Journal of Mathematical Sciences: Advances and Applications
Volume 69, 2022, Pages 43-68
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122237

INTEGRAL POTENTIAL EQUATION FOR STURM-LIOUVILLE OPERATOR WITH DELAY

ISMET KALČO and DRAGANA NEDIĆ

Faculty of Philosophy
University of Zenica
Zenica
Bosnia and Herzegovina
e-mail: hanasim@windowslive.com

Faculty of Transport and Traffic Engineering
University of East Sarajevo
Bosnia and Herzegovina
e-mail: dragana.nedic@sf.ues.rs.ba

Abstract

In this paper, we deal with the construction of an integral equation by potential for the Sturm-Liouville operator with a delay for the case when $z \in R \setminus \{0\}$, $\tau \in \left[\frac{\pi}{3}, \frac{2\pi}{5}\right)$, $q \in L_2[0, \pi]$. We solved the integral equation by the step method, constructed the characteristic functions and solved the inverse problem for the given conditions.

2020 Mathematics Subject Classification: 35Qxx.
Keywords and phrases: integral equation, potential, delay, characteristic function, eigenvalues.
Received October 25, 2021; Revised January 14, 2022

1. Introduction

Border task defined with

$$- y''(x) + q(x)y(x - \tau) = \lambda y(x),\ \lambda = z^2,\ \tau \in \left[\frac{\pi}{3}, \frac{2\pi}{5}\right),\ q \in L_2[0, \pi], \quad (1)$$

$$y(x - \tau) \equiv 0,\ x \in [0, \tau), \quad (2)$$

$$y'(0) - h(0) = 0,\ h \in R, \quad (3)$$

$$y'(\pi) + h(\pi) = 0,\ H \in R. \quad (4)$$

We will denote shorter by $D^2 y = z^2 y$.

The solution (1)-(3) is equivalent to the integral equation

$$y(x, z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\int_{\tau}^{x} q(t_1)\sin(x - t_1)zy(t_1 - \tau, z)dt_1. \quad (5)$$

We will solve Equation (5) by the step method. For a brief record of the solution of Equation (5), we define the following functions:

$$a_{sc}(x, z) = \int_{\tau}^{x} q(t_1)\sin(x - t_1)z\cos(t_1 - \tau)zd\,t_1, \quad (6_1)$$

$$a_{s^2}(x, z) = \int_{\tau}^{x} q(t_1)\sin(x - t_1)z\sin(t_1 - \tau)zd\,t_1, \quad (6_2)$$

$$a_{cs}(x, z) = \int_{\tau}^{x} q(t_1)\cos(x - t_1)z\sin(t_1 - \tau)zd\,t_1, \quad (6_3)$$

$$a_{c^2}(x, z) = \int_{\tau}^{x} q(t_1)\cos(x - t_1)z\cos(t_1 - \tau)zd\,t_1, \quad (6_4)$$

$$a_{s^2c}(x, z) = \int_{2\tau}^{x} q(t_1) \int_{\tau}^{t_1-\tau} q(t_2)\sin(x - t_1)z \sin(t_1 - \tau - t_2)z \cos(t_2 - \tau)zd\, t_2 dt_1, \tag{6_5}$$

$$a_{\mathrm{csc}}(x, z) = \int_{2\tau}^{x} q(t_1) \int_{\tau}^{t_1-\tau} q(t_2)\cos(x - t_1)z \sin(t_1 - \tau - t_2)z \cos(t_2 - \tau)zd\, t_2 dt_1, \tag{6_6}$$

$$a_{s^3}(x, z) = \int_{2\tau}^{x} q(t_1) \int_{\tau}^{t_1-\tau} q(t_2)\sin(x - t_1)z \sin(t_1 - \tau - t_2)z \sin(t_2 - \tau)zd\, t_2 dt_1, \tag{6_7}$$

$$a_{cs^2}(x, z) = \int_{2\tau}^{x} q(t_1) \int_{\tau}^{t_1-\tau} q(t_2)\cos(x - t_1)z \sin(t_1 - \tau - t_2)z \sin(t_2 - \tau)zdt_2 dt_1, \tag{6_8}$$

For $x \in (\tau, 2\tau]$, the solution of Equation (5) is given by

$$y(x, z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z} a_{sc}(x, z) + \frac{h}{z^2} a_{s^2}(x, z). \tag{7_1}$$

For $x \in (2\tau, \pi]$, the solution has the form

$$F(z) = y'(\pi, z) + Hy(\pi, z) \cdot y(x, z) = \cos xz + \frac{h}{z}\sin xz$$

$$+ \frac{1}{z} a_{sc}(x, z) + \frac{h}{z^2} a_{s^2}(x, z) + \frac{1}{z^2} a_{s^2c}(x, z) + \frac{h}{z^3} a_{s^3}(x, z). \tag{7_2}$$

Differentiating the function (7_2) by x, we obtain

$$\frac{d}{dx}y(x, z) = -z\sin xz + h\cos xz + a_{s^2}(x, z) + \frac{h}{z}a_{cs}(x, z)$$

$$+\frac{1}{z}a_{csc}(x, z) + \frac{h}{z^2}a_{cs^2}(x, z). \quad (8)$$

Putting $x = \pi$ in the relations (6_j) and in further notations we omit π, and then we use the boundary condition (4) we obtain the characteristic function of the operator D^2y in the following form:

$$F(z) = \left(-z + \frac{hH}{z}\right)\sin\pi z + (h+H)\cos\pi z + a_{c^2}(z) + \frac{h}{z}a_{cs}(z) + \frac{H}{z}a_{sc}(z)$$

$$+\frac{hH}{z}a_{s^2}(z) + \frac{1}{z}a_{csc}(z) + \frac{h}{z^2}a_{cs^2}(z) + \frac{H}{z^2}a_{s^2c}(z) + \frac{hH}{z^3}a_{s^3}(z). \quad (9)$$

Function transformation *F*

Let us define the following functions:

$$q^1(\theta) = \begin{cases} 0, & \theta \in \left[0, \frac{\tau}{2}\right) \cup \left(\pi - \frac{\tau}{2}, \pi\right], \\ q\left(\vartheta + \frac{\tau}{2}\right), & \theta \in \left[\frac{\tau}{2}, \pi - \frac{\tau}{2}\right], \end{cases} \quad (10_1)$$

$$Q^{(2)}(\theta) = \begin{cases} 0, \quad \theta \in [0, \tau) \cup (\pi - \tau, \pi], \\ q(\theta+\tau)\int_{\tau}^{\theta} q(t_1)dt_1 + \int_{\theta+\tau}^{\pi} q(t_1 - \theta)q(t_1)dt_1 - q(\theta)\int_{\theta+\tau}^{\pi} q(t_1)dt_1, \ \theta \in [\tau, \pi - \tau], \end{cases}$$

