

STRONG STATIONARY DUALITY AND ALGEBRAIC DUALITY FOR CONTINUOUS TIME MÖBIUS MONOTONE MARKOV CHAINS

Pan Zhao

Institute of Fundamental and Interdisciplinary Sciences, Beijing Union
University, Beijing 100101, P. R. China

Abstract

Under the assumption of Möbius monotonicity, we develop the theory of strong stationary duality for continuous time Markov chains on the finite partially ordered state space, we also construct a nonexplosive algebraic duality for continuous time Markov chains on \mathbb{Z}_+^d . Finally, we present an application to the two-dimensional birth and death chain.

Keywords: strong stationary times, strong stationary duality, algebraic duality, Möbius function, Möbius monotonicity.

*Corresponding author.

E-mail address: ldtzaopan@buu.edu.cn (Pan Zhao).

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1. Introduction

A strong stationary time (SST) introduced by Aldous and Diaconis [1, 2] is a randomized stopping time T for the discrete time chain \mathbf{X} such that X_T has the stationary distribution and is independent of T . The SST is a powerful tool to deal with the convergence to stationarity for a Markov chain. In order to get the SST, Diaconis and Fill [4] suggested a way to construct strong stationary duality (SSD), such that the absorption time of the SSD chain is equal in distribution to an SST for \mathbf{X} . Then one can bound the SST in the original chain through the absorption time. Especially, Diaconis and Fill showed a tractable case in [4] (Theorem 4.6) on the linearly ordered state space. In this case, they showed how to construct SSD, under the assumption of stochastic monotonicity. Diaconis and Fill [5] extended the theory of SSD for the discrete time chain on the countable state space. Fill [6] extended the theory of SSD for the continuous time chain.

Lorek and Szekli [9, 10] generalized the above tractable case for the discrete time chain, they considered the finite partially ordered state space instead of the linearly ordered state space. Since stochastic monotonicity is not sufficient in this case, they used Möbius monotone instead of stochastic monotone.

In this paper, we construct SSD for continuous time Möbius monotone Markov chains on the finite partially ordered state space. Also, we construct a nonexplosive algebraic duality for continuous time Markov chains on \mathbb{Z}_+^d . In order to state conveniently, we introduce some symbols and definitions in [6].

Let $\mathbf{X} = (X_t)_{t \geq 0}$ be an ergodic continuous time Markov chain on the finite state space E , with initial distribution π_0 , the conservative Q matrix Q , the stationary distribution π , and the transition function $P(t)$. We write $\mathbf{X} \sim (\pi_0, Q, P(t))$ for short below. Let $\mathbf{X}^* \sim (\pi_0^*, Q^*, P(t))$ on

the finite state space E^* . Assume \mathbf{X}^* has an absorbing state x_a . Let $\Lambda(x, y), x \in E^*, y \in E$, be a link, or a transition kernel such that $\Lambda(x_a, \cdot) = \pi$. From [6], we know that \mathbf{X}^* is an SSD for \mathbf{X} with respect to Λ if and only if the algebraic duality equations

$$\pi_0 = \pi_0^* \Lambda \text{ and } \Lambda Q = Q^* \Lambda \tag{1.1}$$

hold. Moreover, $\Lambda Q = Q^* \Lambda$ is equivalent to $\Lambda P(t) = P^*(t) \Lambda$ for all $t \geq 0$. In this case, the absorbing time $T_{x_a}^*$ that \mathbf{X}^* hits x_a is an SST for \mathbf{X} . For the countable state space E , Fill also gived SSD theory with some assumptions, which are called as “general setting” in [6]. However, if $E = \mathbb{Z}_+^d$, the assumptions are not easy to tractable. But we can construct a nonexplosive algebraic duality, which satisfies (1.1) and can also be used to study the convergence to stationarity for the Markov chain.

We will assume E is a finite partially ordered state space or \mathbb{Z}_+^d below. If E is a finite partially ordered state space, define a partial order \preceq on E . If $E = \mathbb{Z}_+^d$, define the natural partial order \preceq , i.e., for $x, y \in E$, $x \preceq y \Leftrightarrow x_i \leq y_i (\forall 1 \leq i \leq d)$.

