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STEADY-STATE SOLUTIONS FOR SOME MOTIONS OF MAXWELL FLUIDS WITH PRESSURE-DEPENDENCE OF VISCOSITY

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Abstract

Two isothermal motions of incompressible Maxwell fluids with power-law dependence of viscosity on the pressure are investigated when gravity effects are taken into account. The fluid motion, between two infinite horizontal parallel plates, is generated by the lower plate that applies a time-dependent shear stress to the fluid. Exact expressions are established for the steady-state components of the dimensionless start-up velocity, shear stress, and normal stress. They are used to find the needed time to touch the steady-state and to provide corresponding solutions for the motion of the same fluids induced by an exponential shear stress on the boundary. This time is useful for experimentalists who want to eliminate transients from their experiments. It is higher for motions of ordinary fluids as compared to fluids with pressure-dependent viscosity. The variation of starting solutions (numerical solutions) in time and space is graphically represented and some characteristics of the fluid motion are brought to light.

1. Introduction

The fact that the fluid viscosity could increase at high pressures has been early enough remarked by Stokes [1]. Later, the experimental research of Cutler et al. [2], Johnson and Cameron [3], Johnson and Tewaarwerk [4], Bair and Winer [5], Bair et al. [6], Prusa et al. [7] and many others certified this observation. Renardy [8], for instance, remarked that the fluid viscosity increases more than an order of magnitude at a pressure of 1.000 atm and such pressures appear at the polymer processing operations [9], the fluid film lubrication [10], microfluidics [11], pharmaceutical manufacturing, food processing etc. After his experimental research at high pressures, Bridgman [12] observed a meaningful dependence of viscosity on the pressure and nowadays the fluid models with pressure-dependent viscosity are used to describe the behaviour of fluids in different applications. On the other hand, Dowson and Higginson [13] as well as Rajagopal [14] remarked that the variation of fluid density is small enough at changes of the viscosity of the order $10^8\%$. Consequently, these liquids can be treated as incompressible fluids with pressure-dependent viscosity.

In the same time, the gravity effects are important in many flows of the fluids with engineering applications. Its influence on the fluid motion is more pronounced if the pressure varies lengthways the direction in which the gravity acts. The first exact solutions for steady motions between parallel plates of incompressible Newtonian fluids with pressure-dependent viscosity in which the gravity effects are taken into consideration are those of Rajagopal [15]. The Poiseuille flow between parallel plates, as well as the flow down on an inclined plane of the same fluids, was also studied by Rajagopal [16]. Closed form expressions for the steady-state (permanent or long time) solutions corresponding to the modified Stokes' problems of the same fluids have been established by Prusa [17], Fetecau and Agop [18] and Fetecau et al. [19]. Other interesting steady solutions for the flow of such fluids in rectangular domains have been obtained by Akyildiz and Siginer [20] and Housiadas and Georgiou [21].

During the time some geologists tried to refine their models for the viscosity involving non-Newtonian fluids. Wu and Wang [22], for instance, one asked why the dependence of material moduli on pressure is not taken into consideration. A general study concerning unsteady motions of incompressible upper-convected Maxwell (IUCM) fluids with viscosity and relaxation time depending on the pressure was presented by Karra et al. [23]. Steady solutions for pressure driven flows of such fluids have been established by Housiadas [24, 25] in straight and circular tubes. However, the first steady-state solutions for unsteady motions of the IUCM fluids with pressure-dependent viscosity in which the gravity effects are taken in consideration seem to be those of Fetecau and Rauf [26].

The main purpose of this note is to determine the needed time to touch the steady-state for some motions of IUCM fluids with power-law dependence of viscosity on the pressure between infinite horizontal parallel plates. The fluid motion is generated by the lower plate that applies a time-dependent shear stress to the fluid and the no-slip condition on the upper plate is taken into consideration. Exact expressions are determined for the dimensionless steady-state velocity, shear stress and normal stress. The convergence of start-up velocities (numerical solutions) to their steady-state components is graphically proved and the needed time to touch the steady-state is graphically determined. This time is higher for motions of ordinary fluids than for fluids with pressure-dependent viscosity. Corresponding solutions for same motions of ordinary IUCM fluids, as well as the solutions for the motion induced by an exponential shear stress on the lower plate, are acquired as limiting cases of main results.

2. Constitutive and Governing Equations

The constitutive equations of IUCM fluids with pressure-dependent viscosity, as it results from the work of Karra et al. [23], are given by the following relations:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \quad \boldsymbol{S} + \lambda\left(\frac{d\boldsymbol{S}}{dt} - \boldsymbol{LS} - \boldsymbol{SL}^T\right) = \eta(p)(\boldsymbol{L} + \boldsymbol{L}^T), \tag{1}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $-p\boldsymbol{I}$ represents the undetermined spherical stress due to the constraint of incompressibility, $\boldsymbol{S}$ is the constitutively determined extra-stress, $\boldsymbol{L}$ is the gradient of the velocity vector $\mathbf{v}$, λ is the relaxation time, $\eta(\cdot)$ is the viscosity function, and d/dt denotes the material time derivative. If $\lambda = 0$, the constitutive equations (1) define incompressible Newtonian fluids with pressure-dependent viscosity. In the following, we will refer to the Lagrange multiplier p as pressure although, for the rate type fluids, it is not the mean normal stress [23]. The viscosity function $\eta(p)$ to be here used, has the form

$$\eta(p) = \mu[1 + \alpha(p - p_0)]^2, \tag{2}$$

where μ is the fluid viscosity at the reference pressure p_0 and the constant α is dimensional pressure-viscosity coefficient. If $\alpha = 0$ in Equation (2), the constitutive equations (1) define ordinary IUCM fluids and the fact that $\eta(p) \to \infty$ for $p \to \infty$ is in accordance with a property that was experimentally confirmed.

In the following as well as Karra et al. [23], we shall consider motions whose velocity field $\mathbf{v}$ and the pressure p have the forms

$$\mathbf{v} = \mathbf{v}(y, t) = u(y, t)\mathbf{e}_x, \quad p = p(y), \tag{3}$$

where $\mathbf{e}_x$ is the unit vector in the x-direction of a convenient Cartesian coordinate system x, y, and z. Assuming that, as well as the velocity field $\mathbf{v}$, the extra-stress tensor $\boldsymbol{S}$ depends of y and t only and the fluid is at rest at the initial moment, it is not difficult to show that its components S_{xz}, S_{yy}, S_{yz} and S_{zz} are zero while the non-trivial shear and normal stresses $\tau(y, t) = S_{xy}(y, t)$ and $\sigma(y, t) = S_{xx}(y, t)$ have to satisfy the next differential equations

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau(y, t) = \eta(p)\frac{\partial u(y, t)}{\partial y}, \quad \left(1 + \lambda \frac{\partial}{\partial t}\right)\sigma(y, t) = 2\lambda\tau(y, t)\frac{\partial u(y, t)}{\partial y}. \tag{4}$$

The continuity equation is identically satisfied while the motion equations, in the absence of a pressure gradient in the flow direction, reduce to the following relevant partial and ordinary differential equations:

$$\rho \frac{\partial u(y, t)}{\partial t} = \frac{\partial \tau(y, t)}{\partial y}, \quad \frac{dp}{dy} = -\rho g, \tag{5}$$

where ρ is the fluid density and g is the gravitational acceleration.

