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PRODUCT OF FACTORIALS IN THE SEQUENCE $\{g_n\}$

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Abstract

Let $\{g_n\}_{n\geq 0}$ defined as $g_n = g_{n-1} + g_{n-2}$ with $g_1 = 1$ and $g_2 = a$ $(a \in \mathbb{Z}^+)$. We characterize 2-adic valuation of the sequence $\{g_n\}$ for $a \equiv 3, 4, 5, 6 \pmod 8$. Afterwards, we solve the equation $g_n = m_1! m_2! \dots m_k!$ completely.

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1. Introduction

Several mathematicians are interested in finding factorials in special sequences as Fibonacci, Lucas etc. Luca [2] showed that the terms of F_3, F_6, F_{12}, L_0, and L_3 can be written as the products of the factorials where F_n and L_n are n-th Fibonacci and Lucas numbers, respectively. Moreover, the largest product of distinct Fibonacci numbers which is a product of factorials was shown in [3] by Luca. Later, Grossman and Luca [4] proved that the equation

$$F_n = m_1! + m_2! + \dots + m_k!,$$

has finitely many positive integers n for fixed k integer. The case $k \leq 2$ has been determined. The case $k = 3$ was solved by Bollman, Hernandez and Luca.

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. Recently, Marques and Lengyel [5] characterized 2-adic valuation of T_n and showed that T_n is factorial when $n = 1, 2, 3,$ and 7. For other details about the special sequences, we refer the papers [7] and [8].

Let $a \in \mathbb{Z}^+$. For $n \geq 3$, define the sequence $\{g_n\}$ as

$$g_n = g_{n-1} + g_{n-2},$$

with $g_1 = 1$ and $g_2 = a$. We get Fibonacci and Lucas sequence if taking $a = 1$ and $a = 3$, respectively. In this paper, we characterize 2-adic order of g_n for $a \equiv 3, 4, 5, 6 \pmod 8$ and solve the equation $g_n = \prod_{j=1}^{k} m_j$.

Our theorems are following:

Theorem 1. *For* $n \geq 1$, *we have*

$$\nu_2(g_n) = \begin{cases} 0 & \textit{if } n \equiv 1, 2 \pmod 3, \\ \nu_2(a-1) & \textit{if } n \equiv 0 \pmod 6, \\ \nu_2(a+1) & \textit{if } n \equiv 3 \pmod 6, \end{cases}$$

for $a \equiv 3, 5 \pmod 8$.

If $a \equiv 4, 6 \pmod 8$, *then*

$$\nu_2(g_n) = \begin{cases} 0 & \textit{if } n \equiv 0, 1 \pmod 3, \\ \nu_2(a) & \textit{if } n \equiv 2 \pmod 6, \\ \nu_2(a-2) & \textit{if } n \equiv 5 \pmod 6. \end{cases}$$

Remark 1. As seen above theorem, $\nu_2(g_n) \leq 2$ follows for $n \geq 1$.

Theorem 2. *Assume that* $m_i \geq 2$ $(1 \leq i \leq k)$. *Then the solutions of the equation*

$$g_n = \prod_{j=1}^{k} m_j! \tag{1.1}$$

are given as follows:

a	6	5	4	3	12	11	36	35
n	2	3	2	3	2	3	2	3
$\prod_{j=1}^{k} m_j$	3!	3!	$(2!)^2$	$(2!)^2$	2!3!	2!3!	$(3!)^2$	$(3!)^2$

Before proceeding further, some considerations will be needed for the convenience of the reader.

Lemma 1. *Let* m *and* n *be positive integers and* p *is a prime number. If* $\nu_p(n) \neq \nu_p(m)$, *then*

$$\nu_p(m+n) = \inf\{\nu_p(n),\ \nu_p(m)\}$$

holds.

Lemma 2. *For any integer* $k \geq 1$ *and* p *prime, we have*

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_p(k!) \leq \frac{k-1}{p-1},$$

where $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

Proof. We refer Lemma 2.4 in the paper of Marques [6]. □

Lemma 3. *For* n, k *and* s *are positive integers, we get*

$$g_{rn+s} = L_r g_{r(n-1)+s} - (-1)^r g_{r(n-2)+s},$$

where L_r *is r-th Lucas number and* $0 \leq s \leq r-1$.

Proof. It can be proven by the Binet formula of the sequence $\{g_n\}$. □

2. Proof of Theorem 1

We will prove only the case $a \equiv 3 \pmod 8$. The other cases can be proven by using the similar way. In order to show $\nu_2(g_n) = 0$, we need to prove that $g_n \equiv 1 \pmod 2$. To avoid unnecessary repetitions we shall prove only that case $n \equiv 1 \pmod 3$. For that, we shall proceed by induction on n. The base case $n = 1$, $g_1 = 1$. We may suppose that $g_{3n-2} \equiv 1 \pmod 2$ and $g_{3n-5} \equiv 1 \pmod 2$. By Lemma 3, we deduce that

$$g_{3n+1} = 4g_{3n-2} + g_{3n-5}.$$

After taking modulo 2 of both sides, then

$$\begin{aligned} g_{3n+1} &\equiv 4 \cdot 1 + 1 \pmod 2 \\ &\equiv 1 \pmod 2 \end{aligned}$$

follows as claimed.

Now assume that $n \equiv 0 \pmod 6$. Now the base case is $n = 6$. Since $g_6 = 5a + 3$ and $a \equiv 3 \pmod 8$, then

$$\begin{aligned} \nu_2(g_6) &= \nu_2(5a + 3) \\ &= \nu_2(5(8k + 3) + 3) \\ &= \nu_2(40k + 18), \end{aligned}$$

for some $k \in \mathbb{Z}^+$. As $1 = \nu_2(40k + 10) \neq \nu_2(8) = 3$, then we obtain that $\nu_2(g_6) = \nu_2(40k + 10)$ by Lemma 1. It yields that $\nu_2(g_6) = \nu_2(a - 1)$ as claimed. As $\nu_2(a-1) = 1$ for $a \equiv 3 \pmod 8$, we will show $g_n \equiv 2 \pmod 4$ for $n \equiv 0 \pmod 6$. Assume that $g_{6(n-1)} \equiv 2 \pmod 4$ and $g_{6(n-2)} \equiv 2 \pmod 4$. By Lemma 3, we have

$$\begin{aligned} g_{6n} &= L_6 g_{6(n-1)} - g_{6(n-2)} \\ &= 18 g_{6(n-1)} - g_{6(n-2)}. \end{aligned}$$

Then $g_{6n} \equiv 2 \pmod 4$ follows which gives that $\nu_2(g_{6n}) = \nu_2(a-1) = 1$. Since the case $n \equiv 3 \pmod 6$ can be proven similarly, we omit this case. Therefore, we prove the Theorem 1.

3. Proof of Theorem 2

Assume that $k \geq 3$. Then we arrive at a contradiction after taking 2-adic valuation of both sides of the Equation (1.1) since $\nu_2\left(\prod_{j=1}^{k} m_j!\right) \geq 3$ and $\nu_2(g_n) \leq 2$. So, $k = 1$ and $k = 2$ follow. The possible solutions are given in Theorem 2.

4. Open Question

In this paper, we characterize 2-adic order g_n for $a \equiv 3, 4, 5, 6 \pmod 8$. What is 2-adic valuation of the g_n for $a \equiv 0, 1, 2, 7 \pmod 8$? We leave this problem as a question for reader.

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