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# $P_0$ - ALMOST DISTRIBUTIVE FUZZY LATTICE

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**Abstract**

The concept of $P_0$-Almost Distributive Fuzzy Lattice ($P_0$ - *ADFL*) with a finite chain base is introduced and we prove basic properties about $P_0$ - *ADFL*. Necessary and sufficient conditions for characterization of monotone and disjoint representations of an element $x$ in $P_0$ - *ADFL* are investigated.

## 1. Introduction

Swamy and Rao in [10] introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of existing lattice and ring theoretic generalization of a Boolean algebra and observed that the set $PI(R)$ of all principal ideals of an Almost Distributive Lattice $(R, \vee, \wedge, 0, m)$ with a maximal element $m$, form a distributive lattice. Epstein and Horn in [2] introduced the concept of a $P_0$-lattice. Later in [13], Traczyk was studied and explored properties of $P_0$-lattice. has good application in computer and logic theory and the concept of a $P_0$-Almost Distributive Lattice was introduced by Rao and Mihret in [7].

The concept of fuzzy set was introduced by Zadeh in [14] and this concept was adapted by Goguen in [4] and Sanchez in [12] use to define and study fuzzy relations. In this paper, we use fuzzy partial order relation defined in [5] and the kinds of ideals of fuzzy lattice in [6] to extend some important properties of $P_0$-Almost Distributive Lattice to $P_0$-Almost Distributive Fuzzy Lattice.

## 2. Preliminaries

**Definition 2.1** ([9])**.** An algebra $(R, \vee, \wedge, 0)$ is called an Almost Distributive Lattice if it satisfies the following axioms:

(1) $a \vee 0 = a$.

(2) $0 \wedge a = 0$.

(3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

(6) $(a \vee b) \wedge b = b$, for all $a, b, c \in R$.

**Theorem 2.1** ([9]). *Let $m$ be a maximal element in an Almost Distributive Lattice $R$ and $a \in R$. Then the following are equivalent*:

(1) *$m$ is maximal element of a poset $(R, \leq)$.*

(2) $a \wedge m = a$.

(3) $a \vee m = m$.

(4) *$a \vee m$ is maximal.*

**Definition 2.2** ([10]). Let $R$ be an Almost Distributive Lattice with a maximal element $m$ and $B(R) = \{a \in R | a \wedge b = 0$ and $a \vee b$ is maximal for some $b \in R\}$. Then $(B(R), \vee, \wedge)$ is a relatively complemented Almost Distributive Lattice and it is called the *Birkhoff center of $R$.*

**Lemma 2.1** ([9]). *Let $(R, \vee, \wedge, 0)$ be an Almost Distributive Lattice. Then the following conditions hold for all $a, b, c \in R$ :*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$.

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$.

(3) *$\wedge$ is associative.*

(4) $a \wedge b \wedge c = b \wedge a \wedge c$.

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$.

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$.

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$.

$a \wedge a = a$ *and* $a \vee a = a$.

(9) $0 \vee a = a$ *and* $a \wedge 0 = 0$.

**Definition 2.3** ([9])**.** Let $R$ be an Almost Distributive Lattice with 0. Then for all $a, b \in R$, define $a \leq b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$. Then $(R, \leq)$ is a poset.

**Definition 2.4** ([7])**.** Let $R$ be an Almost Distributive Lattice with 0. Then a unary operation $\star$ on $R$ is called a *pseudo-complementation on* $R$ if, for $a, b \in R$:

(1) $a \wedge a^{\star} = 0$.

(2) $a \wedge b = 0 \Rightarrow a^{\star} \wedge b = b$.

(3) $(a \vee b)^{\star} = a^{\star} \wedge b^{\star}$.

So that $a^{\star}$ is called a pseudo-complement of $a \in R$ and $R$ is called a pseudo-complemented Almost Distributive Lattice. An element $a$ in $R$ is said to be *dense* if $a^{\star} = 0$.

**Definition 2.5** ([7])**.** A pseudo-complemented Almost Distributive Lattice $(R, \vee, \wedge, \star, 0, m)$ is called a stone Almost Distributive Lattice if, for any $a \in R$, $a^{\star} \vee a^{\star\star} = 0^{\star}$.

**Definition 2.6** ([9])**.** Let $R$ be an Almost Distributive Lattice with 0. A non empty subset $I$ of $R$ is an ideal of $R$, if it satisfy the following conditions:

(1) $a, b \in I \Rightarrow a \vee b \in I$.

(2) $a \in I$ and $x \in R \Rightarrow a \wedge x \in R$.

**Theorem 2.2** ([9])**.** *Let* $R$ *be an Almost Distributive Lattice with* 0. *Then for any* $a, b \in R$, *we have the following*:

(1) $(a] = \{a \wedge x | x \in R\}$.

(2) $a \in (b] \Leftrightarrow b \wedge a = a \Leftrightarrow (a] \subseteq (b] \Leftrightarrow a \wedge x \leq b \wedge x$, *for all* $x \in R$.

**Lemma 2.2** ([9])**.** *For any* $a, b \in R$, *the following hold*:

(1) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$.

(2) $(a] \vee (b] = (a \vee b] = (b \vee a]$.

**Definition 2.7** ([7])**.** Let $(R, \vee, \wedge, 0, m)$ be an ADL with Birkhoff center $B(R)$ of $R$ is called a pseudo-supplemented Almost Distributive Lattice if, for each $x \in R$, there exists $b \in B(R)$ such that

(1) $x \wedge b = b$.