Let us now assume that an IUCM fluid with power-law dependence of viscosity on the pressure of the form (2) is at rest between two infinite horizontal parallel plates at the distance d apart. We also assume that the lower plate begins to apply at the moment $t = 0^+$ a time-dependent shear stress of the form

$$\tau(0, t) = \left[\frac{\cos(\omega t) + \lambda\omega \sin(\omega t)}{(\lambda\omega)^2 + 1} - \frac{1}{(\lambda\omega)^2 + 1} \exp\left(-\frac{t}{\lambda}\right)\right] S, \tag{6}$$

$$\tau(0, t) = \left[\frac{\sin(\omega t) + \lambda\omega \cos(\omega t)}{(\lambda\omega)^2 + 1} + \frac{\lambda\omega}{(\lambda\omega)^2 + 1} \exp\left(-\frac{t}{\lambda}\right)\right] S, \tag{7}$$

to the fluid. If $\lambda \to 0$, Equations (1) characterize incompressible Newtonian fluids with power-law dependence of viscosity on the pressure and Equations (6) and (7) take to the simple forms

$$\tau(0, t) = S\cos(\omega t), \text{ respectively } \tau(0, t) = S\sin(\omega t). \tag{8}$$

In this case S and ω are the amplitude and the frequency of the oscillations, respectively.

Owing to the shear the fluid begins to move and, since the plates are boundless, it is reasonable to assume that all physical entities which characterize the fluid motion are functions of y and t only. Integrating Equation $(5)_2$ with respect to y between 0 and d, it result that

$$p(y) = p_0 + \rho g(d - y), \text{ where } p_0 = p(d). \tag{9}$$

Eliminating $\tau(y, t)$ between Equations $(4)_1$ and $(5)_1$ and bearing in mind the expressions of $\eta(p)$ and p from the equalities (2), respectively (9) one obtains for the dimensional velocity field $u(y, t)$ the following partial differential equation:

$$\mu[1 + \alpha\rho g(d - y)]^2 \frac{\partial^2 u(y, t)}{\partial y^2} - 2\mu\alpha\rho g[1 + \alpha\rho g(d - y)]\frac{\partial u(y, t)}{\partial y}$$

$$= \rho\left(1 + \lambda \frac{\partial}{\partial t}\right)\frac{\partial u(y, t)}{\partial t}; \quad 0 < y < d, \; t > 0. \tag{10}$$

The corresponding initial and boundary conditions are

$$u(y, 0) = 0, \; \left.\frac{\partial u(y, t)}{\partial t}\right|_{t=0} = 0; \; 0 \le y \le d, \tag{11}$$

$$\tau(0, t) + \lambda \frac{\partial \tau(0, t)}{\partial t} = \mu\left.\left\{[1 + \alpha\rho g(d - y)]^2 \frac{\partial u(y, t)}{\partial y}\right\}\right|_{y=0}$$

$$= S\cos(\omega t), \; u(d, t) = 0, \; t > 0, \tag{12}$$

or the initial conditions (11) together with the boundary conditions

$$\tau(0, t) + \lambda \frac{\partial \tau(0, t)}{\partial t} = \mu\left.\left\{[1 + \alpha\rho g(d - y)]^2 \frac{\partial u(y, t)}{\partial y}\right\}\right|_{y=0}$$

$$= S\sin(\omega t), \; u(d, t) = 0, \; t > 0. \tag{13}$$

We mention that the solutions of the ordinary differential equations (12) and (13) with the initial condition $\tau(0, 0) = 0$ are given by the equalities (6), respectively (7). If the velocity field $u(y, t)$ is known, the corresponding non-trivial stresses $\tau(y, t)$ and $\sigma(y, t)$ can easily be determined solving the next ordinary differential equations with initial conditions

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau(y, t) = \mu[1 + \alpha\rho g(d - y)]^2 \frac{\partial u(y, t)}{\partial y}; \quad \tau(y, 0) = 0, \tag{14}$$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\sigma(y, t) = 2\lambda\tau(y, t)\frac{\partial u(y, t)}{\partial y}; \quad \sigma(y, 0) = 0. \tag{15}$$

Knowing $\tau(y, t)$, we can determine the frictional forces exerted by the fluid to the plates.

To determine solutions that are independent of the flow geometry, we introduce the next non-dimensional variables, functions and parameters

$$y^* = \frac{y}{d},\ t^* = \frac{S}{\mu}t,\ u^* = \frac{\mu}{Sd}u,\ \tau^* = \frac{\tau}{S},\ \sigma^* = \frac{\sigma}{S},\ \omega^* = \frac{\mu}{S}\omega,\ \alpha^* = \alpha\rho g d. \tag{16}$$

In the terms of the new dimensionless physical entities, dropping out the star notation, the equalities (10)-(13) take the simpler forms

$$[1 + \alpha(1 - y)]^2 \frac{\partial^2 u(y, t)}{\partial y^2} - 2\alpha[1 + \alpha(1 - y)]\frac{\partial u(y, t)}{\partial y}$$

$$= \mathrm{Re}\left(1 + \mathrm{We}\frac{\partial}{\partial t}\right)\frac{\partial u(y, t)}{\partial t}; \quad 0 < y < 1,\ t > 0, \tag{17}$$

$$u(y, 0) = 0,\ \left.\frac{\partial u(y, t)}{\partial t}\right|_{t=0} = 0;\ 0 \le y \le 1, \tag{18}$$

$$\left.\frac{\partial u(y, t)}{\partial y}\right|_{y=0} = \frac{1}{(1 + \alpha)^2}\cos(\omega t),$$

or

$$\left.\frac{\partial u(y, t)}{\partial y}\right|_{y=0} = \frac{1}{(1 + \alpha)^2}\sin(\omega t),\ u(1, t) = 0;\ t > 0. \tag{19}$$

In Equation (17), $\mathrm{Re} = Vd/\nu$ ($V = Sd/\mu$ being a characteristic velocity) and $\mathrm{We} = \lambda/t_0$ ($t_0 = \mu/S$ being a characteristic time) are Reynolds number, respectively the Weissenberg number.

