A and B. Other principal difference with the equation without singularity is the fact that despite the linearity of the Equation (1) under certain relationship between parameters, and namely, if

$$q = p - 1,\ A(q + s + 1) - B = 0, \tag{3}$$

the non-homogeneous equation (1) at all is not solvable (in the case $AB \neq 0$).

In this work, we give the complete description of the solutions of Equation (1) in the space K' as for homogeneous equation and as for the non homogeneous equation (in the case of solvability).

The paper is organized as follow: In the beginning, separately we are considering the case when $AB = 0$ in Section 1. In Section 2 in the case $AB = 0$, we describe the solutions of Equation (1), centered at zero. In Section 3, it is constructed the general solution as union of generalized and classical solutions. In Section 4, we describe the classical solutions of the Equation (1) in $\mathbb{R}^1 - \{0\}$. Lastly, in Section 5, we show the possibility of the existence of not centered at zero solutions. Finally, we summarize our work in Section 7.

2. Preliminaries

In this section we briefly review the important notions of Fourier transform, its properties and generalized function centered at a given point (we refer to [7, 14, 15, 17] for a detailed study). We recall that K is denoted the space of test functions, of finite infinitely differentiable on

$\mathbb{R}^1$ functions and K' the space of generalized functions on K. For function $\varphi(t) \in K$, through $F\varphi = \hat{\varphi}$ we denoted the Fourier transform defined by the formula

$$\left(F_{\varphi}\right)(x) = \hat{\varphi}(x) = \int_{-\infty}^{+\infty} \varphi(t) e^{ixt} dt. \tag{4}$$

The Fourier transform of the generalized function $f \in K'$ we define by the rule (Parseval equality)

$$(\hat{f}, \hat{\varphi}) = 2\pi(f, \varphi). \tag{5}$$

For the Fourier transform of generalized function many properties are conserved as those taking place for Fourier transform for test functions, and particularly formulas relationship between differentiability and decreasement.

From them in particular follow that

$$F[\delta^{(s)}(t)] = (-ix)^s, \; s \in \mathbb{N} \cup \{0\}. \tag{6}$$

We need the following assertions which can be found with their proofs in the books for theory of generalized functions, see, for example, [15, 19, 20, 21].

Theorem 2.1. *If* $f, g \in K'$ *and* $f' = g'$, *then* $f - g = c$.

Theorem 2.2. *Let* $A(x) \in C^{\infty}(\mathbb{R}^1)$. *The differential equations* $y' = A(x)y$ *in the space* K' *does not admit other solutions which are not classical solutions.*

Definition 2.1. Generalized function $f \in K'$ is called centered at the point x_0, if $(f, \varphi(x)) = 0$ for all $\varphi(x) \in K$, such $x_0 \in \operatorname{supp}\varphi$.

Theorem 2.3. *Let $f \in K'$ centered at zero. Then there exist $m \in \mathbb{N} \cup \{0\}$ such that*

$$f(x) = \sum_{j=0}^{m} c_j \delta^{(j)}(x), \tag{7}$$

where c_j-are some constants.

Lemma 2.1. *Let $\beta(x) \in C^{\infty}(\mathbb{R}^1)$. Then it hold the formula*

$$\beta(x)\delta^{(l)}(x) = \sum_{j=0}^{l} (-1)^j C_l^j \beta^i(0)\delta^{(l-j)}(x). \tag{8}$$

The proof of Lemma 2.1 is given in [18].

As consequence from Lemma 2.1 when $\beta(x) = x^k$ we obtain the following assertion.

Lemma 2.2. *Let $k, s \in \mathbb{N} \cup \{0\}$. Then*

$$x^k \delta^{(s)}(x) = \begin{cases} 0, & \text{if} \quad s < k, \\ \dfrac{(-1)^k s!}{(s-k)!} \delta^{(s-k)}(x), & \text{if} \quad s \geq k. \end{cases} \tag{9}$$

Sometime we need in our investigation the following expression $\dfrac{g^{(s)}(x)}{x^n}$. We will understand this expression from the following definition.

Definition 2.2. The quotient $\dfrac{\delta^{(s)}(x)}{x^n}$ is called a generalized function $y(x) \in K'$ which satisfy in the space K' the equality

$$x^n y(x) = \delta^{(s)}(x), \text{ i.e.,}$$

$$(x^n y(x), \varphi(x)) = (\delta^{(s)}(x), \varphi(x)), \; \varphi \in K. \tag{10}$$

From Lemma 2.2 and Definition 2.2, we deduce the following formula for the computation of the generalized function $\frac{\delta^{(s)}(x)}{x^n}$.