(2) If $c \in B(R)$ such that $x \wedge c = c$, then $b \wedge c = c$. In this case, $b \wedge m$ is uniquely determined by $x$ and is denoted by $!x$. We call $!x$ the pseudo-supplement of $x$.

**Definition 2.8** ([7])**.** Let $R$ be a bounded distributive lattice and $B(R)$ the center of $R$. A chain base of $R$ is a finite sequence $0 = e_0, e_1, \ldots, e_{n-2}, e_{n-1} = 1$ such that $R$ is generated by $B(R) \cup \{e_0, e_1, \ldots, e_{n-1} = 1\}$. If $R$ has a chain base, then $R$ is called a $P_0$-lattice.

**Definition 2.9** ([7])**.** Let $R$ be an Almost Distributive Lattice with 0 and maximal elements. Then $R$ is called a $P_0$-Almost Distributive Lattice if $(PI(R), \vee, \wedge, 0, R)$ is a $P_0$-Lattice.

**Definition 2.10** ([8])**.** Let $X$ be a set, A function $A : X \times X \to [0, 1]$ is said to be *fuzzy partial order relation* if it satisfies the following condition:

(1) $A(x, x) = 1, \forall x \in X$ that is $A$ is reflexive.

(2) $A(x, y) > 0$, and $A(y, x) > 0$ implies that $x = y$. That is $A$ is antisymmetric.

(3) $A(x, z) \geq \sup_{y \in X} \min[A(x, y), A(y, z)] > 0$. That is $A$ is transitive.

If $A$ is a fuzzy partial order relation in a set $X$, then $(X, A)$ is called a fuzzy partial order relation or fuzzy poset.

**Definition 2.11** ([5])**.** Let $(X, A)$ be a fuzzy poset. Then $(X, A)$ is a fuzzy lattice if and only if $x \vee y$, and $x \wedge y$ exists for all $x, y \in X$.

**Definition 2.12** ([5])**.** Let $(X, A)$ be a fuzzy lattice. Then $(X, A)$ is distributive if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$, for all $x, y, z \in X$.

**Definition 2.13** ([5])**.** Let $(X, A)$ be a fuzzy lattice and $Y \subseteq X$. Then $Y$ is an ideal of $(X, A)$.

(1) If $x \in X$, $y \in Y$ and $A(x, y) > 0$, then $x \in Y$.

(2) If $x, y \in Y$, then $x \vee y \in Y$.

**Definition 2.14** ([6])**.** Let $(X, A)$ be a fuzzy lattice and $x \in X$. Then the set determined by $\downarrow x = \{y \in X : A(y, x) > 0\}$ is called principal ideal of $(X, A)$ generated by $x$. The family of all ideals of a fuzzy lattice $(X, A)$ will be denoted by $I(X)$.

**Definition 2.15** ([2])**.** Let $(R, \vee, \wedge, 0)$ be an algebra, and we call $(R, A)$ is an Almost Distributive Fuzzy Lattice (ADFL) if the following condition satisfied:

(1) $A(a \vee 0, a) > 0$.

(2) $A(0 \wedge a, 0) > 0$.

(3) $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$.

(4) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1.$

(5) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1.$

(6) $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1,$ for all $a, b, c \in R.$

Throughout this paper, we write $(R, A)$ for an Almost Distributive Lattice, and $R$ be an Almost Distributive Lattice $(R, \vee, \wedge, 0)$ with maximal element, $(PI(R), A)$ a principal ideal fuzzy lattice of an Almost Distributive Fuzzy Lattice $(R, A)$ and $B_A(PI(R))$ the Birkhoff center of a principal ideal fuzzy lattice $(PI(R), A)$.

In $(R, A)$ and $(PI(R), A)$ $A$ represents $A : R \times R \to [0, 1]$ and $A : PI(R) \times PI(R) \to [0, 1]$, respectively.

## 3. $P_0$ - Almost Distributive Fuzzy Lattice

**Definition 3.1.** Let $PI(R)$ be a principal ideal of an Almost Distributive Lattice $R$, and $(R, A)$ be an Almost Distributive Fuzzy Lattice with maximal element $m$. Then $(PI(R), A)$ is the set of all principal ideal fuzzy lattice of $(R, A)$.

**Definition 3.2.** Let $(PI(R), A)$ be the principal ideal fuzzy lattice of an Almost Distributive Fuzzy Lattice $(R, A)$. Then $(a]_A = \{x \in R | A(x, a \wedge x) > 0\}$, for all $x \in R$.

**Definition 3.3.** Let $(PI(R), A)$ be a principal ideal fuzzy lattice of an Almost Distributive Fuzzy Lattice $(R, A)$.

Define $B_A(PI(R)) = \{(a] \in PI(R) | (a]_A \cap (b]_A \subseteq (0]_A$ and $R \subseteq (a \vee b]_A$ for some $b \in R\}$. Then $B_A(PI(R))$ is called the Birkhoff center of the principal ideal fuzzy lattice $(PI(R), A)$.

**Lemma 3.1.** *The set of all principal ideal fuzzy lattice of* $(R, A)$ *forms a distributive fuzzy lattice.*

**Proof.** Let $(R, A)$ be an Almost Distributive Fuzzy Lattice, and $(PI(R), A)$ be the principal ideal fuzzy lattice of $(R, A)$.

Let $(0], (a], (b] \in (PI(R), A) \Rightarrow (0], (a], (b] \in PI(R)$.