As soon as the dimensionless velocity field is known, the adequate stresses $\tau(y, t)$ and $\sigma(y, t)$ are obtained solving the next ordinary differential equations with initial conditions

$$\left(1 + \mathrm{We}\frac{\partial}{\partial t}\right)\tau(y, t) = [1 + \alpha(1 - y)]^2 \frac{\partial u(y, t)}{\partial y}, \ \tau(y, 0) = 0, \tag{20}$$

$$\left(1 + \mathrm{We}\frac{\partial}{\partial t}\right)\sigma(y, t) = 2\mathrm{We}\tau(y, t)\frac{\partial u(y, t)}{\partial y}, \ \sigma(y, 0) = 0. \tag{21}$$

3. Solution of the Problem

To avert a possible confusion, we denote by $u_c(y, t)$, $\tau_c(y, t)$, $\sigma_c(y, t)$ and $u_s(y, t)$, $\tau_s(y, t)$, $\sigma_s(y, t)$ the start-up (starting) dimensionless velocity, shear stress and normal stress corresponding to the two distinct motions induced by cosine, respectively sine oscillations of $\partial u(y, t)/\partial y$ on the boundary. Such motions become steady in time and their solutions can be presented as sum of their steady-state and transient components, namely,

$$u_c(y, t) = u_{cp}(y, t) + u_{ct}(y, t), \ \tau_c(y, t) = \tau_{cp}(y, t) + \tau_{ct}(y, t),$$

$$\sigma_c(y, t) = \sigma_{cp}(y, t) + \sigma_{ct}(y, t), \tag{22}$$

$$u_s(y, t) = u_{sp}(y, t) + u_{st}(y, t), \ \tau_s(y, t) = \tau_{sp}(y, t) + \tau_{st}(y, t),$$

$$\sigma_s(y, t) = \sigma_{sp}(y, t) + \sigma_{st}(y, t). \tag{23}$$

Some time after the motion initiation, the fluid moves according to the start-up solutions $u_c(y, t)$, $\tau_c(y, t)$, $\sigma_c(y, t)$ or $u_s(y, t)$, $\tau_s(y, t)$, $\sigma_s(y, t)$. After this time, when the transients disappear or can be neglected, the fluid behaviour is fully characterized by their steady-state components $u_{cp}(y, t)$, $\tau_{cp}(y, t)$, $\sigma_{cp}(y, t)$, respectively $u_{sp}(y, t)$, $\tau_{sp}(y, t)$, $\sigma_{sp}(y, t)$. This is the need time to reach the steady-state. In practice, it is important for the experimentalists which want to eliminate the transients from their experiments. In order to determine this time for a prescribed motion, at least the steady-state or transient components have to be known. However, if the transient components are determined only, there is no manner to verify their correctness. It is not the same inconvenience with the steady-state solutions. This is the reason that, in the following, closed form expressions for the steady-state solutions corresponding to these motions will be determined. These solutions satisfy the governing equations and the boundary conditions but are independent of the initial conditions.

To determine them in the same time and in a simple way, let us define the complex velocity $u_p(y, t)$ and the complex stresses $\tau_p(y, t)$ and $\sigma_p(y, t)$ by the relations

$$u_p(y, t) = u_{cp}(y, t) + iu_{sp}(y, t),\ \tau_p(y, t) = \tau_{cp}(y, t) + i\tau_{sp}(y, t),$$

$$\sigma_p(y, t) = \sigma_{cp}(y, t) + i\sigma_{sp}(y, t), \tag{24}$$

where i is the imaginary unit. According to the equalities (17) and (19) $u_p(y, t)$ has to satisfy the following boundary value problem:

$$[1 + \alpha(1 - y)]^2 \frac{\partial^2 u_p(y, t)}{\partial y^2} - 2\alpha[1 + \alpha(1 - y)] \frac{\partial u_p(y, t)}{\partial y}$$

$$= \mathrm{Re}\left(1 + \mathrm{We}\frac{\partial}{\partial t}\right)\frac{\partial u_p(y, t)}{\partial t}; \quad 0 < y < 1,\ t \in R, \tag{25}$$

$$\left.\frac{\partial u_p(y, t)}{\partial y}\right|_{y=0} = \frac{1}{(1 + \alpha)^2} e^{iwt},\ u_p(1, t) = 0;\ t \in R, \tag{26}$$

while $\tau_p(y, t)$ and $\sigma_p(y, t)$ are solutions of the ordinary differential equations

$$\left(1 + \mathrm{We}\frac{\partial}{\partial t}\right)\tau_p(y, t) = [1 + \alpha(1 - y)]^2 \frac{\partial u_p(y, t)}{\partial y}; \quad 0 < y < 1,\ t \in R, \tag{27}$$

$$\left(1 + \mathrm{We}\frac{\partial}{\partial t}\right)\sigma_p(y, t) = 2\mathrm{We}\tau_p(y, t)\frac{\partial u_p(y, t)}{\partial y};\ 0 < y < 1,\ t \in R. \tag{28}$$

Due to the form of the boundary conditions (26) and of the linearity of the governing equations (25), (27), and (28), we are looking for solutions of the form [23]

$$u_p(y, t) = U(y)\mathrm{e}^{i\omega t},\ \tau_p(y, t) = T(y)\mathrm{e}^{i\omega t},\ \sigma_p(y, t) = W(y)\mathrm{e}^{2i\omega t}, \tag{29}$$

where $U(y)$, $T(y)$ and $W(y)$ are complex functions.

3.1. Calculation of the steady-state velocities $u_{cp}(y, t)$ and $u_{sp}(y, t)$

Making a change of the spatial variable in Equation (25), specifically

$$r = \ln[1 + \alpha(1 - y)] \text{ or equivalently } y = (\alpha + 1 - \mathrm{e}^r)/\alpha, \tag{30}$$

and denoting by $V(r)$ the function defined by the equality

$$V(r) = U\left(\frac{\alpha + 1 - \mathrm{e}^r}{\alpha}, t\right), \tag{31}$$

we attain to the next boundary value problem

$$\frac{d^2V(r)}{dr^2} + \frac{dV(r)}{dr} - \beta^2 V(r) = 0;\ V(0) = 0,\ \left.\frac{dV(r)}{dr}\right|_{r=a} = -\frac{1}{\alpha(\alpha + 1)}, \tag{32}$$

where $\beta = \sqrt{i\omega \operatorname{Re}(1 + i\omega \mathrm{We})}/\alpha$ and $a = \ln(\alpha + 1)$. The solution of this boundary value problem is given by the relation

$$V(r) = -\frac{1}{\alpha(\alpha+1)} \frac{e^{r_2 r} - e^{r_1 r}}{r_2 e^{a r_2} - r_1 e^{a r_1}}; \; 0 < y < a, \, t \in R, \tag{33}$$

where $r_{1,2} = (-1 \pm \sqrt{1 + 4\beta^2})/2$.