From [8] and [9], we define \downarrow -Möbius monotonicity of a nonnegative function and a Q matrix on E as follows.

Definition 1.1. A nonnegative function f on E is \downarrow -Möbius monotone: if there exists a nonnegative function m on E such that $f(x) = \sum_{y \in E: y \succeq x} m(y)$ for any $x \in E$.

Define $\{x\}^\downarrow = \{y \in E : y \preceq x\}$ for any $x \in E$. Let $Q(x, \{y\}^\downarrow) = \sum_{z \in E: z \preceq y} Q(x, z)$ for any $x, y \in E$.

Definition 1.2. A Q matrix Q on $E \times E$ is \downarrow -Möbius monotone: if on $E \times E$, there exists a matrix M which is nonnegative off the diagonal such that

$$Q(x, \{y\}^\downarrow) = \sum_{z \in E: z \succeq x} M(z, y), \quad \forall x, y \in E.$$

Definition 1.3. Let $\mathbf{X} \sim (\pi_0, Q, P(t))$ on the countable state space. Suppose $P(t)$ is the minimal Q process and let τ_n be the n -th jumping time of \mathbf{X} , then Q is called *nonexplosive* if for any initial distribution, $\mathbb{P}(\lim_{n \rightarrow \infty} \tau_n = \infty) = 1$.

Now we can state our main theorems.

Theorem 1.4. Let $\mathbf{X} \sim (\pi_0, Q, P(t))$ be an ergodic continuous time Markov chain on a finite partially ordered state space $E = \{x^1, \dots, x^M\}$, with a unique maximal state x^M and the stationary distribution π . Suppose Q is conservative, and define the time reversal of Q as $\bar{Q} = (\bar{q}_{x,y})$ with $\bar{q}_{x,y} = \frac{\pi(y)}{\pi(x)} q_{y,x}$, let $H(x) = \sum_{y \in E: y \preceq x} \pi(y)$ and $g(x) = \frac{\pi_0(x)}{\pi(x)}$. Then there exists an SSD chain $\mathbf{X}^* \sim (\pi_0^*, Q^*, P^*(t))$ on E with the link kernel: $\Lambda(x, y) = I_{(y \preceq x)} \frac{\pi(y)}{H(x)}$, $x, y \in E$, if and only if the following two conditions hold:

- (i) g is \downarrow -Möbius monotone on E ;
- (ii) \bar{Q} is \downarrow -Möbius monotone on E .

In this case, the SSD chain (for details, see Section 3) is uniquely determined by

$$\frac{\pi_0(x)}{\pi(x)} = \sum_{y \in E: y \succeq x} \frac{\pi_0^*(y)}{H(y)} \quad \text{and} \quad \bar{Q}(y, \{x\}^\downarrow) = \sum_{z \in E: z \succeq y} \frac{H(x)}{H(z)} q_{x,z}^*,$$

for any $x, y \in E$.

Theorem 1.5. Let $\mathbf{X} \sim (\pi_0, Q, P(t))$ be an ergodic continuous time Markov chain on $E = \mathbb{Z}_+^d$, with the stationary distribution π . Suppose Q is conservative and nonexplosive. Define the time reversal of Q as $\bar{Q} = (\bar{q}_{x,y})$ with $\bar{q}_{x,y} = \frac{\pi(y)}{\pi(x)} q_{y,x}$, let $H(x) = \sum_{y \in E: y \leq x} \pi(y)$ and $g(x) = \frac{\pi_0(x)}{\pi(x)}$. Then there exists a nonexplosive algebraic duality chain $\mathbf{X}^* \sim (\pi_0^*, Q^*, P^*(t))$ ($P^*(t)$ is the minimal Q^* process) on E with the link kernel: $\Lambda(x, y) = I_{(y \leq x)} \frac{\pi(y)}{H(x)}$, $x, y \in E$, if and only if the following three conditions hold:

- (i) g is \downarrow -Möbius monotone on E ;
- (ii) \bar{Q} is \downarrow -Möbius monotone on E ;
- (iii) Q^* is nonexplosive on E .