(1) $(a]_A \cap (b]_A = (a \wedge b]_A \supseteq (0]_A$, since 0 is the least element. Imply that $(a]_A \cap (b]_A \supseteq (0]_A$.

$$(a]_A \vee (b]_A = (a \vee b]_A = R.$$

So that we have $R \subseteq (a \vee b]_A$. Hence $(PI(R), A)$ is bounded. Since $PI(R) \subseteq R$ and $R$ is an ADL with 0. Clearly $(PI(R), A)$ is an Almost Distributive Fuzzy Lattice. It remain to show the binary operation $\vee, \wedge$ is commutative and $\vee$ is right distributive over $\wedge$.

(2) $(a]_A \cap (b]_A = (a \wedge b]_A = (b \wedge a]_A = (b]_A \cap (a]_A$. since $(a] \cap (b] = (a \wedge b] = (b \wedge a]$.

Hence $\wedge$ is commutative. $(a]_A \vee (b]_A = (a \vee b]_A = (b \vee a]_A = (b]_A \vee (a]_A$, since $(a] \vee (b] = (a \vee b] = (b \vee a] = (b] \vee (a]$. $\vee$ is commutative.

(3) Let $(a], (b], (c] \in (PI)(R), A) \Rightarrow (a], (b], (c] \in PI(R)$. Now

$$\begin{aligned}
&[(a]_A \cap (b]_A] \vee (c]_A \\
&= [(a \wedge b]_A] \vee (c]_A \\
&= (c]_A \vee [(a \wedge b]_A] \text{ by (2) above} \\
&= (c \vee (a \wedge b)]_A \\
&= (c \vee a]_A \wedge (c \vee b]_A \text{ by LD}\vee \\
&= ((c]_A \vee (a]_A) \wedge ((c]_A \vee (b]_A) \\
&= ((a]_A \vee (c]_A) \wedge ((b]_A \vee (c]_A) \text{ by (2) above.}
\end{aligned}$$

Hence $[(a]_A \wedge (b]_A] \vee (c]_A = ((a]_A \vee (c]_A) \wedge ((b]_A \vee (c]_A)$ holds. So that $(PI(R), A)$ is distributive fuzzy lattice. Thus $(PI(R), A)$ is a bounded distributive fuzzy lattice on $[(0], (a \vee b]]$. □

**Lemma 3.2.** *Let* $(R, A)$ *be an Almost distributive Fuzzy Lattice with maximal element* $m$ *and* $a, b \in R$. *Then the following hold*:

(1) $(a]_A = R$ *if and only if* $a$ *is maximal.*

(2) $(a]_A \subseteq (b]_A$ *if and only if* $A(a, b) > 0$.

(3) $(a]_A = (b]_A$ *if and only if* $A(a \wedge m, b \wedge m) = A(b \wedge m, a \wedge m) = 1$.

(4) *Let* $B_A(R)$ *and* $B_A(PI(R))$ *be the center of an Almost Distributive Fuzzy Lattice* $(R, A)$, *and the set of principal fuzzy ideal of* $(R, A)$, *respectively, and* $a \in R$. *Then* $(a] \in B_A(PI(R))$ *if and only if* $a \in B_A(R)$.

**Proof.** (1) Let $(a]_A = R$, for any $x \in R$ we have $x \in (a]_A \Rightarrow A(x, a \wedge x) > 0$. Since $a \wedge x \leq x$, we have $A(a \wedge x, x) > 0$. Hence $a \wedge x = x$ by antisymmetry property of $A$. Imply that $x \leq a$ by Theorem 2.9. Thus $a$ is maximal element. On the other hand, suppose $a$ is maximal element. Then $a \wedge x = x$, for all $x \in R$. Imply that $x \leq a \wedge x$. So that we have $A(x, a \wedge x) > 0$ and hence $x \in (a]_A$. Therefore $R \subseteq (a]_A$. Clearly $a \in (a]_A \Rightarrow a \in R$ and hence $(a]_A \subseteq R$. Thus $(a]_A = R$.

(2) Suppose $(a]_A \subseteq (b]_A$. We need to show $A(a, b) > 0$. Since $a \in (a]_A \Rightarrow a \in (b]_A \Rightarrow A(a, b \wedge a) > 0$. Since $b \wedge a \leq a$, we have $A(b \wedge a, a) > 0$. Thus $b \wedge a = a$ by antisymmetry property of $A$. Implies that $a \leq b$ by Theorem 2.9. Therefore $A(a, b) > 0$.

On the other hand, suppose $A(a, b) > 0$. Then we show $(a]_A \subseteq (b]_A$. Now, $A(a, b) > 0$ implies $b \wedge a = a \Rightarrow b \wedge a \leq a$ and $a \leq b \wedge a \Rightarrow A(a, b \wedge a) > 0$. So that $a \in (b]_A$ and hence $(a]_A \subseteq (b]_A$.

(3) Let $a, b \in R$ and $m$ is maximal element in $(R, A)$. Then $(a]_A = (b]_A$ if and only if $(a]_A \subseteq (b]_A$ and $(b]_A \subseteq (a]_A \Leftrightarrow A(a, b \wedge a) > 0$ and $A(b, a \wedge b) > 0 \Leftrightarrow A(a \wedge m, b \wedge a \wedge m) = A(b \wedge a \wedge m, a \wedge m) = 1$ and $A(b \wedge m, a \wedge b \wedge m) = A(a \wedge b \wedge m, b \wedge m) = 1$ $A(b \wedge m, a \wedge m) > 0$ and $A(a \wedge m, b \wedge m) > 0$, since $a \wedge b \wedge m \leq a \wedge m$ and $b \wedge a \wedge m \leq b \wedge m$. $\Leftrightarrow a \wedge m = b \wedge m$ by antisymmetry property of $A$.