On the basis of the previous notations and calculi, it results that

$$u_p(y, t) = -\frac{1}{\alpha(\alpha+1)} \frac{[1 + \alpha(1-y)]^{r_2} - [1 + \alpha(1-y)]^{r_1}}{r_2(\alpha+1)^{r_2} - r_1(\alpha+1)^{r_1}} e^{i\omega t}; \; 0 < y < 1, \, t \in R, \tag{34}$$

while $u_{cp}(y, t)$ and $u_{sp}(y, t)$ have the expressions

$$u_{cp}(y, t) = -\frac{1}{\alpha(\alpha+1)} \Re e \left\{ \frac{[1 + \alpha(1-y)]^{r_2} - [1 + \alpha(1-y)]^{r_1}}{r_2(\alpha+1)^{r_2} - r_1(\alpha+1)^{r_1}} e^{i\omega t} \right\};$$
$$0 < y < 1, \, t \in R, \tag{35}$$

$$u_{sp}(y, t) = -\frac{1}{\alpha(\alpha+1)} \operatorname{Im} \left\{ \frac{[1 + \alpha(1-y)]^{r_2} - [1 + \alpha(1-y)]^{r_1}}{r_2(\alpha+1)^{r_2} - r_1(\alpha+1)^{r_1}} e^{i\omega t} \right\};$$
$$0 < y < 1, \, t \in R, \tag{36}$$

where $\Re e$ and Im denotes the real and the imaginary part of that which follows.

3.2. Calculation of steady-state shear stresses $\tau_{cp}(y, t)$ and $\tau_{sp}(y, t)$

Using the relations (27), (29), and (34), it is not difficult to find $\tau_p(y, t)$ in the form

$$\tau_p(y, t) = \frac{1 + \alpha(1-y)}{\alpha+1} \frac{r_2[1 + \alpha(1-y)]^{r_2} - r_1[1 + \alpha(1-y)]^{r_1}}{r_2(\alpha+1)^{r_2} - r_1(\alpha+1)^{r_1}} \frac{e^{i\omega t}}{1 + i\omega \mathrm{We}};$$
$$0 < y < 1, \, t \in R. \tag{37}$$

Consequently, the dimensionless shear stresses $\tau_{cp}(y, t)$ and $\tau_{sp}(y, t)$ have the forms

$$\tau_{cp}(y, t) = \frac{1 + \alpha(1 - y)}{\alpha + 1} \times \Re\mathrm{e}\left\{\frac{r_2[1 + \alpha(1 - y)]^{r_2} - r_1[1 + \alpha(1 - y)]^{r_1}}{r_2(\alpha + 1)^{r_2} - r_1(\alpha + 1)^{r_1}} \frac{\mathrm{e}^{i\omega t}}{1 + i\omega\mathrm{We}}\right\}; \quad 0 < y < 1,\ t \in R, \tag{38}$$

$$\tau_{sp}(y, t) = \frac{1 + \alpha(1 - y)}{\alpha + 1} \times \mathrm{Im}\left\{\frac{r_2[1 + \alpha(1 - y)]^{r_2} - r_1[1 + \alpha(1 - y)]^{r_1}}{r_2(\alpha + 1)^{r_2} - r_1(\alpha + 1)^{r_1}} \frac{\mathrm{e}^{i\omega t}}{1 + i\omega\mathrm{We}}\right\}; \quad 0 < y < 1,\ t \in R. \tag{39}$$

The dimensionless frictional forces per unit area exerted by the fluid on the stationary plate, for instance, are given by the next relations

$$\tau_{cp}(1, t) = -\frac{1}{\alpha + 1} \Re\mathrm{e}\left\{\frac{\sqrt{1 + 4\beta^2}}{r_2(\alpha + 1)^{r_2} - r_1(\alpha + 1)^{r_1}} \frac{\mathrm{e}^{i\omega t}}{1 + i\omega\mathrm{We}}\right\}, \tag{40}$$

$$\tau_{sp}(1, t) = -\frac{1}{\alpha + 1} \mathrm{Im}\left\{\frac{\sqrt{1 + 4\beta^2}}{r_2(\alpha + 1)^{r_2} - r_1(\alpha + 1)^{r_1}} \frac{\mathrm{e}^{i\omega t}}{1 + i\omega\mathrm{We}}\right\}. \tag{41}$$

3.3. Calculation of the normal stresses $\sigma_{cp}(y, t)$ and $\sigma_{sp}(y, t)$

Following the same way as before and using the equalities (28), (29), (34), and (37), we can find that the complex normal stress $\sigma_p(y, t)$ has the expression

$$\sigma_p(y, t) = \frac{2\mathrm{We}}{(\alpha + 1)^2}\left[\frac{r_2[1 + \alpha(1 - y)]^{r_2} - r_1[1 + \alpha(1 - y)]^{r_1}}{r_2(\alpha + 1)^{r_2} - r_1(\alpha + 1)^{r_1}}\right]^2 \times \frac{\mathrm{e}^{2i\omega t}}{(1 + i\omega\mathrm{We})(1 + 2i\omega\mathrm{We})}; \quad 0 < y < 1,\ t \in R, \tag{42}$$

while the normal stresses $\sigma_{cp}(y, t)$ and $\sigma_{sp}(y, t)$ are given by the next relations

$$\sigma_{cp}(y, t) = \frac{2\mathrm{We}}{(\alpha+1)^2} \Re\mathrm{e}\left\{\left[\frac{r_2[1+\alpha(1-y)]^{r_2} - r_1[1+\alpha(1-y)]^{r_1}}{r_2(\alpha+1)^{r_2} - r_1(\alpha+1)^{r_1}}\right]^2 \times \frac{e^{2i\omega t}}{(1+i\omega\mathrm{We})(1+2i\omega\mathrm{We})}\right\}; \; 0 < y < 1, \; t \in R, \tag{43}$$

$$\sigma_{sp}(y, t) = \frac{2\mathrm{We}}{(\alpha+1)^2} \mathrm{Im}\left\{\left[\frac{r_2[1+\alpha(1-y)]^{r_2} - r_1[1+\alpha(1-y)]^{r_1}}{r_2(\alpha+1)^{r_2} - r_1(\alpha+1)^{r_1}}\right]^2 \times \frac{e^{2i\omega t}}{(1+i\omega\mathrm{We})(1+2i\omega\mathrm{We})}\right\}; \; 0 < y < 1, \; t \in R. \tag{44}$$

Finally, taking $\mathrm{We} = 0$ into above relations, we recover the non-dimensional steady-state solutions for the motion of Newtonian fluids with power-law dependence of viscosity on the pressure due to the lower plate that applies oscillatory shear stresses of the form (8) to the fluid. From Equations (35) and (36), for example, we recover the solutions

$$u_{Ncp}(y, t) = -\frac{1}{\alpha(\alpha+1)} \times \Re\mathrm{e}\left\{\frac{[1+\alpha(1-y)]^{r_4} - [1+\alpha(1-y)]^{r_3}}{r_4(\alpha+1)^{r_4} - r_3(\alpha+1)^{r_3}} e^{i\omega t}\right\}; \; 0 < y < 1, \; t \in R, \tag{45}$$

$$u_{Nsp}(y, t) = -\frac{1}{\alpha(\alpha+1)} \times \mathrm{Im}\left\{\frac{[1+\alpha(1-y)]^{r_4} - [1+\alpha(1-y)]^{r_3}}{r_4(\alpha+1)^{r_4} - r_3(\alpha+1)^{r_3}} e^{i\omega t}\right\}; \; 0 < y < 1, \; t \in R, \tag{46}$$

obtained by Fetecau et al. [19, Equations (43) and (44)]. In the above relations the roots r_3 and r_4 are given by the relations $r_{3,4} = [-\alpha \pm \sqrt{\alpha^2 + 4i\omega \mathrm{Re}}]/(2\alpha)$.