Lemma 2.3. *Let* $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$. *It hold the following formula*:

$$\frac{\delta^{(s)}(x)}{x^n} = \frac{(-1)^n s!}{(s+n)!} \delta^{(s+n)}(x) + \sum_{k=1}^{n} c_k \delta^{(k-1)}(x). \tag{11}$$

Proof. We use the Definition 2.2 and let apply the Fourier transform to both sides of the equality $x^n y(x) = \delta^{(s)}(x)$ with consideration that $\widehat{(ix)^n y}(x) = \widehat{y(x)}(\xi)$. Finally, we have

$$\begin{aligned} \hat{y}^n(\xi) = \widehat{(ix)^n y}(x) = i^n \widehat{\delta^{(s)}}(x) \\ = i^n (-i\xi)^s. \end{aligned}$$

From the previous we reach

$$\hat{y}(\xi) = P_{n-1}(\xi) + \frac{(-1)^n s!}{(s+n)!} (-i\xi)^{s+n},$$

where $P_{n-1}(\xi)$ is a polynomial with arbitrary coefficients. Applying the inverse Fourier transform with respect to (6) we arrive to (11). The lemma is proved.

Note that in the result of the division of the generalized function $\delta^{(s)}(x)$ by x^n because of the "union" of singularity, it is not only increasing the order of the derivative of delta function, but it arises some arbitrary depending of n.

3. The Degenerate Case

Let consider at the beginning Equation (1) in the case called degenerate case, when

$$AB = 0. \tag{12}$$

This case is simple and allow us to obtain completed result as consequence of Lemma 2.3.

Theorem 3.1. *Let* $q,\ s \in \mathbb{N} \cup \{0\}$. *The general solution of equation*

$$Bx^q y(x) = \delta^{(s)}(x) \tag{13}$$

has the following form:

$$y(x) = \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) + \sum_{k=1}^{q} C_k \delta^{(k-1)}(x). \tag{14}$$

Here and further we set

$$\sum_{m}^{n} \delta^{(k)}(x) = 0 \text{ if } n < m. \tag{15}$$

It is clear that the Theorem 3.1 is just a reformulation of the Lemma 2.3.

Theorem 3.2. *Let* $p \in \mathbb{N},\ s \in \mathbb{N} \cup \{0\}$. *The general solution of the equation*

$$Ax^p y'(x) = \delta^{(s)}(x) \tag{16}$$

has the following form:

$$y(x) = \frac{(-1)^p s!}{A(s+p)!}\delta^{(s+p-1)}(x) + \sum_{k=1}^{p-1} c_{k+1}\delta^{(k-1)}(x) + c_1\theta(x) + c_0, \tag{17}$$

where $\theta(x)$ *is the Heaviside test function.*

The proof of this theorem also can be deduced from the formula (11). In fact, we have

$$y' = \frac{1}{A}\frac{\delta^{(s)}(x)}{x^p} = \frac{(-1)^p s!}{A(s+p)!}\delta^{(s+p)}(x) + \sum_{k=1}^{p} c_k \delta^{(k-1)}(x),$$

from there by integration, with respect to $\delta = \theta'$ we arrive to (17). The theorem is proved.

4. The Non Degenerate Case

We here forward the consideration of the main case when

$$AB \neq 0, \tag{18}$$

which we call non-degenerate case. The following theorem gives the necessary conditions of the solvability of the Equation (1) in K'.

Theorem 4.1. *Let* $AB \neq 0,\ p \in \mathbb{N},\ q \in \mathbb{N} \cup \{0\}$. *For the solvability of the Equation* (1) *in the space* K' *it necessary and sufficient that*

$$(q - p + 1)^2 + (B - A(q + s + 1)^2) \neq 0. \tag{19}$$

Proof. We just prove only the necessary part of the theorem. The sufficient part of this theorem will be constructed below at the same time with the construction of the solutions.

It is not difficult when proving to understand that the particular solution $y(x)$ of the non-homogeneous equation (1) should be a functional centered in zero. If looking $y(x)$ in the form of such functional

$$y(x) = \sum_{j=0}^{N} c_j \delta^{(j)}(x), \tag{20}$$

where N is sufficient great number and suppose contrary (19), i.e., when

$$q = p - 1,\ B = A(q + s + 1), \tag{21}$$

then simple calculations with consideration of Lemma 2.2 give after substitution (21) in the Equation (1)

$$Ax^p \sum_{j=0}^{N} c_j \delta^{(j+1)}(x) + Bx^q \sum_{j=0}^{N} c_j \delta^{(j)}(x) = \delta^{(s)}(x).$$