(10_2)

$$Q^{(2*)}(\theta) = \begin{cases} 0, \quad \theta \in [0, \tau) \cup (\pi - \tau, \pi], \\ -q(\theta+\tau)\int_{\tau}^{\theta} q(t_1)dt_1 + \int_{\theta+\tau}^{\pi} q(t_1 - \theta)q(t_1)dt_1 + q(\theta)\int_{\theta+\tau}^{\pi} q(t_1)dt_1, \ \theta \in [\tau, \pi - \tau], \end{cases}$$

(10_3)

$$a^{(2)}(z) = \int_{\tau}^{\pi-\tau} Q^{(2)}(\theta)\cos(\pi - 2\theta)zd\theta, \quad b^{(2)}(z) = \int_{\tau}^{\pi-\tau} Q^{(2)}(\theta)\sin(\pi - 2\theta)zd\theta, \tag{10_4}$$

$$a^{(2*)}(z) = \int_{\tau}^{\pi-\tau} Q^{(2*)}(\theta)\cos(\pi - 2\theta)zd\theta, \quad b^{(2*)}(z) = \int_{\tau}^{\pi-\tau} Q^{(2*)}(\theta)\sin(\pi - 2\theta)zd\theta, \tag{10_5}$$

$$a^{1}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} q^{1}(\theta)\cos(\pi - 2\theta)zd\theta, \quad b^{1}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} q^{1}(\theta)\sin(\pi - 2\theta)zd\theta. \tag{10_6}$$

Besides, let's put it

$$I_1 = \int_{\tau}^{\pi} q(t_1)dt_1 = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} q^{1}(\theta)d\theta, \quad I_2 = \int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)dt_2 dt_1. \tag{10_7}$$

Using relations to translate the product of trigonometric functions into sums and using relations (10_l), $l = \overline{1, 7}$, we obtain the following equations:

$$a_{sc}(z) = \frac{1}{2}b^{1}(z) + \frac{I_1}{2}\sin(\pi - \tau)z, \; a_{s^2}(z) = \frac{1}{2}a^{1}(z) - \frac{I_1}{2}\cos(\pi - \tau)z, \tag{11_1}$$

$$a_{cs}(z) = -\frac{1}{2}b^{1}(z) + \frac{I_1}{2}\sin(\pi - \tau)z, \; a_{c^2}(z) = \frac{1}{2}a^{1}(z) + \frac{I_1}{2}\cos(\pi - \tau)z, \tag{11_2}$$

$$a_{s^3}(z) = \frac{1}{4} b^{(2*)}(z) - \frac{I_2}{4} \sin(\pi - 2\tau)z,\ a_{cs^2}(z) = \frac{1}{4} a^{(2*)}(z) - \frac{I_2}{4} \cos(\pi - 2\tau)z, \tag{11_3}$$

$$a_{csc}(z) = -\frac{1}{4} b^{(2)}(z) + \frac{I_2}{4} \sin(\pi - 2\tau)z,\ a_{s^2c}(z) = \frac{1}{4} a^{(2)}(z) - \frac{I_2}{4} \cos(\pi - 2\tau)z. \tag{11_4}$$

Using relations (10_l), $l = \overline{1, 7}$ and (11_j), $j = \overline{1, 4}$ the function F is transformed into the following form:

$$\begin{aligned} F(z) = {} & \left(-z + \frac{hH}{z}\right) \sin \pi z + (h + H) \cos \pi z + \frac{1}{2} \hat{a}(z) + \frac{I_1}{2} \cos(\pi - \tau)z \\ & + \frac{H - h}{2z} \hat{b}(z) + \frac{h + H}{2z} I_1 \sin(\pi - 2\tau)z + \frac{hH}{2z^2} \hat{a}(z) - \frac{hH}{2z^2} I_1 \cos(\pi - \tau)z \\ & - \frac{1}{4z} b^{(2)}(z) + \frac{I_2}{4z} \sin(\pi - 2\tau)z + \frac{h}{4z^2} a^{(2*)}(z) \\ & - \frac{hI_2}{4z^2} \cos(\pi - 2\tau)z + \frac{H}{4z^2} a^{(2)}(z) - \frac{HI_2}{4z^2} \cos(\pi - 2\tau)z \\ & + \frac{hH}{4z^3} b^{(2*)}(z) - \frac{hH}{4z^3} I_2 \sin(\pi - 2\tau)z, \quad z \in R. \end{aligned} \tag{12}$$

We put

$$\xi_0 = \frac{h + H}{\pi},\ \xi_1 = \frac{1}{2} I_1 = \frac{\pi}{4} \hat{a}_0, \quad \hat{a}_{2n} = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}(\theta) \cos 2n\theta d\theta,$$

$$\eta_1 = -\frac{\tau}{\pi} \xi_0 \xi_1,\ \eta_2 = \frac{1}{2\pi} \left[(\pi - \tau)\xi_1^2 - \frac{I_2}{2} \right]. \tag{13}$$

Zero z_n of functions F have the following asymptotic decomposition:

$$z_n = \pm\left\{n + \left(\xi_0 + \xi_1 \cos n\pi + \frac{1}{4}\hat{a}_{2n}\right)\frac{1}{n} + (\eta_1 \sin n\tau + \eta_2 \sin 2n\tau)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right\}.$$

Asymptotics of eigenvalues $\lambda_n = z_n^2$ operator D^2 is given with

$$\lambda_n = n^2 + 2\xi_0 + 2\xi_1 \cos n\pi$$
$$+ \frac{1}{2}\hat{a}_{2n} + \frac{1}{n}(2\eta_1 \sin n\tau + 2\eta_2 \sin 2n\tau) + o\left(\frac{\eta_{2n}}{n}\right), \tag{14}$$

where $\eta_{2n} \in L_2$.

Inverse task setting

If we vary the boundary condition at the right end of the segment $[0, \pi]$ with H on H_j, $j = 1, 2$, we get a pair of operators or a pair of boundary value tasks

$$D_j^2 y = z^2 y, \quad D_j^2 = D^2(\tau, h, H_j, q), \quad j = 1, 2. \tag{15}$$

For a given pair $\lambda_{nj}, n \in N_0$, $j = 1, 2$, eigenvalues determining parameters τ, h, H_j, q operator D_j^2 we call the solution of the inverse spectral problem for the delay operator. Based on zero $z_{nj} = \pm\sqrt{\lambda_{nj}}$, $n \in N_0$, $j = 1, 2$, characteristic functions $F_j(z)$ which are the whole functions of exponential type and unit degree of growth, is simply determined through the Adamar product

$$F_j(z) = \dot{\pi}\lambda_{0j}\prod_{n=1}^{\infty}\frac{\lambda_{nj}}{n^2}\left(1 - \frac{z^2}{\lambda_{nj}}\right)\sin \pi z \cdot \prod_{n=1}^{\infty}\left(1 + \frac{\lambda_{nj} - n^2}{n^2 + z^2}\right), z \in C, \ j = 1, 2. \tag{16}$$

Given own values λ_{nj} have zero asymptotics (16), that is,

$$\lambda_{nj} = n^2 + 2\xi_0 + 2\xi_1 \cos n\pi$$
$$+ \frac{1}{4}\hat{a}_{2n} + \frac{1}{n}\left[\left(2\eta_1 + 2\eta_{1j}\right)\sin n\tau + 2\eta_2 \sin 2n\tau\right] + o\left(\frac{\eta_{2n}}{n}\right), \qquad (17)$$

where is $\hat{a}_{2n}$, $\hat{\eta}_{2n} \in l_2$. Suppose they are arrays $\lambda_{nj} - n^2$ oscillating that is, it is $\xi_1 \neq 0$.