4. Limiting Cases

In this section, for completion with new steady solutions as well as to recover some known results from the existing literature, we consider two limit cases.

4.1. Case $\omega \to 0$ (flow due to an exponential shear stress $[1 - \exp(-t/\lambda)]S$ on the boundary)

Making $\omega \to 0$ in Equations (35), (38) and (43), the non-dimensional steady solutions

$$u_{ep}(y) = \lim_{\omega \to 0} u_{cp}(y, t) = \frac{y - 1}{1 + \alpha(1 - y)}, \ \tau_{ep} = \lim_{\omega \to 0} \tau_{cp}(y, t) = 1,$$

$$\sigma_{ep}(y) = \lim_{\omega \to 0} \sigma_{cp}(y, t) = \frac{2\mathrm{We}}{[1 + \alpha(1 - y)]^2}, \tag{47}$$

for the motion of IUCM fluids with power-law dependence of viscosity on the pressure produced by the lower plate that applies an exponential shear stress $[1 - \exp(-t/\lambda)]S$ to the fluid are obtained.

As expected $u_{ep}(y)$ and τ_{ep} given by Equations (47) are identical to the similar solutions obtained by Fetecau et al. [19] for the flow of Newtonian fluids with power-law dependence of the form (2) of viscosity on the pressure generated by the lower plate that applies a constant shear stress S to the fluid. Certainly, this it is possible because the governing equations corresponding to the steady flows of Newtonian and UCM fluids with/without pressure-dependent viscosity are identical. In the same time the boundary condition $[1 - \exp(-t/\lambda)]S \to S$ for increasing values of the time t.

4.2. Case $\alpha \to 0$ (flows of the ordinary IUCM fluids)

Now, writing the expressions of $u_{cp}(y, t), \tau_{cp}(y, t), \sigma_{cp}(y, t)$, $u_{sp}(y, t)$, $\tau_{sp}(y, t)$, and $\sigma_{sp}(y, t)$ in suitable forms (see Equation (A1) from Appendix for $u_{cp}(y, t)$ only) and taking their limits for $\alpha \to 0$, one obtains the following simple expressions (see also Equations (A2) and (A3)):

$$u_{Ocp}(y, t) = \lim_{\alpha\to 0} u_{cp}(y, t) = \Re\mathrm{e}\left\{\frac{\mathrm{sh}[(y-1)\sqrt{\delta}]}{\mathrm{ch}[\sqrt{\delta}]}\frac{\mathrm{e}^{i\omega t}}{\sqrt{\delta}}\right\}, \tag{48}$$

$$u_{Osp}(y, t) = \lim_{\alpha\to 0} u_{sp}(y, t) = \mathrm{Im}\left\{\frac{\mathrm{sh}[(y-1)\sqrt{\delta}]}{\mathrm{ch}[\sqrt{\delta}]}\frac{\mathrm{e}^{i\omega t}}{\sqrt{\delta}}\right\}, \tag{49}$$

$$\tau_{Ocp}(y, t) = \lim_{\alpha\to 0} \tau_{cp}(y, t) = \Re\mathrm{e}\left\{\frac{\mathrm{ch}[(y-1)\sqrt{\delta}]}{\mathrm{ch}[\sqrt{\delta}]}\frac{\mathrm{e}^{i\omega t}}{1+i\omega\mathrm{We}}\right\}, \tag{50}$$

$$\tau_{Osp}(y, t) = \lim_{\alpha\to 0} \tau_{sp}(y, t) = \mathrm{Im}\left\{\frac{\mathrm{ch}[(y-1)\sqrt{\delta}]}{\mathrm{ch}[\sqrt{\delta}]}\frac{\mathrm{e}^{i\omega t}}{1+i\omega\mathrm{We}}\right\}, \tag{51}$$

$$\sigma_{Ocp}(y, t) = \lim_{\alpha\to 0} \sigma_{cp}(y, t)$$

$$= 2\mathrm{We}\,\Re\mathrm{e}\left\{\frac{\mathrm{ch}^2[(y-1)\sqrt{\delta}]}{\mathrm{ch}^2[\sqrt{\delta}]}\frac{\mathrm{e}^{2i\omega t}}{(1+i\omega\mathrm{We})(1+2i\omega\mathrm{We})}\right\}, \tag{52}$$

$$\sigma_{Osp}(y, t) = \lim_{\alpha\to 0} \sigma_{sp}(y, t)$$

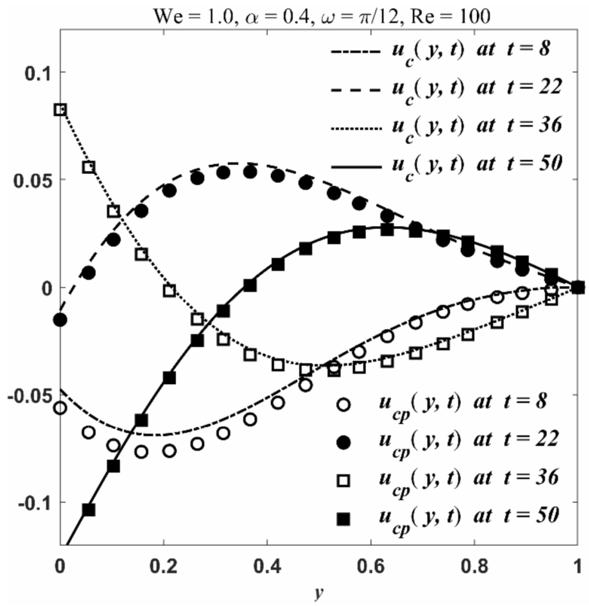
$$= 2\mathrm{We}\,\mathrm{Im}\left\{\frac{\mathrm{ch}^2[(y-1)\sqrt{\delta}]}{\mathrm{ch}^2[\sqrt{\delta}]}\frac{e^{2i\omega t}}{(1+i\omega\mathrm{We})(1+2i\omega\mathrm{We})}\right\}, \tag{53}$$

for dimensionless velocity, shear stress and normal stress fields corresponding to the same motions of the ordinary IUCM fluids. In the above relations, $\delta = i\omega\,\mathrm{Re}(1+i\omega\mathrm{We})$ and the solutions (49) and (51) are in accordance with those obtained by Fetecau et al. [27, Equations (39) with $K = 0$] where a different normalization has been used. Taking $\mathrm{We} = 0$ in Equations (48)-(53), the similar solutions corresponding to motions of ordinary Newtonian fluids produced by the lower plate that applies shear stresses of the form (8) to the fluid are received (see [19, Equations (47) and (48)] for the velocity field only).