From the previous we obtain

$$A \sum_{j=p-1}^{N} \frac{(-1)^p (j+1)!}{(j+1-p)!} c_j \delta^{(j+1-p)}(x) + B \sum_{j=q}^{N} \frac{(-1)^q j!}{(j-q)!} c_j \delta^{(j+q)}(x) = \delta^{(s)}(x), \tag{22}$$

or after arrangement

$$A \sum_{j=0}^{N-(p-1)} c_{j+p-1} \frac{(-1)^p (j+p)!}{j!} \delta^{(j)}(x) + B \sum_{j=0}^{N-q} \frac{(-1)^q (j+q)!}{j!} c_{j+q} \delta^{(j)}(x) = \delta^{(s)}(x). \tag{23}$$

From the previous, if we equalize coefficient under δ-functions and its derivatives we obtain a non-homogeneous linear algebraic system for the determination of the unknown coefficients c_j of the following form:

$$A \frac{(-1)^p (j+p)!}{j!} c_{j+p-1} + B \frac{(-1)^q (j+q)!}{j!} c_{j+q} = \begin{cases} 1, & \text{if} \quad j = s, \\ 0, & \text{if} \quad j \neq s. \end{cases} \tag{24}$$

If we observe the equation in (24) when $j = s$ under the choice $q = p - 1$, then it is not difficult to see that it has the following form:

$$\frac{(-1)^q (q+s)!}{s!} [B - A(q+s+1)] c_{q+s} = 1, \tag{25}$$

and solvable with respect of the second condition in (21). Therefore the algebraic system in (24) is solvable, that was to be prove. We note that the analysis of system (24) in the cases (21) do not take place permit to construct the solution of Equation (1). Nevertheless, we will use also other process for the construction of the general solution of Equation (1) centered in zero. For that matter it is important to distinguish two cases:

(a) $q \neq p-1$

(b) $q = p-1\, B \neq A(q+s+1)$.

The most simple for investigation is the case (b). In this case, the solution obtained when analyzing system (24) with consideration that at this time Equation (25) is solvable.

Theorem 4.2. *Let* $AB \neq 0$, $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{0\}$ *and be realized the condition* $q = p-1$,

$$B \neq A(q+s+1). \tag{26}$$

Then, the centered in zero general solution of the Equation (1) *has the following form*:

$$y(x) = \frac{(-1)^q s!}{(s+q)!\,[B - A(s+q+1)]}\delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x), \tag{27}$$

in the case when

$$B - A(j+q+1) \neq 0,\ j \in \mathbb{Z}_+. \tag{28}$$

And if there exists $j_* \in \mathbb{Z}_+ - \{s\}$, *such that*

$$B - A(j_* + q + 1) = 0,\ j_* \in \mathbb{Z}_+ - \{s\}, \tag{29}$$

then, the solution has the following form:

$$y(x) = \frac{(-1)^q s!}{(s+q)!\,[B - A(s+q+1)]}\delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + c_{j_*+q}\delta^{(j_*+q)}(x). \tag{30}$$

Proof. As we already noticed upstair, in this case we can use system (24) which we can rewrite with respect to the condition $p = q + 1$ in the following form:

$$\frac{(-1)^q (j+q)!}{j!}[B - A(j+q+1)]c_{j+q} = \begin{cases} 1, & \text{if } j = s, \\ 0, & \text{if } j \neq s. \end{cases} \tag{31}$$

Taking into consideration (26) it is clear that

$$c_{s+q} = \frac{(-1)^q s!}{(s+q)!} \frac{1}{B - A(s+q+1)}. \tag{32}$$

Concerning the remaining coefficients c_{j+q}, $j \in \mathbb{Z}_+ - \{s\}$, they are equal to zero, when $B - A(j+q+1) \neq 0$ for all $j \in \mathbb{Z}_+$ and one (and only one of them) will remain free, if there exists $j_* \in \mathbb{Z}_+ - \{s\}$, such that $B - A(j_* + q + 1) = 0$. All these lead us to (27)-(30) with respect to $\delta^{(j)}(x)$, $j = 0, 1, \ldots, q-1$ are solutions of the homogeneous equation.

The proof is realized.

Now, let us move to the investigation of the most difficult case when $q \neq p - 1$.

Theorem 4.3. *Let* $AB \neq 0$, $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{0\}$ *and realized the condition*

$$q < p - 1. \tag{33}$$

Then the centered in zero general solution of the Equation (1) *is defined by the formula*

$$y(x) = \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \frac{(-1)^q s!}{B(s+q)!} \sum_{l=1}^{[\frac{s}{p-1-q}]} (-1)^{(p-1-q)l}$$

$$\times \left(\frac{A}{B}\right)^l \delta^{(s+q-l(p-1-q))}(x) \prod_{m=1}^{l} \frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}. \tag{34}$$