Then let's form a string

$$\mu_{nj} = \left[\lambda_{n+2j} - (n+2)^2 - \lambda_{n-2j} + (n-2)^2\right] / \left[\lambda_{n+1j} - (n+1)^2 - \lambda_{n-1j} + (n-1)^2\right]$$
$$= \frac{\cos(n+2)\pi - \cos(n-2)\pi}{\cos(n+1)\pi - \cos(n-1)\pi} + o(1) = 2\cos\tau + o(1).$$

It follows from here

$$\tau = \arccos\left(\frac{1}{2}\right)\lim_{n\to\infty}\mu_{nj}, \; j = 1, 2. \qquad (18)$$

In what follows, we consider it to be $\tau \in \left[\frac{\pi}{3}, \frac{2\pi}{5}\right)$.

Let's choose substrings $\left\{n_k^{(l)}\right\}$, $l = 1, 2,$ string $\{n_k\}$ for which the conditions apply $\cos n_k^{(l)}\tau \neq 0$ and $\left|\cos n_k^{(2)}\tau - \cos n_k^{(1)}\tau\right| \geq \delta > 0 \quad (\forall k)$.

Then from (17), we have

$$2(\xi_0 + \xi_j) = \lim_{k\to\infty}\left[\left(\lambda_{n_k^{(2)}} - \left(n_k^{(2)}\right)^2\right)\cos n_k^{(1)}\tau - \left(\lambda_{n_k^{(1)}} - \left(n_k^{(1)}\right)^2\right)\cos n_k^{(2)}\tau\right] /$$
$$\cos n_k^{(2)}\tau - \cos n_k^{(1)}\tau.$$

According to (17), we have

$$\frac{2}{\pi}(h + H_j) = 2(\xi_0 + \xi_{0j}),$$
$$h + H_j = \pi(\xi_0 + \xi_{0j}), \quad j = 1, 2. \qquad (19)$$

Further from (17) also follows

$$2\xi_1 = \lim_{k\to\infty}\left[\lambda_{n_k^{(l)},\, j} - \left(n_k^{(l)}\right)^2 - 2(\xi_{0j} + \xi_0)\right]/\cos n_k^{(l)}\tau, \quad j = 1, 2,\ l = 1,$$

we put

$$I_1 = 2\xi_1, \quad \hat{a}_0 = \frac{4}{\pi}\xi_1. \tag{20}$$

From (19) follows

$$H_2 - H_1 = \pi(\xi_{02} + \xi_{01}). \tag{21}$$

Putting in (12) $H = H_j,\ j = 1, 2,$ and then using (21) we get

$$h = \lim_{k\to\infty}\left\{\frac{2k + \frac{1}{2}}{H_2 - H_1}\left[F_2\left(2k + \frac{1}{2}\right) - F_1\left(2k + \frac{1}{2}\right)\right] - \frac{I_1}{2}\sin\left(2k + \frac{1}{2}\right)\tau\right\}. \tag{22}$$

Now from (19) we still have

$$H_j = \pi(\xi_0 + \xi_{0j}) - h, \quad j = 1, 2. \tag{23}$$

In this way we proved the following result.

Theorem 1. *Using two sets of eigenvalues* $\lambda_{nj},\ n \in N_0,\ j = 1, 2,$ *with asymptotics* (17) *such that they are arrays* $\lambda_{nj} - n^2$ *oscillating*, *the parameters are uniquely defined* $\tau,\ h,\ H_1,\ H_2,\ \hat{a}_0$ *operators* $D_j^2,\ j = 1, 2.$

2. Construction of Integral Equation by Potential

Characteristic functions F_j of the requested operator, whose eigenvalues are known are given with (16). The unknown potential q satisfies the identity system

$$F(z)\left(-z+\frac{hH_j}{z}\right)\sin \pi z+(h+H_j)\cos \pi z+\frac{1}{2}\hat{a}(z)+\frac{I_1}{2}\cos(\pi-\tau)z$$

$$+\frac{H_j-h}{2z}\hat{b}(z)+\frac{h+H_j}{2z}I_1\sin(\pi-2\tau)z+\frac{hH_j}{2z^2}\hat{a}(z)$$

$$-\frac{hH_j}{2z^2}I_1\cos(\pi-\tau)z-\frac{1}{4z}b^{(2)}(z)+\frac{I_2}{4z}\sin(\pi-2\tau)z$$

$$+\frac{h}{4z^2}a^{(2*)}(z)-\frac{hI_2}{4z^2}\cos(\pi-2\tau)z+\frac{H_j}{4z^2}a^{(2)}(z)$$

$$-\frac{H_jI_2}{4z^2}\cos(\pi-2\tau)z+\frac{hH_j}{4z^3}b^{(2*)}(z)-\frac{hH_j}{4z^3}I_2\sin(\pi-2\tau)z$$

$$\equiv F_j(z), \quad j=1,2,\ z\in C. \tag{24}$$

Let’s introduce new functions

$$\phi_c(z)=\frac{2}{H_2-H_1}[H_2F_1(z)-H_1F_2(z)]+2z\sin \pi z-I_1\cos(\pi-\tau)z, \tag{25_c}$$

$$\phi_s(z)=\frac{2z}{H_2-H_1}[F_2(z)-F_1(z)]-2z\cos \pi z-2h\sin \pi z-I_1\sin(\pi-\tau)z. \tag{25_s}$$

Now the system (24) got in shape

$$\hat{a}(z)-h\frac{\hat{b}(z)}{z}+hI_1\frac{\sin(\pi-\tau)z}{z}-\frac{b^{(2)}(z)}{z}+I_2\frac{\sin(\pi-2\tau)}{2z}$$

$$+h\frac{a^{(2k)}(z)}{2z^2}-hI_2\frac{\cos(\pi-2\tau)z}{2z^2}=\phi_c(z), \tag{26_c}$$

$$\hat{b}(z)+h\frac{\hat{a}(z)}{z}-hI_2\frac{\cos(\pi-\tau)z}{z}+\frac{a^{(2)}(z)}{z}-I_2\frac{\cos(\pi-2\tau)z}{2z}$$

$$+h\frac{b^{(2k)}(z)}{2z^2}-hI_2\frac{\sin(\pi-2\tau)z}{2z^2}=\phi_s(z). \tag{26_s}$$