In this case, the nonexplosive algebraic duality chain (for details, see Section 3) is uniquely determined by

$$\frac{\pi_0(x)}{\pi(x)} = \sum_{y \in E: y \geq x} \frac{\pi_0^*(y)}{H(y)} \text{ and } \bar{Q}(y, \{x\}^\downarrow) = \sum_{z \in E: z \geq y} \frac{H(x)}{H(z)} q_{x,z}^*$$

for any $x, y \in E$.

Let π_t be the law of X_t . The variation distance is defined by $\|\pi_t - \pi\| = \max_{A \subset E} |\pi_t(A) - \pi(A)|$. One advantage in establishing SSD is that useful bounds on variation distance can be obtained by stopping the process before absorption. Through this method, the early stopping theorem in [6] yielded useful bounds. Moreover, the early stopping theorem can apply to the algebraic duality. An argument similar to the early stopping theorem in [5] (Corollary 2.1) shows the following corollary, which transforms analysis of convergence rate for \mathbf{X} into the first passage time for the dual chain.

Corollary 1.6. For the chain $(\mathbf{X}^*, \mathbf{X})$ of Theorem 1.4 or Theorem 1.5, let T be the first hitting time of $A \subset E$ for \mathbf{X}^* , where $A = \{x \in E : x \succeq x^*\}$, $x^* \in E$. Then

$$\|\pi_t - \pi\| \leq (1 - H(x^*)) + H(x^*)P\{T > t\}, \quad t \geq 0.$$

Here is an outline for the paper. Section 2 of this paper studies the *Möbius function* and Möbius monotonicity. Section 3 of this paper shows the proof of Theorem 1.4 and Theorem 1.5. Section 4 of this paper gives examples of the two-dimensional birth and death chain.

2. Möbius Monotonicity

Define E as a finite partially ordered set or \mathbb{Z}_+^d . The *zeta function* ζ of E is denoted by $\forall x, y \in E$, $\zeta(x, y) = 1$ if $x \preceq y$ and $\zeta(x, y) = 0$ otherwise. The inverse to ζ is the *Möbius function* μ , which satisfies

$$\sum_{z \in E: x \preceq z \preceq y} \mu(x, z)\zeta(z, y) = \sum_{z \in E: x \preceq z \preceq y} \zeta(x, z)\mu(z, y) = \delta(x, y), \quad (2.1)$$

where $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ otherwise. From Proposition 5 in [11], we obtain for any $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbb{Z}_+^d$,

$$\begin{aligned} &\mu((x_1, \dots, x_d), (y_1, \dots, y_d)) \\ &= \begin{cases} (-1)^{\sum_{i=1}^d (y_i - x_i)}, & \text{if each } y_i - x_i = 0 \text{ or } 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.2)$$

With the help of *Möbius function* μ , we will give criteria on the Möbius monotonicity of nonnegative functions and Q matrices. From [9], we can obtain the following proposition:

Proposition 2.1. *If E is a finite partially ordered set, then a nonnegative function f on E is \downarrow -Möbius monotone if and only if $\sum_{z \in E: z \succeq y} \mu(y, z)f(z) \geq 0$ for any $y \in E$.*

Proposition 2.2. *If E is a finite partially ordered set, then a Q matrix Q on $E \times E$ is \downarrow -Möbius monotone if and only if $\sum_{z \in E: z \succeq y} \mu(y, z)Q(z, \{x\}^\downarrow) \geq 0$ for any $x, y \in E$ and $x \neq y$.*

Proof. Suppose Q is \downarrow -Möbius monotone, then from Definition 1.2, there exists a matrix M which is nonnegative off the diagonal such that

$$Q(z, \{x\}^\downarrow) = \sum_{z' \in E: z' \succeq z} M(z', x), \quad \forall x, z \in E.$$

Since E is finite and (2.1), we can get

$$\begin{aligned} \sum_{z \in E: z \succeq y} \mu(y, z)Q(z, \{x\}^\downarrow) &= \sum_{z \in E: z \succeq y} \mu(y, z) \sum_{z' \in E: z' \succeq z} M(z', x) \\ &= \sum_{z' \in E: z' \succeq y} M(z', x) \sum_{z \in E: y \preceq z \preceq z'} \mu(y, z) \\ &= \sum_{z' \in E: z' \succeq y} M(z', x)\delta(y, z') = M(y, x) \geq 0, \quad x \neq y. \end{aligned}$$