Proof. It is clear that when $q \neq p-1$ the homogeneous equation (1) is satisfied only (and only) the function $\delta^{(j)}(x),\ j = 0, 1, \ldots, \min(q,\ p-1)-1$ that is why for the main part of Theorem 4.3 (and to the next Theorem 4.4) it will be the proof of the fact, that the concentrated in zero particular solution has the type described in (34). For the proof of that, we will use this solution in the following form:

$$y(x) = \sum_{l=0}^{[\frac{s}{p-1-q}]} \gamma_l \delta^{(s+q-l(p-1-q))}(x) \tag{35}$$

with unknown coefficients γ_l. Putting that in (1) gives us:

$$Ax^p \sum_{l=0}^{[\frac{s}{p-1-q}]} \gamma_l \delta^{(s+p-(l+1)(p-1-q))}(x) + Bx^q \sum_{l=0}^{[\frac{s}{p-1-q}]} \gamma_l \delta^{(s+q-l(p-1-q))}(x) = \delta^{(s)}(x).$$

Note that $s+q-l-l(p-1-q) \geq q$ for all possible values of l by the virtue of $l \leq [\frac{s}{p-1-q}] \leq \frac{s}{p-1-q}$. This lead us to the fact that when using Lemma 2.2 In the second member of the summation in the left hand side all the members will be conserved, and in the first all, unless the last one when $l = [\frac{s}{p-1-q}]$. That gives

$$A \sum_{l=0}^{[\frac{s}{p-1-q}]-1} \gamma_l \frac{(-1)^p [s+p-(l+1)(p-1-q)]!}{[s-(l+1)(p-1-q)]!} \delta^{(s-(l+1)(p-1-q))}(x)$$

$$+ B \sum_{l=0}^{[\frac{s}{p-1-q}]} \gamma_l \frac{(-1)^q [s+q-l(p-1-q)]!}{[s-l(p-1-q)]!} \delta^{(s-l(p-1-q))}(x) = \delta^{(s)}(x).$$

After changing $l+1 \to l$ in the first summation we obtain:

$$A \sum_{l=1}^{[\frac{s}{p-1-q}]} \frac{(-1)^p [s+p-l(p-1-q)]!}{[s-l(p-1-q)]!} \gamma_{l-1} \delta^{(s-l(p-1-q))}(x) \tag{36}$$

$$+ B \sum_{l=0}^{[\frac{s}{p-1-q}]} \frac{(-1)^q [s+q-l(p-1-q)]!}{[s-l(p-1-q)]!} \gamma_l \delta^{(s-l(p-1-q))}(x) = \delta^{(s)}(x). \tag{37}$$

Therefore, for the unknown coefficient γ_l we obtain the following algebraic system:

$$\begin{cases} B \dfrac{(-1)^q (s+q)!}{s!} \gamma_0 = 1; \\ A \dfrac{(-1)^p [s+p-l(p-1-q)]!}{[s-l(p-1-q)]!} \gamma_{l-1} \\ \quad + B \dfrac{(-1)^q [s+q-l(p-1-q)]!}{[s-l(p-1-q)]!} \gamma_l = 0, \ l = 1, \dots, [\dfrac{s}{p-1-q}], \end{cases}$$

which presents acquire the following recurrent relations:

$$\begin{cases} \gamma_0 = \dfrac{(-1)^q s!}{B(s+q)!}; \\ \gamma_l = \dfrac{(-1)^{p-1-q} A[s+p-l(p-1-q)]!}{B[s+q-l(p-1-q)]!} \gamma_{l-1}, \ l = 1, \dots, [\dfrac{s}{p-1-q}]. \end{cases} \tag{38}$$

Calculating γ_l from these recurrent relationships, it is easy to obtain

$$\gamma_l = \gamma_0 \left(\frac{(-1)^{p-1-q} A}{B}\right)^l \prod_{m=1}^{l} \frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}, \ l = 1, \dots, [\frac{s}{p-1-q}], \tag{39}$$

and the previous lead us to the needed result. The theorem is proved.

Analogously it can be investigated the case in the contrary inequality in (33). Namely, it takes place.

Theorem 4.4. *Let* $AB \neq 0,\ p \in \mathbb{N},\ q \in \mathbb{N} \cup \{0\}$ *and realized the condition*

$$q > p - 1. \tag{40}$$

Then, the centered in zero general solution of Equation (1) *is given by the following formula*:

$$y(x) = \sum_{j=0}^{p-2} c_j \delta^{(j)}(x) + \frac{(-1)^p s!}{A(s+p)!} \delta^{(s+p-1)}(x)$$

$$+ \frac{(-1)^p s!}{A(s+p)!} \sum_{l=1}^{[\frac{s}{q-p+1}]} \left(\frac{A}{B}\right)^l (-1)^{l(q-p+1)} \delta^{(s+p-1-l(q-p-1))}(x)$$