We will now take the fractions $\frac{\hat{b}(z)}{z}$ and $\frac{\hat{a}(z)}{z}$ once partially integrated on the interval $\left[\frac{\tau}{2}, \theta\right]$, and fractions $\frac{b^{(2)}(z)}{z}$ and $\frac{a^{(2)}(z)}{z}$, we will also partially integrate once, but at intervals $[\tau, \theta]$. Next, fractions $\frac{a^{(2k)}(z)}{2z^2}$ and $\frac{b^{(2k)}(z)}{2z^2}$ we will partially integrate twice on the interval $[\tau, \theta]$. For a brief note, we introduce the following notations:

$$a^{I^1\hat{q}}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)\, d\theta_1 \right) \cos(\pi - 2\theta)\, z d\theta, \tag{27_c}$$

$$b^{I^1\hat{q}}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)\, d\theta_1 \right) \sin(\pi - 2\theta)\, z d\theta, \tag{27_s}$$

$$a^{I^1Q^{(2)}}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\tau}^{\theta} Q^{(2)}(\theta_1)\, d\theta_1 \right) \cos(\pi - 2\theta)\, z d\theta, \tag{28_c}$$

$$b^{I^1Q^{(2)}}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\tau}^{\theta} Q^{(2)}(\theta_1)\, d\theta_1 \right) \cos(\pi - 2\theta)\, z d\theta, \tag{28_s}$$

$$a^{I^2Q^{(2*)}}(z) = \int_{\tau}^{\pi-\tau} \left(\int_{\tau}^{\theta} \int_{\tau}^{\theta_1} Q^{(2*)}(\theta_2)\, d\theta_2 d\theta_1 \right) \cos(\pi - 2\theta)\, d\theta, \tag{29_c}$$

$$b^{I^2Q^{(2*)}}(z) = \int_{\tau}^{\pi-\tau} \left(\int_{\tau}^{\theta} \int_{\tau}^{\theta_1} Q^{(2*)}(\theta_2)\, d\theta_2 d\theta_1 \right) \sin(\pi - 2\theta)\, d\theta. \tag{29_s}$$

By applying partial integration, we get

$$- h\frac{\hat{b}(z)}{z} + hI_1\frac{\sin(\pi-\tau)z}{z} = -2ha^{I^1\hat{q}}(z) + 2hI_1\frac{\sin(\pi-\tau)z}{z}, \qquad (30_c)$$

$$- h\frac{\hat{a}(z)}{z} - hI_1\frac{\cos(\pi-\tau)z}{z} = -2hb^{I^1\hat{q}}(z), \qquad (30_s)$$

$$- \frac{b^{(2)}(z)}{2z} + I_2\frac{\sin(\pi-2\tau)z}{2z} = -a^{I^1Q^{(2)}}(z) + I_2\frac{\sin(\pi-2\tau)z}{z}, \qquad (31_c)$$

$$\frac{a^{(2)}(z)}{2z} + I_2\frac{\cos(\pi-2\tau)z}{2z} = -b^{I^2Q^{(2)}}(z), \qquad (31_s)$$

$$h\frac{a^{(2*)}(z)}{2z^2} - hI_2\frac{\cos(\pi-2\tau)z}{2z^2} = -2ha^{I^2Q^{(2*)}}(z)$$

$$+ h\int_{\tau}^{\pi-\tau}\int_{\tau}^{\theta} Q^{(2*)}(\theta_1)d\theta_1 d\theta\cdot\frac{\sin(\pi-2\tau)z}{z}, \qquad (32_c)$$

$$h\frac{b^{(2*)}(z)}{2z^2} - hI_2\frac{\sin(\pi-2\tau)z}{2z^2} = -2hb^{I^2Q^{(2*)}}(z)$$

$$+ h\int_{\tau}^{\pi-\tau}\int_{\tau}^{\theta} Q^{(2*)}(\theta_1)d\theta_1 d\theta\cdot\frac{\cos(\pi-2\tau)z}{z} - hI_2\frac{\sin(\pi-2\tau)z}{z^2}. \qquad (32_s)$$

Using relations (27_c)-(31_s) and (27_s)-(31_s) system (26_c), receives a form

$$\hat{a}(z) - 2ha^{I^1\hat{q}}(z) - a^{I^1Q^{(2)}}(z) - 2ha^{I^2Q^{(2*)}}(z)$$

$$+ \left[2hI_1\frac{\sin(\pi-\tau)z}{z} + I_2\frac{\sin(\pi-2\tau)z}{z} + h\int_{\tau}^{\pi-\tau}\int_{\tau}^{\theta} Q^{(2*)}(\theta_1)d\theta_1 d\theta\cdot\frac{\sin(\pi-2\tau)z}{z}\right]$$

$$= \phi_c(z), \qquad (32_c)$$

and

$$\hat{b}(z) - 2hb^{I^1\hat{q}}(z) - b^{I^1Q^{(2)}}(z) - 2hb^{I^2Q^{(2*)}}(z)$$

$$+\left[h\int_{\tau}^{\pi-\tau}\int_{\tau}^{\theta} Q^{(2*)}(\theta_1)d\theta_1 d\theta \cdot \frac{\cos(\pi-2\tau)z}{z} - hI_2\frac{\sin(\pi-2\tau)z}{z^2}\right] = \phi_s(z). \tag{32_s}$$

How is it

$$\lim_{z\to 0}\phi_s(z) = \lim_{z\to 0}\frac{a_{s^2(z)}}{z} = \lim_{z\to 0}\frac{a_{s^2c(z)}}{z} = \lim_{z\to 0}\frac{a_{s^3(z)}}{z^2} = 0.$$

Using the first relation in (11_3) and (32_s), we come to the conditions

$$\lim_{z\to 0}\frac{\int_{\tau}^{\pi-\tau}\int_{\tau}^{\theta} Q^{(2*)}(\theta_1)d\theta_1 d\theta \cdot z\cos(\pi-2\tau)z - I_2\sin(\pi-2\tau)z}{z^2} = 0,$$

and hence follows

$$\int_{\tau}^{\pi-\tau}\int_{\tau}^{\theta} Q^{(2*)}(\theta_1)d\theta_1 d\theta = (\pi-2\tau)I_2, \tag{33}$$

Let it be

$$A(z) = \phi_c(z) - 2hI_1\frac{\sin(\pi-\tau)z}{z},\ B(z) = \phi_s(z), \tag{34}$$

$$\alpha(z) = [1 + h(\pi-2\tau)]\frac{\sin(\pi-2\tau)z}{z},$$

$$\beta(z) = h\left[(\pi-2\tau)\frac{\cos(\pi-2\tau)z}{z} - \frac{\sin(\pi-2\tau)z}{z^2}\right].$$

System (32_c) and (32_s) we rewrite in the following form:

$$\hat{a}(z) - 2ha^{I^1 \hat{q}}(z) - a^{I^1 Q^{(2)}}(z) - 2ha^{I^2 Q^{(2*)}}(z) + I_2 \alpha(z) = A(z), \qquad (35_c)$$

$$\hat{b}(z) - 2hb^{I^1 \hat{q}}(z) - b^{I^1 Q^{(2)}}(z) - 2hb^{I^2 Q^{(2*)}}(z) + I_2 \beta(z) = B(z). \qquad (35_s)$$

Let it be $\in R \setminus \{0\}$. In what follows, we will use the following sets of functions on the variable y.