$$A(a \wedge m, b \wedge m) = A(b \wedge m, a \wedge m) = 1.$$

(4) Let $B_A(PI(R)) = B_1$, suppose $(a] \in B_1$. Then there exist $(b] \in (PI(R), A)$ such that $(a]_A \cap (b]_A = (a \wedge b]_A \subseteq (0]_A$. Since $(0]_A \subseteq (a \wedge b]_A$. We have $(a \wedge b]_A = (0]_A$. Again, $R \subseteq (a \vee b]_A$. As $(a \vee b]_A \subseteq R$. We get $(a]_A \vee (b]_A = (a \vee b]_A = R$. Hence $a \wedge b = 0$ and $a \vee b$ is maximal. Implies $A(a \wedge b, 0) > 0$ and $A((a \vee b) \vee x, a \vee b) > 0$, for all $x \in R$. Thus $a \in B_A(R)$.

Conversely, suppose $a \in B_A(R)$. Then there exist $b \in R$ such that $A(a \wedge b, 0) > 0$ and $A((a \vee b) \vee x, a \vee b) > 0$. Since 0 is the least element, $0 \leq a \wedge b$ and hence $A(0, a \wedge b) > 0$. Imply that $a \wedge b = 0$ by antisymmetry property of $A$. Again for any $x \in R$, $a \leq (a \vee b) \vee x$, $0 \leq a \Rightarrow b = 0 \vee b \leq (a \vee b) \vee x. \Rightarrow a \vee b \leq (a \vee b) \vee x$ and hence $A(a \vee b, (a \vee b) \vee x) > 0$.

Therefore $(a \vee b) \vee x = a \vee b$. Thus $a \vee b$ is maximal. Now $(a]_A \cap (b]_A = (a \wedge b]_A = (0]_A$, since $a \wedge b = 0$. Imply that $(a \wedge b]_A \subseteq (0]_A$. $(a]_A \vee (b]_A = (a \vee b]_A = R$ since $a \vee b$ is maximal. So that we have $R \subseteq (a \vee b]_A$. Thus $(a] \in B_A(PI(R)) = B_1$. □

**Lemma 3.3.** *Let* $(R, A)$ *be an Almost Distributive Fuzzy Lattice with Birkhoff center* $B_A(R)$ *and* $\{e_1, e_2, \ldots, e_{n-1}\} \subseteq R$. *Then* $T = \{\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) : b_i \in B_A(R)\}$, *for* $1 \leq i \leq n-1$ *is a sub Almost Distributive Fuzzy Lattice of* $(R, A)$.

**Definition 3.4.** The sub Almost Distributive Fuzzy Lattice $T$ in Lemma 3.6 is called the sub Almost Distributive Fuzzy Lattice of $(R, A)$ generated by $B_A(R) \cup \{e_1, e_2, \ldots, e_{n-1}\}$.

**Definition 3.5.** Let $(R, A)$ be a bounded distributive fuzzy lattice $0, 1$, and $B_A(R)$ the Birkhoff center of $(R, A)$. A chain base of $(R, A)$ is a finite sequence $A(e_{i-1}, e_i) > 0$, for $1 \leq i \leq n-1$ such that $(R, A)$ is generated by $B_A(R) \cup \{e_1, e_2, \ldots, e_{n-1}\}$ in which every element $x \in R$ satisfy $A(x, \vee_{i=1}^{n-1}(b_i \wedge e_i)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i), x) = 1$, where $b_i \in B_A(R)$. If $(R, A)$ has a chain base ,then $(R, A)$ is called a $P_0$ - fuzzy lattice.

**Theorem 3.1.** $R$ *is a* $P_0$ *- lattice if and only if* $(R, A)$ *is a* $P_0$ *- fuzzy lattice.*

**Proof.** Let $R$ be a $P_0$ - lattice and $x \in R$. Then $x = \vee_{i=1}^{n-1}(b_i \wedge e_i)$, for $b_i \in B(R)$ and $\{0 = e_0, e_1, e_2, \ldots, e_{n-1}\}$, for $1 \leq i \leq n-1$ is a chain base. $R$ is generated by $B(R) \cup \{e_0, e_1, \ldots, e_{n-1}\}$ with $0 = e_0 \leq e_1 \leq e_2 \leq \ldots \leq e_{n-1} = 1$. Let $(R, A)$ be an Almost Distributive Fuzzy Lattice. Then $A(x, \vee_{i=1}^{n-1}(b_i \wedge e_i)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i), x) = 1$. Since $0 = e_0 \leq e_1 \leq e_2 \leq \ldots \leq e_{n-1} = 1$. We have $A(e_0, e_1) > 0$, $A(e_1, e_2) > 0, \ldots, A(e_{n-2},$

$e_{n-1}) > 0$. Hence $A(e_{i-1}, e_i) > 0$ for $1 \leq i \leq n-1$ and $(R, A)$ is generated by $B_A(R) \cup \{e_0, e_1, \ldots, e_{n-1}\}$, $b_i \in B_A(R)$. Hence $(R, A)$ is a $P_0$-fuzzy lattice. On the other hand, suppose $(R, A)$ be $P_0$-fuzzy lattice. For $x \in R$, we have $A(x, \vee_{i=1}^{n-1}(b_i \wedge e_i)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i), x) = 1 > 0$, $\Rightarrow A(x, \vee_{i=1}^{n-1}(b_i \wedge e_i) > 0$ and $A(\vee_{i=1}^{n-1}(b_i \wedge e_i), x) > 0$. So that we have $x = \vee_{i=1}^{n-1}(b_i \wedge e_i)$ by antisymmetry property of $A$.