$$\times \prod_{m=1}^{l} \frac{[s+q-m(q-p+1)]!}{[s+p-m(q-p+1)]!}. \tag{41}$$

Proof. It is sufficient to find the particular solution of Equation (1). For this matter set

$$y(x) = \sum_{l=0}^{[\frac{s}{q-p+1}]} \gamma_l \delta^{(s+p-1-l(q-p+1))}(x),$$

with unknown coefficient γ_l. Putting into (1) we obtain

$$Ax^p \sum_{l=0}^{[\frac{s}{q-p+1}]} \gamma_l \delta^{(s+p-l(q-p+1))}(x)$$

$$+ Bx^q \sum_{l=0}^{[\frac{s}{q-p+1}]} \gamma_l \delta^{(s+q-1-(l+1)(q-p+1))}(x) = \delta^{(s)}(x).$$

From the previous we now reach to the following recurrent relationships:

$$\begin{cases} \gamma_0 = \dfrac{(-1)^p s!}{B(s+q)!}; \\ \gamma_l = \dfrac{(-1)^{q-p+1} B[s+q-l(q-p+1)]!}{A[s+p-l(q-p+1)]!} \gamma_{l-1}, \ l = 1, 2, \dots, [\dfrac{s}{p-1-q}]. \end{cases} \tag{42}$$

This lead us to the result

$$\gamma_l = \gamma_0 (\frac{(-1)^{q-p+1} B}{A})^l \prod_{m=1}^{l} \frac{[s+q-m(q-p+1)]!}{[s+p-m(q-p+1)]!}, \ l = 1, \dots, [\frac{s}{q-p+1}]. \tag{43}$$

The theorem is proved.

5. Analysis of the Classical Solutions of the Homogeneous Equation (1)

We suppose here that $AB \neq 0$. The homogeneous Equation (1) has the following form:

$$y' = -\frac{B}{A} x^{q-p} y. \tag{44}$$

Let find the ordinary solutions of Equation (44): under condition $p \neq q+1$, we have:

$$y(x) = \begin{cases} k_1 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}}; \ x < 0, \\ k_2 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}}, \ x > 0. \end{cases}$$

The corresponding solutions of Equation (44) in the sense of K' have the following type:

$$y(x) = k_1 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}} \theta(x) + k_2 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}} \theta(-x), \tag{45}$$

where $\theta(x)$ the Heaviside step function. From the different relationships between the parameters p, q, A and B and taking into consideration local integration of functions on $\mathbb{R}$, we can bring out those of these functions which are solutions of Equation (44).

Lemma 5.1. *It is taking place the following assertions*:

(1) *If* $q > p-1$, *then equation* (44) *has solution in the form of* (45) *for all values of parameters* A, B, k_1, k_2 *from* $\mathbb{R}$.

(2) *If* $q < p-1$, *then we can distinguish different cases with dependence of the parity of the number* $q-p$ *and the sign of* $\frac{B}{A}$.

(a) *Let* $\frac{B}{A} < 0$ *and* $q-p$ *odd number then the equation* (44) *admit a solution of the form* (45) *with all values of numbers* $A, B, k_1, k_2 \in \mathbb{R}$.

(b) *Let* $\frac{B}{A} < 0$ *and* $q-p$ *even number then the solution of the Equation* (44) *has the following*:

$$y(x) = k_1 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}} \theta(x), \tag{46}$$

with arbitrary values of parameters $k_1, A, B \in \mathbb{R}$.

(c) *Let* $\frac{B}{A} > 0$ *and* $q-p$ *odd number then the solution of Equation* (44) *is defined by* $y(x) = 0$.

(d) *Let* $\frac{B}{A} > 0$ *and* $q-p$ *even number then the solution of Equation* (44) *has the following form*:

$$y(x) = k_2 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}} \theta(-x), \tag{47}$$

with arbitrary value of parameters $k_2, A, B \in \mathbb{R}$.

Next, let consider the case when $p = q + 1$. *In this case the solution in the sense of* K' *of Equation* (44) *has the following form defined by*:

$$y(x) = k_1 x^{-\frac{B}{A}} \theta(x) + k_2 |x|^{-\frac{B}{A}} \theta(-x), \tag{48}$$

with arbitrary value of parameters k_2, A, $B \in \mathbb{R}$.

Taking into consideration the idea of local integrability of functions in $\mathbb{R}$, we arrive to the two situations describe below.

Lemma 5.2. *It hold the next assertions*:

(a) *Let* $\frac{B}{A} < 1$, *then the Equation* (44) *admit solution of the form* (48) *with arbitrary values* A, B, k_1, $k_2 \in \mathbb{R}$.

(b) *Let* $\frac{B}{A} \geq 1$, *then the Equation* (44) *admit only trivial solution* $y(x) = 0$.