Conversely, let $M(z, y) = \sum_{z' \in E: z' \succeq z} \mu(z, z')Q(z', \{y\}^\downarrow)$ for any $z, y \in E$, then M is nonnegative off the diagonal. Since E is finite and (2.1), we can obtain

$$\begin{aligned} \sum_{z \in E: z \succeq x} M(z, y) &= \sum_{z \in E: z \succeq x} \sum_{z' \in E: z' \succeq z} \mu(z, z')Q(z', \{y\}^\downarrow) \\ &= \sum_{z' \in E: z' \succeq x} Q(z', \{y\}^\downarrow) \sum_{z \in E: x \preceq z \preceq z'} \mu(z, z') \\ &= \sum_{z' \in E: z' \succeq x} Q(z', \{y\}^\downarrow)\delta(x, z') = Q(x, \{y\}^\downarrow). \end{aligned}$$

Thus Q is \downarrow -Möbius monotone. □

Note that, if $E = \mathbb{Z}_+^d$, then $\lim_{x \rightarrow \infty} f(x) = 0$ is to say for every $\epsilon > 0$, there is a finite set $A \subset E$ such that $|f(x)| \leq \epsilon$ for any $x \in E \setminus A$. Then from [7] (Proposition 2.4 and Remark 2.5), we can get the following proposition.

Proposition 2.3. *If $E = \mathbb{Z}_+^d$, then a nonnegative function f on E is \downarrow -Möbius monotone if and only if $\lim_{x \rightarrow \infty} f(x) = 0$ and $\sum_{z \in E: z \succeq y} \mu(y, z)f(z) \geq 0$ for any $y \in E$.*

Proposition 2.4. *If $E = \mathbb{Z}_+^d$, then a Q matrix Q on $E \times E$ is \downarrow -Möbius monotone if and only if $\lim_{x \rightarrow \infty} q_{x,y} = 0$ for any $y \in E$, and $\sum_{z \in E: z \succeq y} \mu(y, z)Q(z, \{x\}^\downarrow) \geq 0$ for any $x, y \in E$ and $x \neq y$.*

Proof. Suppose Q is \downarrow -Möbius monotone, then from Definition 1.2, there exists a matrix M which is nonnegative off the diagonal such that

$$Q(x, \{z\}^\downarrow) = \sum_{y \in E: y \succeq x} M(y, z), \quad \forall x, z \in E.$$

We will prove $\lim_{x \rightarrow \infty} Q(x, \{z\}^\downarrow) = \lim_{x \rightarrow \infty} \sum_{y \in E: y \succeq x} M(y, z) = 0$. Since for fixed $z \in E$,

$$\sum_{\substack{y \in E: y \succeq (0, \dots, 0) \\ y \neq z}} M(y, z) = Q(x, \{z\}^\downarrow) - M(z, z) < \infty,$$

we have $\lim_{n \rightarrow \infty} \sum_{y \neq z, y \in E \setminus A_n} M(y, z) = 0$, where $A_n = \{(n, \dots, n)\}^\downarrow$, that is to say for any $\epsilon > 0$, there exists $N \geq 0$, such that when $n \geq N$, we have $\sum_{y \neq z, y \in E \setminus A_n} M(y, z) \leq \epsilon$. Then, for any $\epsilon > 0$, for enough large $M \geq 0$, there exists A_M such that when $x \in E \setminus A_M$, we have $z \not\preceq x$ and

$$\sum_{y \in E: y \succeq x} M(y, z) \leq \sum_{y \neq z, y \in E \setminus A_M} M(y, z) \leq \epsilon.$$

Therefore, $\lim_{x \rightarrow \infty} \sum_{y \in E: y \succeq x} M(y, z) = \lim_{x \rightarrow \infty} Q(x, \{z\}^\downarrow) = 0$. Also, since

$$\begin{aligned} \sum_{z \in E: z \preceq y} Q(x, \{z\}^\downarrow) \mu(z, y) &= \sum_{z \in E: z \preceq y} \sum_{z' \in E: z' \preceq z} q_{x, z'} \mu(z, y) \\ &= \sum_{z' \in E: z' \preceq y} q_{x, z'} \sum_{z \in E: z' \preceq z \preceq y} \mu(z, y) \\ &= \sum_{z' \in E: z' \preceq y} q_{x, z'} \delta_{z', y} = q_{x, y}. \end{aligned}$$