Since $(R, A)$ is a $P_0$-fuzzy lattice, we have $A(0 = e_0, e_1) > 0$, $A(e_1, e_2) > 0, \ldots, A(e_{n-2}, e_{n-1} = 1) > 0 \Rightarrow A(e_0, e_1) > 0 \Leftrightarrow e_0 = e_0 \wedge e_1 \leq e_1 A(e_1, e_2) > 0 \Leftrightarrow e_1 = e_1 \wedge e_2 \leq e_2 \Rightarrow e_0 \leq e_1 \leq e_2$ proceeding in the same manner, we have $A(e_{n-2}, e_{n-1}) > 0 \Leftrightarrow e_{n-2} = e_{n-2} \wedge e_{n-1} \leq e_{n-1} \Rightarrow 0 = e_0 \leq e_1 \leq e_2 \leq \ldots \leq e_{n-1} = 1$. Hence $\{0 = e_0, e_1, \ldots, e_{n-1}\}$ form a chain base of $R$ and $b_i \in B(R)$. Imply that $R$ is generated by $B(R) \cup \{e_0, e_1, \ldots, e_{n-1}\}$. Thus $R$ is a $P_0$-lattice. □

**Definition 3.6.** If $(R, A)$ be an Almost Distributive Fuzzy Lattice with maximal elements, then $(R, A)$ is called $P_0$-Almost Distributive Fuzzy Lattice if, $(PI(R), A)$ is a $P_0$-fuzzy lattice.

In the following theorem we give elementwise characterization of a $P_0$-Almost Distributive Fuzzy Lattice.

**Theorem 3.2.** *If $(R, A)$ be an Almost Distributive Fuzzy Lattice with maximal element $m$ and Birkhoff center $B_A(R)$, then $(R, A)$ is a $P_0$-Almost Distributive Fuzzy Lattice if and only if there exist elements $\{0 = e_0, e_1, \ldots, e_{n-1}\}$ in $(R, A)$ such that*

(1) $A(m, e_{n-1} \wedge m) > 0$.

(2) $A(e_{i-1}, e_i) > 0$, *for* $1 \leq i \leq n-1$ *and*

(3) *For any* $x \in R$, *there exist* $b_i \in B_A(R)$ *such that* $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$.

**Proof.** Let $(R, A)$ be a $P_0$-Almost Distributive Fuzzy Lattice with maximal element $m$ and Birkhoff center $B_A(R)$. Then $(PI(R), A)$ with maximal element $R$ is a $P_0$-fuzzy lattice and there exist elements $e_0, e_1, \ldots, e_{n-1} \in R$ such that $(0]_A = (e_0]_A \subseteq (e_1]_A, (e_1]_A \subseteq (e_2]_A, \ldots,$ $(e_{n-2}]_A \subseteq (e_{n-1}]_A$ and for any $x \in R$, $(x]_A = \vee_{i=1}^{n-1}((b_i]_A \wedge (e_i]_A) = \vee_{i=1}^{n-1}$ $(b_i \wedge e_i)]_A = (\vee_{i=1}^{n-1}(b_i \wedge e_i)]_A$, with $(b_i]_A \in B_A(PI(R))$ the Birkhoff center of $(PI(R), A)$. Since $(e_{i-1}]_A \subseteq (e_i]_A$, for $1 \leq i \leq n-1$. We get $A(e_{i-1}, e_i) > 0$, for $1 \leq i \leq n-1$ and hence condition (2) holds. Now, $(e_{n-1}]_A = R$ imply that $m \in (e_{n-1}]_A$ and hence $A(m, e_{n-1} \wedge m) > 0$, since $(m] = R$ for $m$ is maximal element of $R$. So that condition (1) holds. If $x \in R$, then $(x]_A = \vee_{i=1}^{n-1}(b_i]_A \wedge (e_i]_A) = (\vee_{i=1}^{n-1}(b_i \wedge e_i)]_A$, for $1 \leq i \leq n-1$, $b_i \in B_A(R)$ and since $x = \vee_{i=1}^{n-1}(b_i \wedge e_i)$. So that condition (3) holds. Therefore, the three condition holds. On the other hand, assume $(R, A)$ be an ADFL with maximal element $m$ satisfying condition (1), (2) and (3) above. From (2), $A(e_{i-1}, e_i) > 0$. Hence $(e_{i-1}]_A \subseteq (e_i]_A$, for $1 \leq i \leq n-1$. Let $(x] \in (PI(R), A)$. Then by (3), there exist $b_1, b_2, \ldots, b_{n-1} \in B_A(R)$ such that $(x \wedge m]_A = (\vee_{i=1}^{n-1}(b_i \wedge e_i)]_A$, for $b_i \in B_A(PI(R))$, $1 \leq i \leq n-1$. From (1), we have $A(m, e_{n-1} \wedge m) > 0$ and hence $(e_{n-1}]_A = R$, since $(m] = R$ for a maximal element $m$ in $R$.

Therefore $(PI(R), A)$ is a $P_0$-fuzzy lattice generated by $B_A(PI(R))$ $\cup \{(0] = (e_0], (e_1], \ldots, (e_{n-1}]\}$. Hence $(R, A)$ is $P_0$-Almost Distributive Fuzzy Lattice. □

**Definition 3.7.** Let $(R, A)$ be an Almost Distributive Fuzzy Lattice with Birkhoff center $B_A(R)$ is a $P_0$-Almost Distributive Fuzzy Lattice if and only if there exist $\{0 = e_0, e_1, \dots, e_{n-1} = 1\} \subseteq R$ such that $A(e_{i-1}, e_i) > 0$, for $1 \le i \le n-1$, $A(m, e_{n-1} \wedge m) > 0$ and $(R, A)$ is generated by $B_A(R) \cup \{e_1, e_2, \dots, e_{n-1}\} \subseteq (R, A)$.