First of all we prove that (45) is the solution of Equation (44) under $q > p - 1$ and under arbitrary values of parameters A, B, k_1, $k_2 \in \mathbb{R}$.

In fact putting $y(x)$ in the Equation (44) we obtain $\forall \varphi \in K$

$$(Ax^p y'(x) + Bx^q y(x), \varphi(x)) = (Ax^p (k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}} \theta(x)$$

$$+ k_2 e^{\frac{-Bx^{q-p+1}}{A(q-p+1)}} \theta(-x))' + Bx^q (k_1 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}} \theta(x)$$

$$+ k_2 e^{-\frac{B}{A(q-p+1)} x^{q-p+1}} \theta(-x), \varphi(x)) = 0.$$

Taking into consideration that $\forall f \in K'$, $\forall \varphi \in K(f'(x), \varphi(x)) = (f(x), -\varphi(x))$ we have

$$(Ax^p y'(x) + Bx^q y(x), \varphi(x)) = -k_1 \int_0^\infty e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} (Ax^p \varphi(x))' dx$$

$$+ Bk_1 \int_0^\infty x^q e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}} \varphi(x) dx$$

$$- k_2 \int_{-\infty}^0 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} (Ax^p \varphi)' dx$$

$$+ Bk_2 \int_{-\infty}^0 \frac{-B}{A(q-p+1)} x^{q-p+1} \varphi(x) dx.$$

Integrating part by part we obtain

$$(Ax^p y'(x) + Bx^q y(x), \varphi(x)) = -k_1 Ax^p \varphi(x) e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} \Big|_0^\infty$$

$$- k_1 \int_0^\infty Ax^p \varphi(x) \left(\frac{B}{A} x^{q-p} e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\right) dx$$

$$- k_1 B \int_0^\infty x^q e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} \varphi(x) dx$$

$$- k_2 Ax^p e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} \varphi(x) \Big|_{-\infty}^0$$

$$- Ak_2 \int_{-\infty}^0 \frac{B}{A} x^{q-p} e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} x^p \varphi(x) dx$$

$$+ k_2 B \int_{-\infty}^0 x^q e^{-\frac{B}{A(q-p+1)}x^{q-p+1}} \varphi(x) dx.$$

Simple calculations lead us to the end of the proof. In the same way, we can consider the case when $q < p - 1.$ Further, let verify that in the case when $p = q + 1$ the solution of the equation defined by (44) and given by the formula (48) when $\frac{B}{A} < 1$ is true. In fact when putting (48) into (44) for every $\varphi(x) \in K,$ we obtain:

$$(Ax^p y'(x) + Bx^q y(x),\ \varphi(x)) = (Ax^p (k_1 x^{-\frac{B}{A}} \theta(x) + k_2 |x|^{-\frac{B}{A}} \theta(-x))'$$

$$+ Bx^q (k_1 x^{-\frac{B}{A}} \theta(x) + k_2 |x|^{-\frac{B}{A}} \theta(-x),\ \varphi(x))$$

$$= -\int_0^\infty k_1 x^{-\frac{B}{A}} (Ax^p \varphi(x))' dx + Bk_1 \int_0^\infty x^{q-\frac{B}{A}} \varphi(x) dx$$

$$- k_2 \int_{-\infty}^0 - x^{-\frac{B}{A}} (Ax^p \varphi(x))' dx - Bk_2 \int_{-\infty}^0 x^{q-\frac{B}{A}} \varphi(x) dx$$

$$= - Ak_1 x^{p-\frac{B}{A}} \varphi(x)|_0^\infty + Ak_1 (-\frac{B}{A}) \int_0^\infty x^{p-(\frac{B}{A}-1)} \varphi(x) dx$$

$$+ Bk_1 \int_0^\infty x^{q-\frac{B}{A}} \varphi(x) dx + Ak_2 x^{p-\frac{B}{A}} \varphi(x)|_{-\infty}^0$$

$$- Ak_2 (-\frac{B}{A}) \int_{-\infty}^0 x^{p-(\frac{B}{A}-1)} \varphi(x) dx - Bk_2 \int_{-\infty}^0 x^{q-\frac{B}{A}} \varphi(x) dx.$$

After calculation and simplification we obtain what we need.

The verification in the second case when $p = q + 1$ and when $\frac{B}{A} \geq 1$ is obvious.

Definition 5.1. Equation (1) is called particular when the conditions $p = q + 1$ and $B = A(q + s + 1)$ are realized and in the contrarily case it is called non particular.