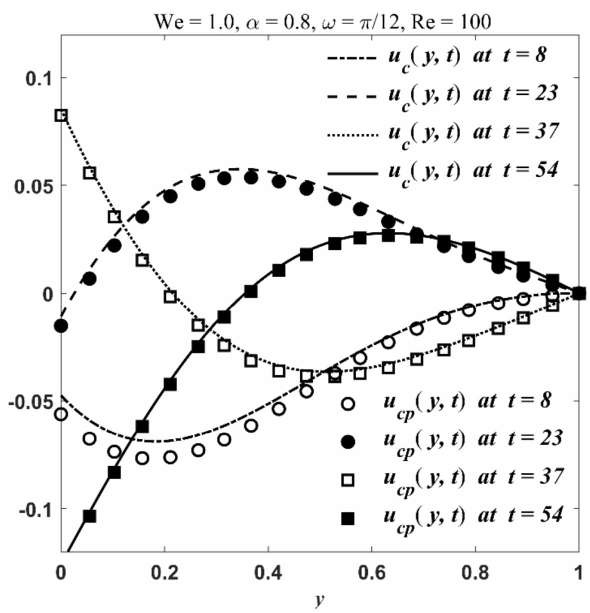
5. Numerical Results, Discussions and Conclusions

In this note two mixed initial-boundary value problems are analytically and numerically investigated. They correspond to isothermal unidirectional motions of IUCM fluids with power-law dependence of viscosity on the pressure between infinite horizontal parallel plates. The gravity effects are taken into consideration and closed form expressions are established for the dimensionless steady-state components $u_{cp}(y, t)$, $\tau_{cp}(y, t)$, $\sigma_{cp}(y, t)$ and $u_{sp}(y, t)$, $\tau_{sp}(y, t)$, $\sigma_{sp}(y, t)$ of the corresponding starting solutions. The steady solutions $u_{ep}(y)$, τ_{ep} and $\sigma_{ep}(y)$ corresponding to the motion of the same fluids induced by a shear stress of the form $[1 - \exp(-t/\lambda)]S$ on the boundary are acquired as limiting cases of $u_{cp}(y, t)$, $\tau_{cp}(y, t)$, respectively $\sigma_{cp}(y, t)$ when the oscillations' frequency $\omega \to 0$. As expected, these solutions are identical to the similar solutions corresponding to the motion of Newtonian fluids with power-law dependence of viscosity on the pressure generated by the lower plate that induces a constant shear stress S to the fluid.

As application of the results that have been obtained, as well as for the validation of their correctness, the convergence of start-up solutions $u_c(y, t)$ and $u_e(y, t)$ (numerical solutions) to their steady-state components $u_{cp}(y, t)$, respectively $u_{ep}(y)$ has been graphically proved in Figures 1 and 2 for two different values of the pressure-viscosity coefficient α and increasing values of the time t. It is interesting to observe from these graphical representations that the need time to touch the permanent state is a decreasing function with regard to this parameter for the motion due to an exponential shear stress on the boundary but increases in the case of the oscillatory motion. From Figures 2(a)-(b) it also results that, as it was to be expected, the fluid velocity in absolute value increases in time and smoothly diminishes from the maximum value one on the lower wall to the zero value on the upper plate.

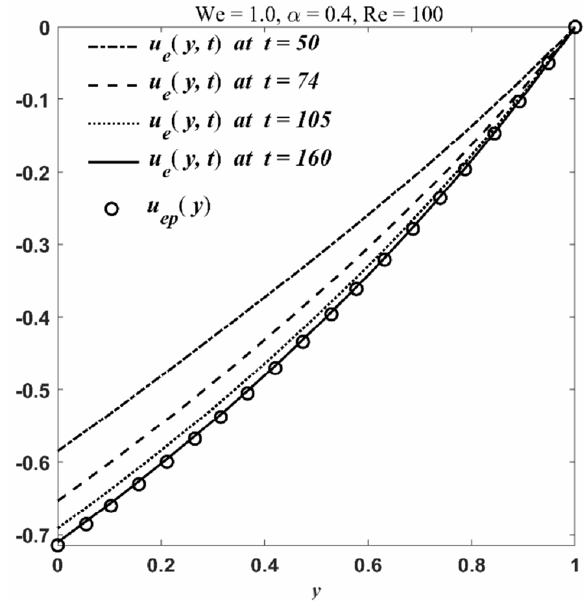


(a) $\alpha = 0.4$

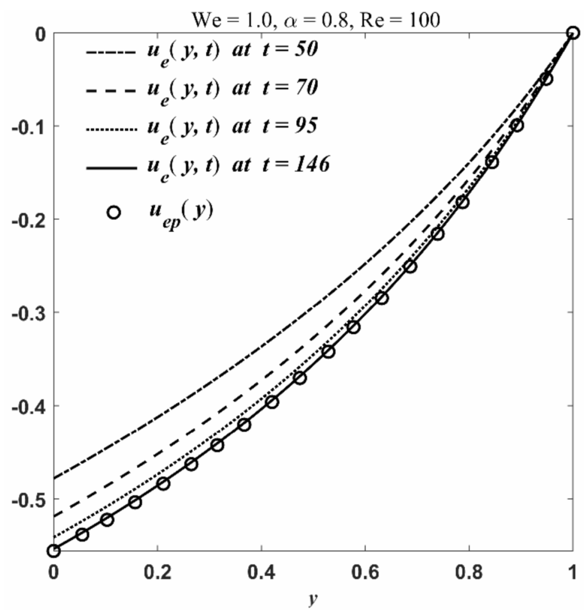


(b) $\alpha = 0.8$

Figure 1. Convergence of the start-up velocity $u_c(y, t)$ to its steady-state component $u_{cp}(y, t)$ for increasing values of the time t.