We get $\lim_{x \rightarrow \infty} q_{x, y} = 0$. By using (2.1), we obtain

$$\begin{aligned} \sum_{z \in E: z \succeq y} \mu(y, z) Q(z, \{x\}^\downarrow) &= \sum_{z \in E: z \succeq y} \mu(y, z) \sum_{z' \in E: z' \succeq z} M(z', x) \\ &= \sum_{z' \in E: z' \succeq y} M(z', x) \sum_{z \in E: y \preceq z \preceq z'} \mu(y, z) \\ &= \sum_{z' \in E: z' \succeq y} M(z', x) \delta(y, z') = M(y, x) \geq 0, \quad x \neq y. \end{aligned}$$

The second equality above is followed by the dominated convergence theorem. Therefore, $\sum_{z \in E: z \succeq y} \mu(y, z) Q(z, \{x\}^\downarrow) \geq 0$ for any $x \neq y$.

Conversely, let $M(z, y) = \sum_{z' \in E: z' \succeq z} \mu(z, z') Q(z', \{y\}^\downarrow)$, for any $z, y \in E$. We need to prove $Q(x, \{y\}^\downarrow) = \sum_{z \in E: z \succeq x} M(z, y)$ for any $x, y \in E$. For convenience, we take $x = (0, \dots, 0)$, and the proof is similar for any other $x \in E$. Define $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the i -th coordinate, for convenience, we define $\sum_{y=x}^z = \sum_{y: x \preceq y \preceq z}$. Then by using (2.2), we get

$$\begin{aligned}
\sum_{z \in E: z \succeq (0, \dots, 0)} M(z, y) &= \lim_{n \rightarrow \infty} \sum_{z=(0, \dots, 0)}^{(n, \dots, n)} M(z, y) \\
&= \lim_{n \rightarrow \infty} \sum_{z=(0, \dots, 0)}^{(n, \dots, n), z+(1, \dots, 1)} \sum_{z'=z} \mu(z, z') Q(z', \{y\}^\downarrow) \\
&= \lim_{n \rightarrow \infty} \left(Q((0, \dots, 0), \{y\}^\downarrow) + \sum_{i=1}^d (-1) Q((n+1)e_i, \{y\}^\downarrow) \right. \\
&\quad + \sum_{\substack{i \neq j \\ 1 \leq i, j \leq d}} (-1)^2 \cdot Q((n+1)e_i + (n+1)e_j, \{y\}^\downarrow) \\
&\quad \left. + \dots + (-1)^d Q((n+1, \dots, n+1), \{y\}^\downarrow) \right) \\
&= Q((0, \dots, 0), \{y\}^\downarrow).
\end{aligned}$$

A similar proof of Lemma 2.4 in [12] shows that, for any $y \in E$, $\lim_{x \rightarrow \infty} q_{x, y} = 0$ implies $\lim_{x \rightarrow x^*} q_{x, y} = 0$ for any $x^* \in \overline{\mathbb{Z}_+^d} \setminus E$ with $\overline{\mathbb{Z}_+} = \mathbb{Z}_+ \cup \{\infty\}$. Thus, the last equality above is followed by $\lim_{x \rightarrow \infty} q_{x, y} = 0$ for any $y \in E$. \square

3. Proof of Theorem 1.4 and Theorem 1.5

With the help of Proposition 2.1 and Proposition 2.2, the proof of Theorem 1.4 is similar to that of Theorem 1.5. Thus, we now only prove Theorem 1.5, using the Propositions 2.3 and 2.4.