**Definition 3.8.** A set $\{0 = e_0, e_1, \dots, e_{n-1}\}$ of elements in a $P_0$-Almost Distributive Fuzzy Lattice $(R, A)$ satisfying conditions (1), (2) and (3) of Theorem 3.11 is called a chain base of $(R, A)$. From now on wards, when we write $((R, A); e_0, e_1, \dots, e_{n-1})$ is a $P_0$-Almost Distributive Fuzzy Lattice with a chain base $\{0 = e_0, e_1, \dots, e_{n-1} = 1\}$ and Birkhoff center $B_A(R)$. Where $(R, \vee, \wedge, 0, m)$ is a $P_0$-Almost Distributive Lattice.

**Definition 3.9.** Let $((R, A); e_0, e_1, \dots, e_{n-1})$ be a $P_0$-Almost Distributive Fuzzy Lattice and $x \in R$ such that $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m))$ $= A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1(*)$.

(1) If $A(b_{i+1}, b_i) > 0$, for $1 \le i \le n-1$. Then $(*)$ is called a monotone representation of $x$, abbreviated as mon. rep.

(2) If $A(b_i \wedge b_j, 0) > 0$, for $i \ne j$, then $(*)$ is called a disjoint representation of $x$, abbreviated as disj. rep.

**Theorem 3.3.** *Every element in a $P_0$-Almost Distributive Fuzzy Lattice, $(R, A)$ has both a monotone and a disjoint representation.*

**Proof.** Let $(R, A)$ be an Almost Distributive Fuzzy Lattice and $((R, A); e_0, e_1, \dots, e_{n-1})$ be a $P_0$-Almost Distributive Fuzzy Lattice.

(1) $x \in (R, A)$ and $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$ be a disjoint representation of $x$, where $b_i \in B_A(R)$, for each $i$. Define $A(c_i, \vee_{i=1}^{n-1}(b_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge m), c_i) = 1$, for $1 \leq i \leq n-1$. Then $c_i \in B_A(R)$, for each $i$ and

$$A(c_{i+1} \wedge c_i, c_{i+1}) = A(\vee_{i=1}^{n-1}(b_i \wedge m) \wedge \vee_{j=i+1}^{n-1}(b_j \wedge m), \vee_{j=i+1}^{n-1}(b_j \wedge m))$$

$$= A(\vee_{j=i+1}^{n-1}(b_j \wedge m), \vee_{j=i+1}^{n-1}(b_j \wedge m)) = 1.$$

Hence $A(c_{i+1} \wedge c_i, c_{i+1}) > 0$. Similarly $A(c_{i+1}, c_{i+1} \wedge c_i) > 0$. Therefore $c_{i+1} \wedge c_i = c_{i+1}$. So that we get $c_{i+1} \leq c_i$. Hence $A(c_{i+1}, c_i) > 0$, for $1 \leq i \leq n-1$. Also

$$A(c_i \wedge e_i \wedge m, \vee_{j=i}^{n-1}(b_j \wedge e_j \wedge m))$$

$$= A([\vee_{j=i}^{n-1}(b_j \wedge m) \wedge e_i \wedge m], \vee_{j=i}^{n-1}(b_j \wedge e_i \wedge m)) = 1.$$

Hence $A(c_i \wedge e_i \wedge m, \vee_{j=i}^{n-1}(b_j \wedge e_j \wedge m)) = 1$. Similarly $A(\vee_{j=i}^{n-1}(b_j \wedge e_i \wedge m), c_i \wedge e_i \wedge m) = 1$.

Therefore $A(c_i \wedge e_i \wedge m, \vee_{j=i}^{n-1}(b_j \wedge e_j \wedge m)) = A(\vee_{j=i}^{n-1}(b_j \wedge e_i \wedge m), c_i \wedge e_i \wedge m) = 1$. Then for $1 \leq j \leq n-1$, we get $A((c_i \wedge e_i \wedge m) \vee (c_{i+1} \wedge e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m)) = A(\vee_{j=i}^{n-1}(b_j \wedge e_i \wedge m) \vee \vee_{j=i+1}^{n-1}(b_j \wedge e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_j \wedge e_{i+1} \wedge m)) = A((b_i \wedge e_i \wedge m) \vee \vee_{j=i+1}^{n-1}(b_j \wedge e_i \vee e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m)) = A((b_i \wedge e_i \wedge m) \vee (\vee_{j=i+1}^{n-1} b_j) \wedge e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m)) = A((b_{i+1} \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m)) = 1.$

Hence $A((c_i \wedge e_i \wedge m) \vee (c_{i+1}e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m)) = 1$. Similarly $A(b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m), (c_i \wedge e_i \wedge m) \vee (c_{i+1} \wedge e_{i+1} \wedge m)) = 1$. Therefore $A((c_i \wedge e_i \wedge m) \vee (c_{i+1} \wedge e_{i+1} \wedge m), (b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m)) = A((b_i \wedge e_i \wedge m) \vee (b_{i+1} \wedge e_{i+1} \wedge m), (c_i \wedge e_i \wedge m) \vee (c_{i+1} \wedge e_{i+1} \wedge m)) = 1$.

$$A(\vee_{j=1}^{n-1}(c_i \wedge e_i \wedge m), x \wedge m) = A(\vee_{j=i}^{n-1}(\vee_{i=1}^{n-1}(b_i \wedge m) \wedge e_i \wedge m), x \wedge m)$$

$$= A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m)$$

$$= A(x \wedge m, x \wedge m) = 1.$$

Similarly $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) = 1$. Thus $A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), x \wedge m) = A(x \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m)) = 1$ is a monotone representation of $x$ in $(R, A)$.