Theorem 5.1. *The general solution of the non particular Equation* (1) *when* $A.B \neq 0,\ p \in \mathbb{N},\ q \in \mathbb{N} \cup \{0\}$ *has the following form*:

(1) *If* $q > p-1$

$$y(x) = k_1 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\theta(x) + k_2 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\theta(-x) + \sum_{j=0}^{p-2} C_j \delta^{(j)}(x)$$

$$+ \frac{(-1)^p s!}{A(s+p)!}\delta^{(s+p-1)}(x) + \frac{(-1)^p s!}{A(s+p)!}\sum_{l=1}^{[\frac{s}{q-p+1}]}\left(\frac{B}{A}\right)^l (-1)^{l(q-p+1)}$$

$$\times \prod_{m=1}^{l} \frac{[s+q-m(q-p+1)]!}{[s+p-m(q-p+1)]!}\delta^{(s+p-1-l(q-p+1))}(x),$$

where $k_1,\ k_2,\ A,\ B,\ c_0,\ \ldots,\ c_{p-2}$ *are arbitrary constants.*

(2) *If* $q < p-1,\ \dfrac{B}{A} < 0$ *and* $q-p$ *odd number.*

$$y(x) = k_1 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\theta(x) + k_2 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\theta(-x) + \frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x)$$

$$+ \sum_{j=0}^{q-1} C_j \delta^{(j)}(x) + \frac{(-1)^q s!}{A(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(-1)^{l(p-1-q)}\left(\frac{A}{B}\right)^l$$

$$\times \prod_{m=1}^{l} \frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x),$$

where $k_1,\ k_2,\ A,\ B,\ c_0,\ \ldots,\ c_{q-1}$ *are arbitrary constants.*

(3) *If* $q < p-1$, $\frac{B}{A} > 0$ *and* $q-p$ *even number*

$$y(x) = k_1 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\theta(x) + \sum_{j=0}^{q-1} C_j \delta^{(j)}(x) + \frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x)$$

$$+ \frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(-1)^{l(p-1-q)}(\frac{A}{B})^l$$

$$\times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x),$$

where k_1, k_2, A, B, $c_0, \ldots, c_{q-1}$ *are arbitrary constants.*

(4) *If* $q < p-1$, $\frac{B}{A} > 0$ *and* $q-p$ *odd number*

$$y(x) = \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} C_j \delta^{(j)}(x)$$

$$+ \frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(-1)^{l(p-1-q)}(\frac{B}{A})^l$$

$$\times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x),$$

where k_1, k_2, A, B, $c_0, \ldots, c_{q-1}$ *are arbitrary constants.*

(5) *If* $q < p-1$, $\frac{B}{A} > 0$ *and* $q-p$ *even number*

$$y(x) = k_2 e^{-\frac{B}{A(q-p+1)}x^{q-p+1}}\theta(-x) + \frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} C_j \delta^{(j)}(x)$$

$$+ \frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(-1)^{l(p-1-q)}\left(\frac{A}{B}\right)^l$$

$$\times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x),$$

where k_2, A, B, $c_0, \ldots, c_{q-1}$ *are arbitrary constants.*

Finally, we move to the next following case.

6. The Case $q = p-1$, $B \neq A(q+s+1)$

(1) If $B - A(j+q+1) \neq 0$ $\forall_j \in \mathbb{Z}_+$ and $\frac{B}{A} < 1$

$$y(x) = \frac{(-1)^q s!}{(s+q)!\,[B - A(q+s+1)]}\delta^{(s+q)}(x)$$

$$+ \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + k_1 x^{-\frac{B}{A}}\theta(x) + k_2 |x|^{-\frac{B}{A}}\theta(-x),$$

where k_1, k_2, A, B, $c_0, \ldots, c_{q-1}$ are arbitrary constants.

(2) If $\exists\, j_* \in \mathbb{Z}_+ - \{s\}$ such that, $B - A(j_* + q + 1) = 0$ and $\frac{B}{A} < 1$,

$$y(x) = \frac{(-1)^q s!}{(s+q)!\,[B - A(q+s+1)]}\delta^{(s+q)}(x) + c_{j_*+q}\delta^{(j_*+q)}(x)$$

$$+ \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + k_1 x^{-\frac{B}{A}}\theta(x) + k_2 |x|^{-\frac{B}{A}}\theta(-x),$$

where k_1, k_2, c_{j+q}, A, B, $c_0, \ldots, c_{q-1}$, c_{j_*+q} are arbitrary constants.

(3) If $B - A(j + q + 1) \neq 0$, $\forall j \in \mathbb{Z}_+$ and $\frac{B}{A} \geq 1$,

$$y(x) = \frac{(-1)^q s!}{(s+q)!\,[B - A(q + s + 1)]}\,\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x),$$

where A, B, $c_0, \ldots, c_{q-1}$ are arbitrary constants.