Proof of Theorem 1.5. Let $E = \mathbb{Z}_+^d$, from [6], to prove the existence of nonexplosive algebraic duality, we need to check the algebraic duality equations

$$\pi_0 = \pi_0^* \Lambda, \quad \Lambda Q = Q_0^* \Lambda.$$

Also, we need to check that Q^* is nonexplosive on E . As for $\pi_0 = \pi_0^* \Lambda$, we have

$$g(x) = \frac{\pi_0(x)}{\pi(x)} = \sum_{y \in E: y \succeq x} \frac{\pi_0^*(y)}{H(y)}, \quad \forall x \in E.$$

Then we get $\pi_0 = \pi_0^* \Lambda$ has a nonnegative solution if and only if g is \downarrow -Möbius monotone, which from Proposition 2.3 is equivalent to $\lim_{x \rightarrow \infty} g(x) = 0$ and $\sum_{z \in E: z \succeq y} \mu(y, z)g(z) \geq 0$, for any $y \in E$. Also, in this case, we get

$$\pi_0^*(y) = H(y) \sum_{z \in E: z \succeq y} \mu(y, z)g(z) \geq 0, \quad \forall y \in E.$$

From $\pi_0 = \pi_0^* \Lambda$, it is clear that the solution sums to unity. Therefore, π_0^* is a probability vector.

As for $\Lambda Q = Q^* \Lambda$, we have

$$\sum_{z \in E} \Lambda(x, z)q_{z, y} = \sum_{z \in E} q_{x, z}^* \Lambda(z, y), \quad \forall x, y \in E,$$

that is,

$$\frac{1}{H(x)} \sum_{z \in E: z \preceq x} \frac{\pi(z)}{\pi(y)} q_{z, y} = \sum_{z \in E: z \succeq y} \frac{q_{x, z}^*}{H(z)}.$$

Using $\frac{\pi(z)}{\pi(y)} q_{z, y} = \overleftarrow{q}_{y, z}$, we have

$$\overleftarrow{Q}(y, \{x\}^\downarrow) = \sum_{z \in E: z \succeq y} \frac{H(x)}{H(z)} q_{x, z}^*, \quad \forall x, y \in E.$$

Then we get $\Lambda Q = Q^* \Lambda$ has a solution which is nonnegative off the diagonal, if and only if Q is \downarrow -Möbius monotone, which from Proposition

2.4 is equivalent to $\lim_{x \rightarrow \infty} \bar{q}_{x,y} = 0$ for any $y \in E$, and $\sum_{z \in E: z \succeq y} \mu(y, z) \bar{Q}(z, \{x\}^\downarrow) \geq 0$ for any $x \neq y$. Also, in this case, we get

$$q_{x,y}^* = \frac{H(y)}{H(x)} \sum_{z \in E: z \succeq y} \mu(y, z) \bar{Q}(z, \{x\}^\downarrow), \quad \forall x, y \in E. \quad (3.1)$$

We now prove Q^* is a conservative Q matrix. From $\Lambda Q = Q^* \Lambda$, we have

$$\sum_{z \in E} \Lambda(x, z) q_{z,y} = \sum_{z \in E} q_{x,z}^* \Lambda(z, y), \quad \forall x, y \in E.$$

If $x \in E$, since $\{y \in E : y \preceq x\}$ is finite and Q is conservative, we get

$$\sum_{y \in E} \sum_{z \in E} \Lambda(x, z) q_{z,y} = \sum_{z \in E} \Lambda(x, z) \sum_{y \in E} q_{z,y} = 0,$$

and

$$\sum_{y \in E} \sum_{z \in E} q_{x,z}^* \Lambda(z, y) = \sum_{z \in E} q_{x,z}^* \sum_{y \in E} \Lambda(z, y) = \sum_{z \in E} q_{x,z}^*.$$