(2) Let $x \in R$ and $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$ be a monotone representation of $x$. Where $b_i \in B_A(R)$ for each $i$. Define $A(c_i, b_i \wedge b_{i+1}^m) = A(b_i \wedge b_{i+1}^m, c_i) = 1$, for $1 \leq i \leq n-2$ and $A(c_{n-1}, c_{n-1}) > 0$. Where $b_{i+1}^m$ is the complement of $b_{i+1} \wedge m$ in $[0, m]$. That is $A(b_{i+1}^m \wedge (b_{i+1} \wedge m), 0) > 0$ and $A(m, b_{i+1}^m \vee (b_{i+1} \wedge m)) > 0$. Where $m$ is a maximal element in $(R, A)$. $A(c_i \wedge c_j, 0) > 0$, for $i \neq j \Rightarrow i+1 \neq j+1$.

$= A((b_i \wedge b_{i+1}^m) \wedge (b_j \wedge b_{j+1}^m), 0)$ since $c_i = b_i \wedge b_{i+1}^m$, $c_j = b_j \wedge b_{j+1}^m$,

$= A((b_{i+1}^m \wedge b_i \wedge b_j) \wedge (b_{j+1}^m), 0) = A(b_{i+1}^m \wedge 0 \wedge b_{j+1}^m, 0) = A(0, 0) = 1 > 0$.

Hence $A(c_i \wedge c_j, 0) > 0$, for $i \neq j$. Now, we show $A(\vee_{k=j}^{n-1}(c_k \wedge e_k \wedge m), \vee_{k=j}^{n-1}(b_k \wedge e_k \wedge m)) = A(\vee_{k=j}^{n-1}(b_k \wedge e_k \wedge m), \vee_{k=j}^{n-1}(b_k \wedge e_k \wedge m)) = 1(*)$.

By using induction on $j$. Suppose $j = n-1$. Then $A(c_{n-1}, b_{n-1}) = A(b_{n-1}, c_{n-1}) = 1$. So that $A(c_{n-1} \wedge e_{n-1} \wedge m, b_{n-1} \wedge e_{n-1} \wedge m) = A(b_{n-1} \wedge e_{n-1} \wedge m, c_{n-1} \wedge e_{n-1} \wedge m) = 1$. Thus (*) is true for $j = n-1$. Assume $i > 1$ and (*) holds for $j = i$. That is $A(\vee_{k=i}^{n-1}(c_k \wedge e_k \wedge m), \vee_{k=i}^{n-1}(b_k \wedge e_k \wedge m)) = A(\vee_{k=i}^{n-1}(b_k \wedge e_k \wedge m), \vee_{k=i}^{n-1}(c_k \wedge e_k \wedge m)) = 1$. To prove (*) holds for $j = i-1$.

$$A(\vee_{k=i-1}^{n-1}(c_k \wedge e_k \wedge m), \vee_{k=i-1}^{n-1}(b_k \wedge e_k \wedge m))$$

$$= A((b_{i-1} \wedge e_{i-1} \wedge m) \vee (b_i \wedge e_i \wedge m) \vee \vee_{k=i+1}^{n-1}(b_k \wedge e_k \wedge m),$$

$$\vee_{k=i-1}^{n-1}(b_k \wedge e_k \wedge m)) = A(\vee_{k=i-1}^{n-1}(b_k \wedge e_k \wedge m),$$

$$\vee_{k=i-1}^{n-1}(b_k \wedge e_k \wedge m)) = 1 > 0.$$

Hence $A(\vee_{k=i-1}^{n-1}(c_k \wedge e_k \wedge m), \vee_{k=i-1}^{n-1}(b_k \wedge e_k \wedge m)) > 0$. Similarly $A(\vee_{k=i-1}^{n-1}(b_k \wedge e_k \wedge m), \vee_{k=i-1}^{n-1}(c_k \wedge e_k \wedge m)) > 0$. Now,

$$A(x \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m))$$

$$= A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m))$$

$$= A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m))$$

$$= \mathrm{A}(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = 1 > 0.$$

Hence $A(x \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), x \wedge m) = 1$ is a disjoint representation of $x$ in $(R, A)$. □

**Corollary 3.1.** *Let* $(R, A)$ *be an Almost Distributive Fuzzy Lattice with a maximal element* $m$ *and Birkhoff center* $B_A(R)$, $\{0 = e_0, e_1, \ldots, e_{n-1}\} \subseteq R$ *such that* $A(e_{i-1}, e_i) > 0$, *for* $1 \leq i \leq n-1$ *and* $A(m, e_{n-1}, \wedge m) > 0$ *and* $(R_0, A)$ *be the sub Almost Distributive Fuzzy Lattice of* $(R, A)$ *generated by* $B_A(R) \cup \{e_0, e_1, e_2, \ldots, e_{n-1}\}$. *Then every element of* $(R_0, A)$ *has both a monotone and disjoint representation.*