$$a_{2m}^{ch}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}(\theta)\, ch(\pi - 2\theta)\, y \cos 2m\theta d\theta,$$

$$b_{2m}^{ch}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}(\theta)\, ch(\pi - 2\theta)\, y \sin 2m\theta d\theta,$$

$$a_{2m}^{sh}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}(\theta)\, sh(\pi - 2\theta)\, y \cos 2m\theta d\theta,$$

$$b_{2m}^{sh}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}(\theta)\, sh(\pi - 2\theta)\, y \sin 2m\theta d\theta,$$

$$a_{2m}^{(2)ch}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} Q^{(2)}(\theta)\, ch(\pi - 2\theta)\, y \cos 2m\theta d\theta,$$

$$b_{2m}^{(2)ch}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q^{(2)}(\theta)\, ch(\pi - 2\theta)\, y \sin 2m\theta d\theta,$$

$$a_{2m}^{I^1 ch}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)\, d\theta_1 \right) ch(\pi - 2\theta)\, y \cos 2m\theta d\theta,$$

$$b_{2m}^{I^1 ch}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)\, d\theta_1 \right) ch(\pi - 2\theta)\, y \sin 2m\theta d\theta,$$

$$a_{2m}^{I^1 sh}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)\, d\theta_1 \right) sh(\pi - 2\theta)\, y \cos 2m\theta d\theta,$$

$$b_{2m}^{I^1 sh}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)\, d\theta_1 \right) sh(\pi - 2\theta)\, y \sin 2m\theta d\theta,$$

$$a_{2m}^{(2)sh}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q^{(2)}(\theta)\, sh(\pi - 2\theta)\, y \cos 2m\theta d\theta,$$

$$b_{2m}^{(2)sh}(y) = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q^{(2)}(\theta)\, sh(\pi - 2\theta)\, y \sin 2m\theta d\theta,$$

$$a_{2m}^{(I_2^1)ch}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}(\theta)^{(2)}(\theta_1)\,d\theta_1\right)ch(\pi-2\theta)\,y\cos 2m\theta d\theta,$$

$$b_{2m}^{(I_2^1)ch}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}(\theta)^{(2)}(\theta_1)\,d\theta_1\right)ch(\pi-2\theta)\,y\sin 2m\theta d\theta,$$

$$a_{2m}^{(I_2^1)sh}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}(\theta)^{(2)}(\theta_1)\,d\theta_1\right)sh(\pi-2\theta)\,y\cos 2m\theta d\theta,$$

$$b_{2m}^{(I_2^1)ch}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}(\theta)^{(2)}(\theta_1)\,d\theta_1\right)sh(\pi-2\theta)\,y\sin 2m\theta d\theta,$$

$$a_{2m}^{(a_{2*}^2)ch}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}\int\limits_{\tau}^{\theta_1}\theta^{(2*)}(\theta_2)\,d\theta_2 d\theta_1\right)ch(\pi-2\theta)y\cos 2m\theta d\theta,$$

$$b_{2m}^{(I_{2*}^2)ch}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}\int\limits_{\tau}^{\theta_1}\theta^{(2*)}(\theta_2)\,d\theta_2 d\theta_1\right)ch(\pi-2\theta)y\sin 2m\theta d\theta,$$

$$a_{2m}^{(a_{2*}^2)sh}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}\int\limits_{\tau}^{\theta_1}\theta^{(2*)}(\theta_2)\,d\theta_2 d\theta_1\right)sh(\pi-2\theta)y\cos 2m\theta d\theta,$$

$$b_{2m}^{(I_{2*}^2)sh}(y) = \frac{2}{\pi}\int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int\limits_{\frac{\tau}{2}}^{\theta}\int\limits_{\tau}^{\theta_1}\theta^{(2*)}(\theta_2)\,d\theta_2 d\theta_1\right)sh(\pi-2\theta)y\sin 2m\theta d\theta.$$

Using the introduced sequences we come to relations

$$\hat{a}(m+iy) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)\left[\cos(\pi-2\theta)\cos iy(\pi-2\theta) - \sin(\pi-2\theta)\sin iy(\pi-2\theta)\right]d\theta$$

$$= (-1)^m \frac{\pi}{2}\left[a_{2m}^{ch}(y) + ib_{2m}^{sh}(y)\right],$$

$$\hat{b}(m+iy) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)\left[\sin(\pi-2\theta)\cos iy(\pi-2\theta) + \cos(\pi-2\theta)\sin iy(\pi-2\theta)\right]d\theta$$

$$= (-1)^m \frac{\pi}{2}\left[-b_{2m}^{ch}(y) + ia_{2m}^{sh}(y)\right].$$

Analogous to the existing one, the relations also apply

$$a^{I^1q^1}(m+iy) = (-1)^m \frac{\pi}{2}\left[-b_{2m}^{I^1ch}(y) + ib_{2m}^{I^1sh}(y)\right],$$

$$b^{I^1q^1}(m+iy) = (-1)^m \frac{\pi}{2}\left[-b_{2m}^{I^1ch}(y) + ia_{2m}^{I^1sh}(y)\right],$$

$$a^{I^1Q^{(2)}}(m+iy) = (-1)^m \frac{\pi}{2}\left[a_{2m}^{(I_2^1)ch}(y) + ib_{2m}^{(I_2^1)sh}(y)\right],$$

$$b^{I^1Q^{(2)}}(m+iy) = (-1)^m \frac{\pi}{2}\left[-b_{2m}^{(I_2^1)ch}(y) + ia_{2m}^{I^1sh}(y)\right],$$

$$a^{I^2Q^{(2*)}}(m+iy) = (-1)^m \frac{\pi}{2}\left[a_{2m}^{(I_{2*}^2)ch}(y) + ib_{2m}^{(I_2^2*)sh}(y)\right],$$

$$b^{I^2Q^{(2*)}}(m+iy) = (-1)^m \frac{\pi}{2}\left[-b_{2m}^{(I_{2*}^2)ch}(y) + ia_{2m}^{(I_{2*}^2)sh}(y)\right].$$

Next, let's put it on

$$\gamma = h(\pi - 2\tau),\ b_1(m,\ y) = (-1)^m \frac{(1+\gamma)m}{m^2 + y^2} \sin 2m\tau,$$

$$a_1(m,\ y) = (-1)^m \frac{(1+\gamma)m}{m^2 + y^2} \cos 2m\tau,$$

$$b_2(m,\ y) = (-1)^m \frac{1+\gamma}{m^2 + y^2} \sin 2m\tau,\ \ a_2(m,\ y) = (-1)^m \frac{1+\gamma}{m^2 + y^2} \cos 2m\tau.$$

Then we have

$$\alpha(m + iy) = -\ b_1(m,\ y)ch(\pi - 2\tau)y + a_1(m,\ y)sh(\pi - 2\tau)y$$
$$+\ i[b_2(m,\ y)ch(\pi - 2\tau)y + a_2(m,\ y)sh(\pi - 2\tau)y]. \qquad (36_c)$$