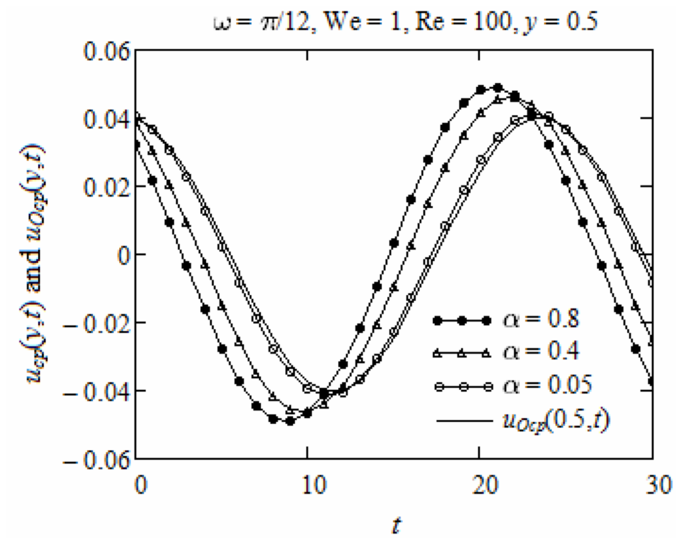
(a) $\alpha = 0.4$



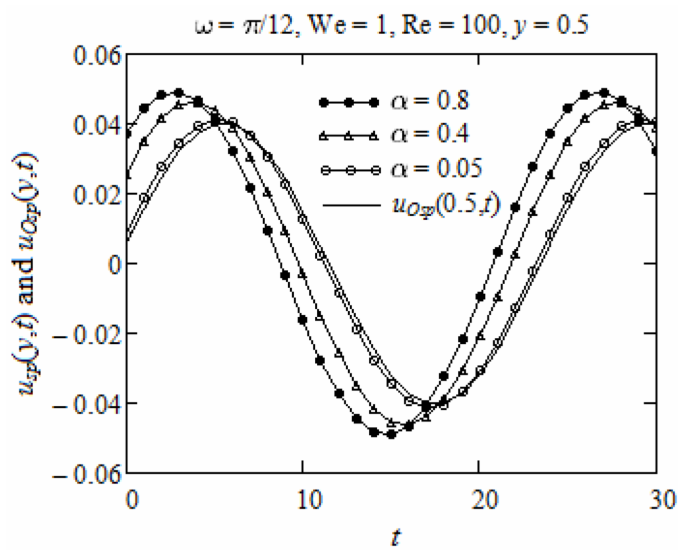
(b) $\alpha = 0.8$

Figure 2. Convergence of the start-up velocity $u_e(y, t)$ to its steady-state component $u_{ep}(y)$ for increasing values of the time t.

In order to underline the oscillatory behaviour of the first two motions, Figures 3-6 were prepared for different values of the pressure-viscosity coefficient α. In Figure 3(a)-(b) are presented time variations of mid plane non-dimensional velocities $u_{cp}(0.5, t)$ and $u_{sp}(0.5, t)$ at decreasing values of this parameter. The last value $\alpha = 0$ corresponds to the ordinary IUCM fluids. Frictional forces $\tau_{cp}(1, t)$ and $\tau_{sp}(1, t)$ per unit area exerted by the fluid on the fixed wall are depicted in Figure 4(a)-(b) while the normal stresses $\sigma_{cp}(0.5, t)$ and $\sigma_{sp}(0.5, t)$ in the median plane are presented in Figure 5(a)-(b) for the same values of physical parameters. From these graphs it clearly results that the amplitude of the oscillations diminishes for decreasing values of α and the diagrams corresponding to fluids with pressure-dependent viscosity tend to superpose over those of the ordinary fluids if α goes to zero. Finally, the spatial-temporal distribution of the non-dimensional start-up velocities $u_c(y, t)$ and $u_s(y, t)$ is presented in Figure 6(a)-(b) for comparison. Here, as well as in Figures 3-4, the phase difference between the two motions and their oscillatory features are clearly observed.

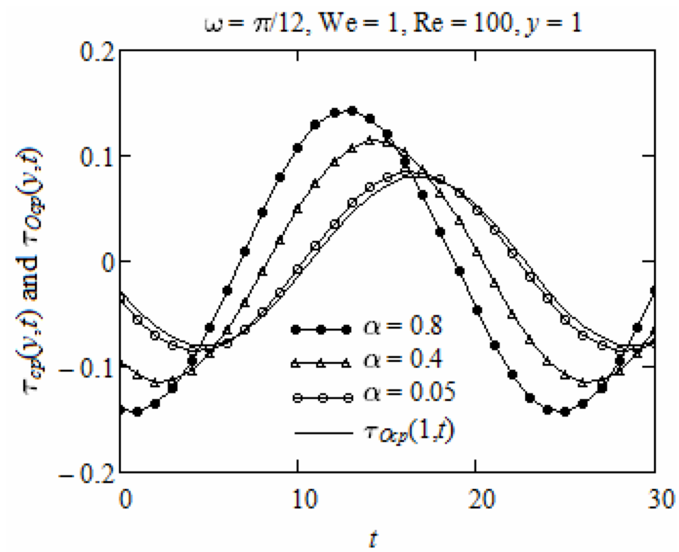


(a) $u_{cp}(0.5, t)$

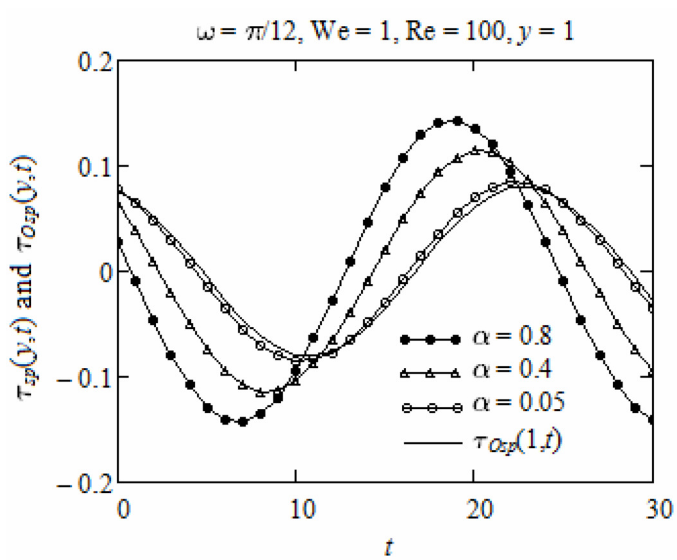


(b) $u_{sp}(0.5, t)$

Figure 3. Time variation of the mid plane velocities $u_{cp}(0.5, t)$, $u_{sp}(0.5, t)$ (at three decreasing values of α) and $u_{Ocp}(0.5, t)$, $u_{Osp}(0.5, t)$.