**Theorem 3.4.** *Let* $(R, A)$ *be an Almost Distributive Fuzzy Lattice and* $((R, A); e_0, e_1, \ldots, e_{n-1})$ *be a* $P_0$*-Almost Distributive Fuzzy Lattice. Let* $x \in R$, $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$ *be a disjoint representation of* $x \in R$. *Define* $A(c_i \wedge m, (b_i \vee b_{i+1} \vee \ldots \vee b_{n-1}) \wedge m) = A((b_i \vee b_{i+1} \vee \ldots \vee b_{n-1}) \wedge m, c_i \wedge m) = 1$, *for* $1 \leq i \leq n-2$. *Then* $A(x \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), x \wedge m) = 1$ *is a monotone representation of x.*

**Lemma 3.4.** *Let* $((R, A); e_0, e_1, \ldots, e_{n-1})$ *be a* $P_0$*-Almost Distributive Fuzzy Lattice, and* $x, y \in (R, A)$. *If* $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$ *and* $A(y \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), y \wedge m) = 1$ *are mono. rep. of x and y respectively, then*

(1) $A(x \wedge y \wedge m, \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m), x \wedge y \wedge m) = 1$ *is a mono. rep. of* $x \wedge y$.

(2) $A((x \vee y) \wedge m, \vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m), (x \vee y) \wedge m) = 1$ *is a mono. rep. of* $x \vee y$.

**Proof.** (1) Since $A(b_{i+1}, b_i) > 0$ and $A(c_{i+1}, c_i) > 0$, we get $A(b_{i+1} \wedge c_{i+1}, b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}) = A(b_i \wedge b_{i+1} \wedge c_i \wedge c_{i+1}, b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1})$, since $(b_{i+1} = b_i \wedge b_{i+1}$ and $c_{i+1} = c_i \wedge c_{i+1} = A(b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}, b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}) = 1 > 0$, since $b_i \wedge b_{i+1} \wedge c_i \wedge c_{i+1} = b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}$. Hence $A(b_{i+1} \wedge c_{i+1}, b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}) > 0$. Similarly $A(b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}, b_{i+1} \wedge c_{i+1}) > 0$. Therefore $A(b_{i+1} \wedge c_{i+1}, b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}) = A(b_i \wedge c_i \wedge b_{i+1} \wedge c_{i+1}, b_{i+1} \wedge c_{i+1}) = 1$.

$A(x \wedge y \wedge m, \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m))$

$= A(x \wedge m \wedge y \wedge m, \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m)$

$= A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) \wedge \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m))$

$= A(\vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m), \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m)) = 1 > 0.$

Hence $A(x \wedge y \wedge m, \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m)) > 0$. Similarly $A(\vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m), x \wedge y \wedge m) > 0$. Therefore $A(x \wedge y \wedge m, \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge c_i \wedge e_i \wedge m), x \wedge y \wedge m) = 1$ are mono. rep. of $x \wedge y$.

(2) $A((b_i \vee c_i) \wedge (b_{i+1} \vee c_{i+1}), b_{i+1} \vee c_{i+1})$

$= A([(b_i \vee c_i) \wedge b_{i+1}] \vee [(b_i \vee c_i) \wedge c_{i+1}] \wedge t, b_{i+1} \vee c_{i+1})$

$= A([(b_i \wedge b_{i+1}) \vee (b_i \wedge c_{i+1}) \vee (c_i \wedge b_{i+1}) \vee (c_i \wedge c_{i+1})] \wedge t, b_{i+1} \vee c_{i+1})$

$= A(b_{i+1} \vee (b_i \wedge c_{i+1}) \vee (c_i \wedge b_{i+1}) \vee c_{i+1}] \wedge t, b_{i+1} \vee c_{i+1})$

$= A([b_{i+1} \vee c_{i+1} \vee s] \wedge t, b_{i+1} \vee c_{i+1})$, for $s = (b_i \wedge c_{i+1}) \vee (c_i \wedge b_{i+1})$ and $t = b_{i+1} \vee c_{i+1} = A((t \vee s) \wedge t, t) = A(t, t) = 1$. Hence $A((b_i \vee c_i) \wedge (b_{i+1} \vee c_{i+1}), b_{i+1} \vee c_{i+1}) = 1 > 0$. Similarly $A(b_{i+1} \vee c_{i+1}, (b_i \vee c_i) \wedge (b_{i+1} \vee c_{i+1})) = 1$. Therefore $A((b_i \vee c_i) \wedge (b_{i+1} \vee c_{i+1}), b_{i+1} \vee c_{i+1}) = A(b_{i+1} \vee c_{i+1}, (b_i \vee c_i) \wedge (b_{i+1} \vee c_{i+1})) = 1$. Now,

$A((x \vee y) \wedge m, \vee_{i=1}^{n-1}(b_i \vee c_i) \wedge e_i \wedge m))$

$= A((x \wedge m) \vee (y \wedge m), \vee_{i=1}^{n-1}(b_i \vee c_i) \wedge e_i \wedge m))$

$= A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) \vee \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), \vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m))$

$= A(\vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m), \vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m)) = 1 > 0.$

Hence $A((x \vee y) \wedge m, \vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m) > 0$. Similarly $A(\vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m, (x \vee y) \wedge m) > 0$. Therefore $A((x \vee y) \wedge m, \vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m) = A(\vee_{i=1}^{n-1}((b_i \vee c_i) \wedge e_i \wedge m, (x \vee y) \wedge m) = 1$ are mono. rep. of $x \vee y$. □

## 4. Conclusion

The concept of $P_0$-Almost Distributive Fuzzy Lattice with finite chain base has been introduced and we prove basic properties of $P_0$-Almost Distributive Fuzzy Lattice and the set of all principal ideal fuzzy lattice also discussed. Every element of a $P_0$-Almost Distributive Fuzzy Lattice has both a monotone and a disjoint representation.

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