Also, let's introduce tags

$$\alpha_{r_e}^{(1)}(m,\ y) = \frac{\gamma m}{m^2 + y^2} \cos 2m\tau,\ \beta_{r_e}^{(1)}(m,\ y) = \frac{\gamma(m^2 - y^2)}{(m^2 + y^2)^2} \sin 2m\tau,$$

$$\beta_{r_e}^{(2)}(m,\ y) = \frac{\gamma y}{m^2 + y^2} \sin 2m\tau,\ \alpha_{r_e}^{(2)}(m,\ y) = \frac{2\gamma m y}{(m^2 + y^2)^2} \cos 2m\tau,$$

$$\beta_{i_m}^{(1)}(m,\ y) = \frac{\gamma m}{m^2 + y^2} \sin 2m\tau,\ \alpha_{i_m}^{(1)}(m,\ y) = \frac{\gamma(m^2 - y^2)}{(m^2 + y^2)^2} \cos 2m\tau,$$

$$\alpha_{i_m}^{(2)}(m,\ y) = \frac{\gamma y}{m^2 + y^2} \cos 2m\tau,\ \beta_{i_m}^{(2)}(m,\ y) = \frac{2\gamma m y}{(m^2 + y^2)^2} \sin 2m\tau.$$

By performing elementary operations we come to a relation

$$\beta(m + iy) = (-1)^m \Big[\Big(\alpha_{r_e}^{(1)}(m,\ y) + \beta_{r_e}^{(1)}(m,\ y)\Big)ch(\pi - 2\tau)y$$
$$+ \Big(\beta_{r_e}^{(2)}(m,\ y) + \alpha_{r_e}^{(2)}(m,\ y)\Big)sh(\pi - 2\tau)y\Big]$$
$$+\ i(-1)^m \Big[\Big(\beta_{i_m}^{(1)}(m,\ y) - \alpha_{i_m}^{(1)}(m,\ y)\Big)sh(\pi - 2\tau)y$$
$$-\ \Big(\alpha_{i_m}^{(2)}(m,\ y) + \beta_{i_m}^{(2)}(m,\ y)\Big)ch(\pi - 2\tau)y\Big]. \qquad (36_s)$$

From (16), we have

$$F_j(m+iy) = R_e(F(m+iy)) + i\,\mathrm{Im}(F(m+iy)),\ j = 1, 2,$$

and then we have

$$A(m+iy) = R_e(A(m+iy)) + i\,\mathrm{Im}(A(m+iy)),$$

$$B(m+iy) = R_e(B(m+iy)) + i\,\mathrm{Im}(B(m+iy)).$$

Based on the latest transformations, we come to new relations

$$a_{2m}^{ch}(y) - 2ha_{2m}^{I^1ch}(y) - a_{2m}^{(I_2^1)ch}(y) - 2ha_{2m}^{(I_{2*}^2)ch}(y)$$

$$+ I_2\left[-\frac{2}{\pi}b_1(m, y)ch(\pi - 2\tau)y + \frac{2}{\pi}a_1(m, y)sh(\pi - 2\tau)y\right]$$

$$= (-1)^m \frac{2}{\pi} R_e(A(m+iy)), \qquad (37_c)$$

$$b_{2m}^{sh}(y) - 2hb_{2m}^{I^1sh}(y) - b_{2m}^{(I_2^1)sh}(y) - 2hb_{2m}^{(I_{2*}^2)sh}(y)$$

$$+ I_2\left[-\frac{2}{\pi}b_2(m, y)ch(\pi - 2\tau)y + \frac{2}{\pi}a_2(m, y)sh(\pi - 2\tau)y\right]$$

$$= (-1)^m \frac{2}{\pi} I_m(A(m+iy)), \qquad (37_s)$$

$$b_{2m}^{ch}(y) - 2hb_{2m}^{I^1ch}(y) - b_{2m}^{(I_2^1)ch}(y) - 2hb_{2m}^{(I_{2*}^2)sh}(y)$$

$$+ I_2\Big[\left(\alpha_{r_e}^{(1)}(m, y) + \beta_{r_e}^{(1)}(m, y)\right)ch(\pi - 2\tau)y$$

$$+ \left(\beta_{r_e}^{(2)}(m, y) + \alpha_{r_e}^{(2)}(m, y)\right)sh(\pi - 2\tau)y\Big] = (-1)^{m+1}\frac{2}{\pi}R_e(B(m+iy)),$$

$$(38_s)$$

$$a_{2m}^{sh}(y) - 2ha_{2m}^{I^1 sh}(y) - a_{2m}^{(I_2^1)sh}(y) - 2ha_{2m}^{(I_{2*}^2)sh}(y)$$

$$+ I_2\Big[\Big(\beta_{i_m}^{(1)}(m,\ y) + \alpha_{i_m}^{(1)}(m,\ y)\Big)sh(\pi - 2\tau)y$$

$$- \Big(\alpha_{i_m}^{(2)}(m,\ y) + \beta_{i_m}^{(2)}(m,\ y)\Big)ch(\pi - 2\tau)y\Big] = (-1)^m \frac{2}{\pi} I_m(B(m + iy)). \tag{38_c}$$

By the relation (37_c), a connection is established between the trigonometric cosine Fourier coefficients of unknown functions

$$\hat{q}(\theta)\, ch\,(\pi - 2\tau)\, y,\ \left(\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)d\theta_1\right) ch(\pi - 2\tau)\, y,\ \left(\int_{\tau}^{\theta} Q^{(2)}(\theta_1)\,d\theta_1\right) ch\,(\pi - 2\tau)\, y,$$

$$\left(\int_{\tau}^{\theta}\int_{\tau}^{\theta_1} Q^{(2*)}(\theta_2)\,dQ_2 dQ_1\right) ch\,(\pi - 2\tau) y,$$

on the one hand, and trigonometric Fourier coefficients in cosines $(-1)^m \frac{2}{\pi} R_e(A(m + iy))$ known functions $f(\theta,\ y)$ and product of number I_2 which is not determined with the cosine coefficients of the known function $\varphi(\theta,\ y)$. The relatio (38_s) establishes a connection between the sine Fourier coefficients of the same functions as in (37_c).