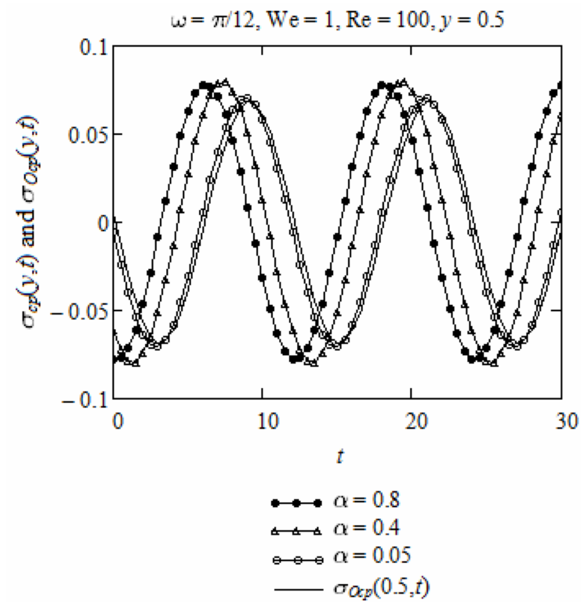


(a) $\tau_{cp}(1, t)$

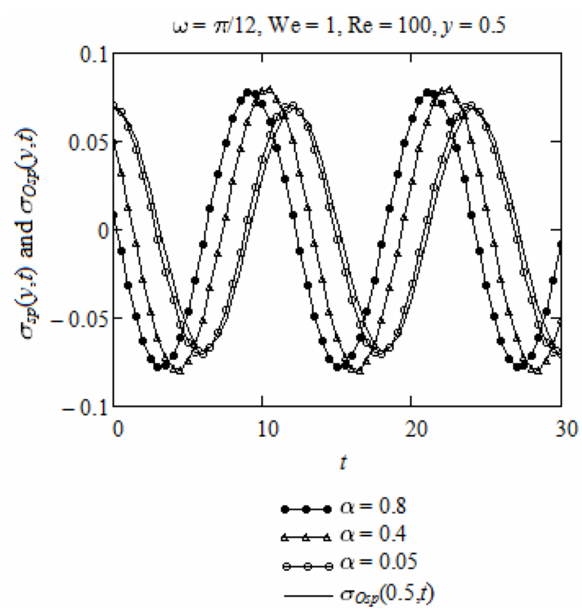


(b) $\tau_{sp}(1, t)$

Figure 4. Time variation of the frictional forces $\tau_{cp}(1, t)$, $\tau_{sp}(1, t)$ (at three decreasing values of α) and $\tau_{Ocp}(1, t)$, $\tau_{Osp}(1, t)$.



(a) $\sigma_{cp}(0.5, t)$



(b) $\sigma_{sp}(0.5, t)$

Figure 5. Time variation of the normal stresses $\sigma_{cp}(0.5, t)$, $\sigma_{sp}(0.5, t)$ (at three decreasing values of α) and $\sigma_{Ocp}(0.5, t)$, $\sigma_{Osp}(0.\ 5, t)$.

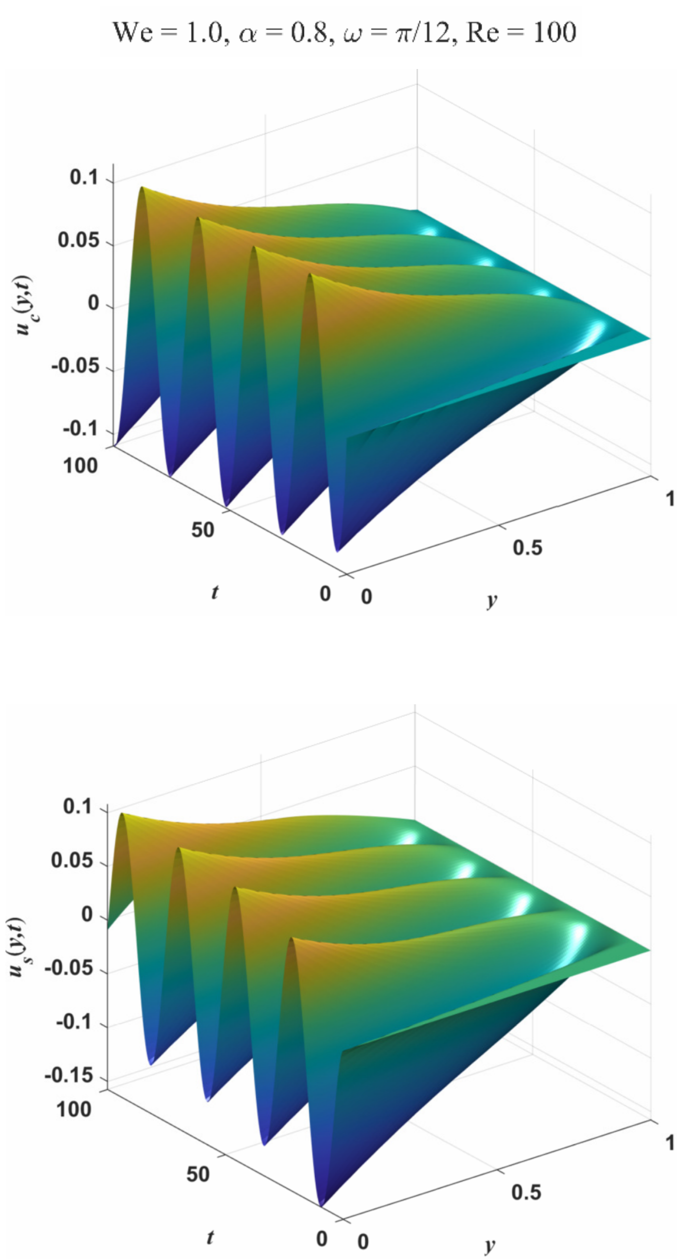


Figure 6. Spatial-temporal distribution of the dimensionless start-up velocities $u_c(y, t)$ and $u_s(y, t)$ (numerical solutions).

The main outcomes that have been obtained by this study are:

– Steady-state solutions were established for two mixed initial-boundary value problems describing motions of IUCM fluids with power-law dependence of viscosity on the pressure.

– These solutions have been used to determine the necessary time to touch the steady-state.

– They were also used to provide steady solutions for the motion of same fluids induced by a time-exponential shear stress. The shear stress τ_{ep}, unlike $u_{ep}(y)$ and $\sigma_{ep}(y)$, is constant.

– Steady-state is later obtained for motions of the ordinary IUCM fluids as compared to IUCM fluids with power-law dependence of viscosity on the pressure

– Steady-state solutions for motions of ordinary IUCM fluids due to oscillatory shear stresses were acquired as limiting cases of the general expressions that were previously determined.

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■

Appendix

$$u_{cp}(y,\ t) = -\frac{1}{\alpha+1}\,\Re\mathrm{e}\left\{\frac{1}{\alpha r_2}\,\frac{[1+\alpha(1-y)]^{r_2}}{(1+\alpha)^{r_2}}\,\frac{1-[1+\alpha(1-y)]^{r_1-r_2}}{1-(1+\alpha)^{r_1-r_2}\,r_1/r_2}\,\mathrm{e}^{i\omega t}\right\}, \tag{A1}$$

$$\lim_{\alpha\to 0}(\alpha r_2) = -\sqrt{\delta}, \quad \lim_{\alpha\to 0}(1+\alpha)^{r_2} = \mathrm{e}^{-\sqrt{\delta}}, \quad \lim_{\alpha\to 0}(1+\alpha)^{r_1-r_2} = \mathrm{e}^{2\sqrt{\delta}},$$

$$\lim_{\alpha\to 0}(r_1/r_2) = -1, \tag{A2}$$

$$\lim_{\alpha\to 0}[1+\alpha(1-y)]^{r_2} = \mathrm{e}^{(y-1)\sqrt{\delta}}, \ \lim_{\alpha\to 0}[1+\alpha(1-y)]^{r_1-r_2} = \mathrm{e}^{2(1-y)\sqrt{\delta}}. \tag{A3}$$