(4) If $\exists j_* \in \mathbb{Z}_+ - \{s\}$ such that, $B - A(j_* + q + 1) = 0$ and $\frac{B}{A} \geq 1$,

$$y(x) = \frac{(-1)^q s!}{(s+q)!\,[B - A(q + s + 1)]}\,\delta^{(s+q)}(x)$$

$$+ c_{j_*+q}\delta^{(j_*+q)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x),$$

where c_{j_*+q}, A, B, $c_0, \ldots, c_{q-1}$ are arbitrary constants.

7. Conclusion

In this paper, we completely investigated the solvability of the linear singular differential equation of the first order in the space of generalized functions K' with a second right hand side in the form of an s-order derivative of the Dirac delta function. All the results obtained are described in the theorems we formulated. It would be very interesting to further these investigations in a more widely generalized space considering the solvability of the homogeneous equation.

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References

[1] F. S. Aliev, On the solutions of certain systems of ordinary differential equations in the space of generalized functions, Vestnik Moscov. Gos. Univ. Ser. Mat. Meh. 5 (1973). 3-10.

[2] J. Wiener, Generalized-function solutions of ordinary differential equations, in Third All-Union Conference on the Qualitative Theory of Differential Equations, Samarkand, USSR, Oct. 1973, Abstracts, p. 51.

[3] J. Wiener, Solutions of linear systems in generalized functions, Differencial'nye Uravnenija 6 (1975), 1128-l130.

[4] J. Wiener, A retarded type system with infinitely differentiable coeflicients has solutions in generalized functions, Uspehi Mat. Nauk. 31 (5) (1976), 227.

[5] J. Wiener, Existence of solutions of differential equations with deviating argument in the space of generalized functions, Sibirsk. Mat. Z. 6 (1976), 1403-1405.

[6] J. Wiener, Generalized-function solutions of differential equations with a countable number of argument deviations, Differencial'nye Uravnenija 2 (1978), 355-358.

[7] I. M. Gel'fand and G. E. Shilov, Generalized Functions, Spaces of Fundamental and Generalized Functions, Academic Press, New York. Vol. II, 1968.

[8] B. S. Mityagin, On the infinitely differentiable function with the given values of the derivatives at a point, Doklady Akademii Nauk SSSR 138(2) (1961), 289-292.

[9] E. Kamke, Differentialgleichungen: Losungsmethoden und Losungen, I. Gewohnliche Differentialgleichungen, Leipzig, 1959.

DOI: http://dx.doi.org/10.1007/978-3-663-05925-7

[10] A. Liangprom and K. Nonlaopon, On the generalized solutions of a certain fourth order Euler equations, Journal of Nonlinear Sciences and Applications 10(8) (2017), 4077-4084.

DOI: http://dx.doi.org/10.22436/jnsa.010.08.04

[11] S. Jhanthanam, K. Nonlaopon and S. Orankitjaroen, Generalized solutions of the third-order Cauchy-Euler equation in the space of right-sided distributions via Laplace transform, Mathematics 7(4) (2019); Article 376.

DOI: https://doi.org/10.3390/math7040376

[12] Abdourahman, On a linear differential equation with singular coefficients, Rostov on Don. 1999. Collection of papers, Integro-Differential Operators and their Applications, Rostov-na-Donu: Izdat. DGTU, 1999. No.5. P.4-7.

Available at: http://chat.ru/

[13] Abdourahman, On a linear differential equation in the spaces of generalized functions, Rostov on Don. Preprint at VINITI 02.08.2000.No.2035-2000. 27 pages.

[14] Abdourahman, On a linear singular differential equation with singular coefficients, Collection of papers, Integro-Differential Operators and their Applications, Rostov-na-Donu: Izdat. DGTU, 5 (2001), 4-10.

[15] I. M. Gelfand and G. E. Chylov, Spaces of test and generalized functions, V.1.M.: Phyzmatgiz, 1958. 307 pages, Russian Edition.

[16] Laurent Schwartz, Analyse: Théorie des Ensembles et Topologie, I, Hermann (1991), 404 pages.

[17] Laurent Schwartz, Théorie de Distributions, Hermann (1998), 418 pages.

[18] R. P. Kanwal, Generalized Functions: Theory and Technique, 3rd Edition, Springer: New York, NY, USA, 2004.

[19] I. M. Gel'fand and G. E. Shilov, Generalized Functions and Operations on Them, T.1. M.: Phyzmatgiz (1959), 470 pages (Russian Edition).

[20] G. E. Shilov, Mathematical Analysis, Second Special Course, Nauka, Moscow, 1965.

[21] V. S. Vladimirov, Generalized Functions in Mathematical Physics, 1979.

[22] I. M. Gelfand and G. E. Shilov, Generalized Functions, Vol. II, Spaces of Fundamental and Generalized Functions, Academic Press, New York, 1968.

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