So $\sum_{z \in E} q_{x,z}^* = 0$ for any $x \in E$. Since for any $x \neq y$,

$$q_{x,y}^* = \frac{H(y)}{H(x)} \sum_{z \in E: z \succeq y} \mu(y, z) \bar{Q}(z, \{x\}^\downarrow) \geq 0,$$

we get Q^* is a conservative Q matrix. □

4. A Two-Dimensional Birth and Death Chain

Assume $E = \{0, 1, \dots, N\}^2$ or \mathbb{Z}_+^2 , $e_1 = (1, 0)$, $e_2 = (0, 1)$. The Q matrix of the two-dimensional birth and death chain $\mathbf{X} = (X_t)_{t \geq 0}$ is given as follows; for each $x = (x_1, x_2)$, $y = (y_1, y_2) \in E$, let

$$q_{x, y} = \begin{cases} b_i, & \text{if } y = x + e_i, \\ a_i, & \text{if } y = x - e_i, \\ -\sum_{z \neq x} q_{x, z}, & \text{if } y = x, \\ 0 & \text{else,} \end{cases} \quad (4.1)$$

where $a_i > 0$, $b_i > 0$, and $a_i \neq b_i$, $i = 1, 2$. As is well known, \mathbf{X} is time reversible, that is $\overline{Q} = Q$, and it has the unique stationary distribution: for any $x = (x_1, x_2) \in E$,

$$\pi(x_1, x_2) = c \left(\frac{b_1}{a_1}\right)^{x_1} \left(\frac{b_2}{a_2}\right)^{x_2},$$

with c a normalizing constant.

Theorem 4.1. *Assume $E = \{0, 1, \dots, N\}^2$, let $\mathbf{X} \sim (\pi_0, Q, P(t))$ be a two-dimensional birth and death chain with Q given in (4.1), $a_i > 0$, $b_i > 0$, and $a_i \neq b_i$, $i = 1, 2$. Assume \mathbf{X} starts at $(0, 0)$, then there exists an SSD chain $\mathbf{X}^* \sim (\pi_0^*, Q^*, P^*(t))$ ($P^*(t)$ is the Q^* process) starting at $(0, 0)$ with the link kernel: $\Lambda(x, y) = I_{(y \leq x)} \frac{\pi(y)}{H(x)}$, which is an absorbing Markov chain (with (N, N) being the absorbing state) on E , with the following Q matrix (for $x = (x_1, x_2)$, $y = (y_1, y_2) \in E$),*

$$q_{x,y}^* = \begin{cases} \frac{1 - (b_i/a_i)^{x_i+2}}{1 - (b_i/a_i)^{x_i+1}} a_i, & \text{if } y = x + e_i, \\ \frac{1 - (b_i/a_i)^{x_i}}{1 - (b_i/a_i)^{x_i+1}} b_i, & \text{if } y = x - e_i, x_i \neq N, \\ -\sum_{i=1}^2 (a_i + b_i), & \text{if } y = x, x_1 \neq N, x_2 \neq N, \\ -a_2 - b_2, & \text{if } y = x, x_1 = N, x_2 \neq N, \\ -a_1 - b_1, & \text{if } y = x, x_1 \neq N, x_2 = N, \\ 0 & \text{else.} \end{cases}$$

Theorem 4.2. Assume $E = \mathbb{Z}_+^2$, let $\mathbf{X} \sim (\pi_0, Q, P(t))$ be a two-dimensional birth and death chain with Q given in (4.1), $a_i > 0, b_i > 0$, and $a_i \neq b_i, i = 1, 2$. Suppose Q is nonexplosive. Assume \mathbf{X} starts at $(0, 0)$, then there exists a nonexplosive algebraic duality chain $\mathbf{X}^* \sim (\pi_0^*, Q^*, P^*(t))$ ($P^*(t)$ is the minimal Q^* process) on E with the link kernel: $\Lambda(x, y) = I_{(y \leq x)} \frac{\pi(y)}{H(x)}$, which starts at $(0, 0)$, with the following Q matrix (for $x = (x_1, x_2), y = (y_1, y_2) \in E$),

$$q_{x,y}^* = \begin{cases} \frac{1 - (b_i/a_i)^{x_i+2}}{1 - (b_i/a_i)^{x_i+1}} a_i, & \text{if } y = x + e_i, \\ \frac{1 - (b_i/a_i)^{x_i}}{1 - (b_i/a_i)^{x_i+1}} b_i, & \text{if } y = x - e_i, \\ -\sum_{i=1}^2 (a_i + b_i), & \text{if } y = x, \\ 0 & \text{else.} \end{cases} \quad (4.2)$$

We will only prove Theorem 4.2. With the help of Theorem 1.4, Theorem 4.1 can be proved similarly.