Let us estimate the sine Fourier coefficients of the series and we will from (37_c) and (38_s) get the following equation:

$$\hat{q}(\theta) - 2h\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)d\theta_1 - \int_{\tau}^{\theta} Q^{(2)}(\theta_1)\,d\theta_1 - 2h\int_{\tau}^{\theta}\int_{\tau}^{\theta_1} Q^{(2*)}(\theta_2)\,dQ_2 dQ_1$$

$$= \frac{1}{ch\,(\pi - 2\tau)\, y}\,[f_1\,(\theta,\ y) - I_2\varphi_1\,(\theta,\ y)]. \tag{39_1}$$

Analogously to proceed with (37_s) and (38_c) so far $y \neq 0$ we will come to the equation

$$\hat{q}(\theta) - 2h\int_{\frac{\tau}{2}}^{\theta} \hat{q}(\theta_1)d\theta_1 - \int_{\tau}^{\theta} Q^{(2)}(\theta_1)\,d\theta_1 - 2h\int_{\tau}^{\theta}\int_{\tau}^{\theta_1} Q^{(2*)}(\theta_2)\,dQ_2 dQ_1$$

$$= \frac{1}{sh\,(\pi - 2\tau)\,y}\left[f_2\,(\theta,\ y) - I_2\varphi_2\,(\theta,\ y)\right]. \qquad (39_2)$$

Since the left sides in (39_l), $l = 1, 2$ does not depend on y it is clear that right sides do not depend on y so we have

$$\frac{1}{ch(\pi - 2\tau)y}\left[f_1(\theta,\ y) - I_2\varphi_1(\theta,\ y)\right] = f_1^*(\theta) - I_2\varphi_1^*(\theta),$$

$$\frac{1}{sh(\pi - 2\tau)y}\left[f_2(\theta,\ y) - I_2\varphi_2(\theta,\ y)\right] = f_2^*(\theta) - I_2\varphi_2^*(\theta).$$

And since the left sides (39_l), $l = 1, 2,$ in equal, that is true

$$f_1^*(\theta) - I_2\varphi_1^*(\theta) \equiv f_2^*(\theta) - I_2\varphi_2^*(\theta),\ \theta \in [\tau,\ \pi - \tau]. \qquad (40)$$

Therefore,

$$I_2(\varphi_2^*(\theta) - \varphi_1^*(\theta)) = f_2^*(\theta) - f_1^*(\theta),$$

$$I_2 = \frac{1}{\varphi_2^*(\theta) - \varphi_1^*(\theta)}\,(f_2^*(\theta) - f_1^*(\theta)),\ \forall\theta \in [\tau,\ \pi - \tau]. \qquad (41)$$

We put

$$f(\theta) = f_1^*(\theta) - I_2\varphi_1^*(\theta) = f_2^*(\theta) - I_2\varphi_2^*(\theta),\ \theta \in [\tau,\ \pi - \tau]. \qquad (42)$$

In a special case for $h = 0$, the calculation is significantly simplified, so let's calculate directly funkcije $\varphi_1^*(\theta)$ and $\varphi_2^*(\theta)$, $y \neq 0$. Relation (36_c) at $h = 0$ becomes

$$\alpha(m + iy) = \frac{\sin(\pi - 2\tau)(m + iy)}{m + iy} = \frac{m - iy}{m^2 + y^2} = [\sin(\pi - 2\tau)m$$

$$\cos(\pi - 2\tau)iy + \cos(\pi - 2\tau)m \sin(\pi - 2\tau)iy] = \frac{m - iy}{m^2 + y^2}[-(-1)^m \sin 2m\tau ch$$

$$(\pi - 2\tau)y + i \cos 2m\tau sh(\pi - 2\tau)y] = \frac{(-1)^m}{m^2 + y^2}\{[-m \sin 2m\tau ch(\pi - 2\tau)y + y$$

$$\cos 2m\tau sh(\pi - 2\tau)y] + i[y \sin 2m\tau ch(\pi - 2\tau)y + m \cos 2m\tau sh(\pi - 2\tau)y]\}.$$

From (36_s) we see that it is $\beta(m + iy) \equiv 0$. Relations (37_c) and (37_s) become

$$a_{2m}^{ch}(y) - a_{2m}^{(I_2^1)ch}(y) + \frac{2I_2}{\pi(m^2 + y^2)}[-m \sin 2m\tau ch(\pi - 2\tau)y$$
$$+ y \cos 2m\tau sh(\pi - 2\tau)y] = A_{2m}(y), \qquad (43_c)$$

$$b_{2m}^{sh}(y) - b_{2m}^{(I_2^1)sh}(y) + \frac{2I_2}{\pi(m^2 + y^2)}[y \sin 2m\tau ch(\pi - 2\tau)y$$
$$+ m \cos 2m\tau sh(\pi - 2\tau)y] = B_{2m}(y), \qquad (43_s)$$

relations (38_s) and (38_c) take shape

$$b_{2m}^{ch}(y) - b_{2m}^{(I_2^1)ch}(y) = B_{2m}^{(1)}(y), \qquad (44_s)$$

$$a_{2m}^{sh}(y) - a_{2m}^{(I_2^1)sh}(y) = A_{2m}^{(1)}(y). \qquad (44_c)$$

Zero Fourier coefficient of the function $\varphi_1(\theta, y)$ is $\frac{2}{\pi} \cdot \frac{sh(\pi - 2\tau)y}{y}$ so it is

$$\varphi_1(\theta,\ y) = \frac{sh(\pi - 2\tau)y}{y}$$

$$+ \frac{2}{\pi}\sum_{m=1}^{\infty}\left[\frac{-\ m\sin 2m\tau}{m^2 + y^2}ch(\pi - 2\tau)y + \frac{\cos 2m\tau}{m^2 + y^2}ysh(\pi - 2\tau)y\right]\cos 2m\theta$$

$$= \frac{sh(\pi - 2\tau)y}{\pi y} + \frac{2}{\pi}\sum_{m=1}^{\infty}\left\{ch(\pi - 2\tau)y\frac{-\ m[\sin 2m(\tau + \theta) + y\sin 2m(\tau - \theta)]}{m^2 + y^2}\right.$$

$$\left. +\ ysh(\pi - 2\tau)y\frac{\cos 2m(\tau + \theta) + \sin 2m(\tau - \theta)}{m^2 + y^2}\right\}.$$

We now use known sums

$$\sum_{m=1}^{\infty}\frac{\cos 2m(\theta + \tau)}{m^2 + y^2} = \frac{\pi}{2y}\frac{ch(\pi - 2(\theta + \tau))y}{sh\pi y} - \frac{1}{2y^2}, \quad 2(\theta + \tau) \in [0,\ 2\pi], \tag{45_1}$$

$$\sum_{m=1}^{\infty}\frac{\cos 2m(\theta - \tau)}{m^2 + y^2} = \frac{\pi}{2y}\frac{ch(\pi - 2(\theta - \tau))y}{sh\pi y} - \frac{1}{2y^2}, \quad 2(\theta - \tau) \in [0,\ 2\pi], \tag{45_2}$$

$$\sum_{m=1}^{\infty}\frac{m\sin 2m(\theta - \tau) - m\sin 2m(\theta + \tau)}{m^2 + y^2}$$

$$= \frac{d}{2d\theta}\sum_{m=1}^{\infty}\frac{-\cos 2m(\theta - \tau) + \cos 2m(\theta + \tau)}{m^2 + y^2}$$

$$= \frac{d}{2d\theta}\left[-\frac{\pi}{2y}\frac{ch(\pi - 2(\theta - \tau))y}{sh\pi y} + \frac{1}{2y^2} + \frac{\pi}{2y}\frac{ch(\pi - 2(\theta + \tau))y}{sh\pi y} - \frac{1}{2y}\right]$$

$$= \frac{\pi}{2sh\pi y}sh(\pi - 2(\theta - \tau))y - \frac{\pi}{sh\pi y}sh(\pi - 2(\theta + \tau))y$$

$$= \frac{\pi}{2sh\pi y}[sh(\pi - 2(\theta - \tau))y - sh(\pi - 2(\theta + \tau))y]. \tag{45_3}$$