Proof of Theorem 4.2. Since $H(x) = \sum_{y \in E: y \preceq x} \pi(y)$, we have $H(x) = c(1 - b_1/a_1)^{-1}(1 - b_2/a_2)^{-1}$. Define $g(x) = \pi_0(x)/\pi(x)$. Assume the chain starts at $(0, 0)$, then $g(x) = \pi_0(x)/\pi(x)$ is \downarrow -Möbius monotone. To prove the existence of the nonexplosive algebraic duality chain, from the proof of Theorem 1.5, we need to show that

$$\left\{ \begin{array}{l} \text{(i) } \lim_{x \rightarrow \infty} g(x) = 0, \text{ and } \sum_{z \in E: z \succeq y} \mu(y, z)g(z) \geq 0, \quad y \in E, \\ \text{(ii) } \lim_{x \rightarrow \infty} q_{x, y} = 0, \text{ and } \sum_{z \in E: z \succeq y} \mu(y, z)Q(z, \{x\}^\downarrow) \geq 0, \quad x \neq y, \\ \text{(iii) } Q^* \text{ is nonexplosive on } E. \end{array} \right.$$

In this case, there exists a nonexplosive algebraic duality chain with the link kernel: $\Lambda(x, y) = I_{(y \preceq x)} \frac{\pi(y)}{H(x)}$, $x, y \in E$, and Q^* in (3.1). Since the chain starts at $(0, 0)$, (i) is satisfied and the dual chain starts at $(0, 0)$. We will prove Q in (4.1) also satisfies (ii). Obviously, $\lim_{x \rightarrow \infty} q_{x, y} = 0$ for any $y \in E$. Let

$$M(y, x) = \sum_{z \in E: z \succeq y} \mu(y, z)Q(z, \{x\}^\downarrow).$$

Using (2.2), we will calculate $M(y, x)$ for any $x, y \in E$, and show $M(y, x) \geq 0$ for any $x \neq y$.

(1) If $y = x + e_i$,

$$\begin{aligned} M(x + e_i, x) &= \sum_{z \in E: z \succeq x + e_i} \mu(x + e_i, z)Q(z, \{x\}^\downarrow) \\ &= \mu(x + e_i, x + e_i)Q(x + e_i, \{x\}^\downarrow) = a_i. \end{aligned}$$

(2) If $y = x$,

$$\begin{aligned}
 M(x, x) &= \sum_{z \in E: z \geq x} \mu(x, z)Q(z, \{x\}^\downarrow) \\
 &= \mu(x, x)Q(x, \{x\}^\downarrow) + \sum_{i=1}^2 \mu(x, x + e_i)Q(x + e_i, \{x\}^\downarrow) \\
 &= -\sum_{i=1}^2 b_i + \sum_{i=1}^2 (-1)a_i = -\sum_{i=1}^2 (a_i + b_i).
 \end{aligned}$$

(3) If $y = x - e_i$, for $j \neq i$,

$$\begin{aligned}
 M(x - e_i, x) &= \sum_{z \in E: z \geq x - e_i} \mu(x - e_i, z)Q(z, \{x\}^\downarrow) \\
 &= \mu(x - e_i, x - e_i)Q(x - e_i, \{x\}^\downarrow) \\
 &\quad + \sum_{k=1}^2 \mu(x - e_i, x - e_i + e_k) \\
 &\quad \cdot Q(x - e_i + e_k, \{x\}^\downarrow) + \mu(x - e_i, x + e_j)Q(x + e_j, \{x\}^\downarrow) \\
 &= 1 \cdot (-b_j) + \mu(x - e_i, x)Q(x, \{x\}^\downarrow) \\
 &\quad + \mu(x - e_i, x - e_i + e_j) \cdot Q(x - e_i + e_j, \{x\}^\downarrow) + a_j \\
 &= -b_j + (-1)(-b_1 - b_2) + (-1)a_j + a_j = b_i.
 \end{aligned}$$

Similarly, we can calculate $M(y, x) = 0$ for any other $x, y \in E$. Then we get Q^* in (4.2), which is bounded obviously. Thus, Q^* in (4.2) is nonexplosive and satisfies (iii). \square

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