Therefore,

$$\varphi_1(\theta,\ y) = \frac{sh(\pi - 2\tau)y}{ch(\pi - 2\tau)y\pi y} + \frac{ch(\pi - 2\tau)y}{2sh\pi y}[sh(\pi - 2(\theta - \tau))y - sh(\pi - 2(\theta + \tau))y]$$

$$- \frac{sh(\pi - 2\tau)y}{ch(\pi - 2\tau)y\pi y} + \frac{1}{2ch(\pi - 2\tau)ysh\pi y}[sh(\pi - 2\tau)ych(\pi - 2(\theta + \tau))y$$

$$+ sh(\pi - 2\tau)ych(\pi - 2(\theta - \tau))y],$$

that is

$$\varphi_1(\theta,\ y) = \frac{1}{2ch(\pi - 2\tau)ysh\pi y}[sh(\pi - 2\tau)ych(\pi - 2(\theta + \tau))y$$

$$+ sh(\pi - 2\tau)ych(\pi - 2(\theta - \tau))y + ch(\pi - 2\tau)ysh(\pi - 2(\theta - \tau))y$$

$$- ch(\pi - 2\tau)ysh(\pi - 2(\theta + \tau))y$$

$$= \frac{1}{2ch(\pi - 2\tau)ysh\pi y}(sh2\theta y + sh2(\pi - \theta)y). \tag{45_4}$$

From (44_c), it is obvious that the zero Fourier coefficient of the function $\varphi_2(\theta,\ y)$ equal to zero. Therefore

$$\varphi_2(\theta,\ y) = \frac{2}{\pi}\sum_{m=1}^{\infty}\left[\frac{y\sin 2m\tau}{m^2 + y^2}ch(\pi - 2\tau)y + \frac{m\cos 2m\tau}{m^2 + y^2}sh(\pi - 2\tau)y\right]\sin 2m\theta$$

$$= \frac{1}{\pi}\sum_{m=1}^{\infty}\left\{ych(\pi - 2\tau)y\left[\frac{\cos 2m(\theta - \tau)}{m^2 + y^2} - \frac{\cos 2m(\theta + \tau)}{m^2 + y^2}\right]\right.$$

$$\left. sh(\pi - 2\tau)y\left[\frac{m\sin 2m(\theta + \tau)}{m^2 + y^2} + \frac{m\sin 2m(\theta - \tau)}{m^2 + y^2}\right]\right\}.$$

Now we use relations (45_l), $l = \overline{1, 3}$, so we get

$$\varphi_2(\theta, y) = \frac{ych(\pi - 2\tau)y}{\pi}\left[\frac{\pi}{2ysh\pi y} ch(\pi - 2(\theta - \tau))y - \frac{\pi}{2ysh\pi y} ch(\pi - 2(\theta + \tau))y\right]$$

$$- \frac{sh(\pi - 2\tau)y}{2\pi} \frac{d}{d\theta}\left[\sum_{m=1}^{\infty} \frac{\cos 2m(\theta + \tau)}{m^2 + y^2} + \sum_{m=1}^{\infty} \frac{\cos 2m(\theta - \tau)}{m^2 + y^2}\right]$$

$$= \frac{ch(\pi - 2\tau)y}{2sh\pi y}[ch(\pi - 2(\theta - \tau))y - ch(\pi - 2(\theta + \tau))y]$$

$$- \frac{sh(\pi - 2\tau)y}{2\pi} \frac{d}{d\theta}\left[\frac{\pi}{2y} \frac{ch(\pi - 2(\theta + \tau))y}{sh\pi y} + \frac{\pi}{2y} \frac{ch(\pi - 2(\theta - \tau))y}{sh\pi y} - \frac{1}{2y^2}\right]$$

$$= \frac{ch(\pi - 2\tau)y}{2sh\pi y}[ch(\pi - 2(\theta - \tau))y - ch(\pi - 2(\theta + \tau))y]$$

$$- \frac{sh(\pi - 2\tau)y}{2\pi}\left[- \frac{\pi}{sh\pi y} sh(\pi - 2(\theta + \tau))y - \frac{\pi}{sh\pi y} sh(\pi - 2(\theta - \tau))y\right]. \tag{45_5}$$

Therefore,

$$\varphi_2(\theta, y) = \frac{1}{sh(\pi - 2\theta)ysh\pi y}[ch(\pi - 2\tau)ych(\pi - 2(\theta - \tau))y$$

$$- ch(\pi - 2\tau)ych(\pi - 2(\theta + \tau))y + sh(\pi - 2\tau)ysh(\pi - 2(\theta + \tau))y$$

$$+ sh(\pi - 2\tau)ysh(\pi - 2(\theta - \tau))y]$$

$$= \frac{1}{sh(\pi - 2\theta)ysh\pi y}[ch2(\pi - \theta)y - 2ch2\theta y]. \tag{45_6}$$

From (45_5) and (45_6), we see that it is $\varphi_2(\theta, y) \neq \varphi_1(\theta, y)$, $y \neq 0$. This condition is also met $h \neq 0$, but the account requires long calculations that are elementary character.

Conclusion

In this paper, we construct an integral equation by potential for an inverse Sturm-Liouville type problem with a delay for the case when z is a real number other than zero. It remains to construct an integral equation of potentials for the case when z is a complex number in the following considerations and works.

References

[1] V. Ambarzumjan, Uber eine Frage der Eigenwerttheorie, Zeitschrift für Physik 53(9-10) (1929), 690-695.

DOI: https://doi.org/10.1007/BF01330827

[2] G. Borg, Eine Eine Umkehrung der Sturm-Liouvilleschen eigenwertaufgabe: Bestimmung der differentialgleichung durch die eigenwerte, Acta Mathematica 78 (1946), 1-96.

DOI: https://doi.org/10.1007/BF02421600

[3] G. Freiling and V. Yurko, Inverse problems for Sturm-Liouville differential operators with a constant delay, Applied Mathematics Letters 25(11) (2012), 1999-2004.

DOI: https://doi.org/10.1016/j.aml.2012.03.026

[4] M. Pikula and N. Mihaljević, The inverse Sturm-Liouville type problem with changeable delay, Mathematica Montisnigri 16 (2003), 41-68.

[5] I. Kalćo, Construction solutions differential operator type Sturm-Liouville border task with linear delay, Bulletin of the International Mathematical Virtual Institute 4(2) (2014), 123-128.

[6] M. Pikula, I. Kalćo and F. Destović, Direct and inverse spectral assignment for the operator Sturm-Liouville type with linear delay, Mathematica Montisnigri 37 (2016), 91-104.

[7] I. Kalćo, M. Pikula and F. Destović, Direct and inverse task for the operator Sturm-Liouville type with linear delay, Journal of Mathematical Sciences: Advances and Applications 44 (2017), 51-71.

DOI: https://doi.org/10.18642/jmsaa_7100121759

■