Journal of Pure and Applied Mathematics: Advances and Applications
Volume 24, Number 1, 2021, Pages 21-57
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jpamaa_7100122208

THE PATTERN AND CONDUCT OF THE SOLUTIONS OF CERTAIN NONLINEAR DIFFERENCE EQUATIONS

E. M. ELSAYED[1,2] and N. H. ALOTAIBI[1]

[1]Department of Mathematics
Faculty of Science
King Abdulaziz University
P.O. Box 80203
Jeddah 21589
Saudi Arabia
e-mail: emmelsayed@yahoo.com

[2]Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516
Egypt
e-mail: n.h.ob@hotmail.com

2021 Mathematics Subject Classification: 39A10.
Keywords and phrases: stability, boundedness, solution of difference equations.
Received June 24, 2021; Revised August 11, 2021.

Abstract

In this article, we investigate the dynamics and behaviour of the difference equation

$$x_{n+1} = \alpha x_{n-2} + \frac{\beta x_{n-1} x_{n-2}}{\gamma x_{n-1} + \delta x_{n-4}}, \quad n = 0, 1, \ldots.$$

Since $\alpha, \beta, \gamma, \delta$ are constants and the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}$ and x_0 are arbitrary positive real numbers. Moreover, we obtain the expressions of the solutions of four special cases of that equation.

1. Introduction

In this paper, we deal with the behaviour of the solutions such as boundedness, local and global stability of the following difference equation:

$$x_{n+1} = \alpha x_{n-2} + \frac{\beta x_{n-1} x_{n-2}}{\gamma x_{n-1} + \delta x_{n-4}}, \quad n = 0, 1, \ldots, \tag{1}$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1},$ and x_0 are arbitrary positive real numbers, and $\alpha, \beta, \gamma, \delta$ are constants. Moreover, we get the form of the solution of some special cases of this equation.

Let us introduce some basic definitions and some theorems that we need in the sequel we refer to the reference [26].

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I,$ the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \; n = 0, 1, \ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 (Equilibrium point)**.** A point $\overline{x} \in I$ is called an equilibrium point of Equation (2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Equation (2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2 (Stability)**.** (i) The equilibrium point $\overline{x}$ of Equation (2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \text{ for all } n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Equation (2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Equation (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Equation (2) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Equation (2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Equation (2).

(v) The equilibrium point $\bar{x}$ of Equation (2) is unstable if $\bar{x}$ is not locally stable.

The linearized equation of Equation (2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(x_n, x_{n-1}, \ldots, x_{n-k})}{\partial x_{n-i}}\bigg|_{x_n = x_{n-1} = \ldots = x_{n-k} = \bar{x}} y_{n-i}. \tag{3}$$

Theorem A ([26])**.** *Assume that* $p, q \in R$ *and* $k \in \{0, 1, 2, \ldots\}$. *Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \ldots.$$

Remark. Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, \ldots, \tag{4}$$

where $p_1, p_2, \ldots, p_k \in R$ and $k \in \{1, 2, \ldots\}$. Then Equation (4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation:

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}). \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B ([26]). *Let* $[p, q]$ *be an interval of real numbers and assume that*

$$g : [p, q]^3 \to [p, q],$$

is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ *is non-decreasing in* x *and* y *in* $[p, q]$ *for each* $z \in [p, q]$, *and is non-increasing in* $z \in [p, q]$ *for each* x *and* y *in* $[p, q]$.

(b) *If* $(m, M) \in [p, q] \times [p, q]$ *is a solution of the system*

$$M = g(M, M, m) \text{ and } m = g(m, m, M),$$

then

$$m = M.$$

Then Equation (5) *has a unique equilibrium* $\overline{x} \in [p, q]$ *and every solution of Equation* (5) *converges to* $\overline{x}$.

Definition 3 (Fibonacci sequence). The sequence $\{f_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, \ldots\}$, i.e., $f_m = f_{m-1} + f_{m-2}$, $m \geq 0$, $f_{-2} = 0$, $f_{-1} = 1$ is called *Fibonacci sequence*.

Many researchers have investigated the behaviour of the solution of difference equations for instance, the solutions of the following difference equation has been obtained as found in Abo-Zeid [5]

$$x_{n+1} = \frac{Ax_{n-1}}{B - Cx_n x_{n-1}}.$$

Ahmed et al. [7] studied the periodicity character and the global stability. Consequently, the solution of special case of the following recursive sequence has been obtained.

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}.$$

Elabbasy et al. [14] examined the boundedness, the global stability, and periodicity character. Therefore, the solution of some special cases of the difference equation has been determined

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

El-Moneam et al. [18] analyzed the periodicity character, the global stability of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-i} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-i}}.$$

The following problem was solved by Saleh and Aloqeili [27]:

$$x_{n+1} = A + \frac{x_n}{x_{n-k}}.$$

The global attractivity for the recursive sequence was investigated as found in Yalcinkaya et al. [33]

$$x_{n+1} = \frac{\alpha + x_{n-m}}{x_n^k}.$$

The global asymptotic stability of the difference equation was studied by Zayed and El-Moneam [35]

$$x_{n+1} = ax_n - (bx_n)/(cx_n + dx_{n-k}).$$

See also [1], [5], [34]. Other related results on rational difference equations can be found in [1-33].

The study of difference equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

2. Dynamics of Equation (1)

Here we study the dynamics and behaviour of Equation (1) when the constants $\alpha, \beta, \gamma,$ and δ are positive real numbers.

2.1. Local stability of Equation (1)

Here, we study the local stability character of the solutions of Equation (1). Equation (1) has a unique equilibrium point

$$\bar{x} = \alpha\bar{x} + \frac{\beta\bar{x}^2}{\gamma\bar{x} + \delta\bar{x}},$$

or,

$$\bar{x}^2(1-\alpha)(\gamma+\delta) = \beta\bar{x}^2,$$

if $(1-\alpha)(\gamma+\delta) \neq \beta,$ then the unique equilibrium point is $\bar{x} = 0.$

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v, w) = \alpha v + \frac{\beta uv}{\gamma u + \delta w}. \quad (6)$$

Accordingly, it follows that

$$f_u(u, v, w) = \frac{\beta\delta vw}{(\gamma u + \delta w)^2},$$

$$f_v(u, v, w) = \alpha + \frac{\beta u}{\gamma u + \delta w},$$

$$f_w(u, v, w) = \frac{-\beta\delta uv}{(\gamma u + \delta w)^2}.$$

Thus

$$f_u(\bar{x}, \bar{x}, \bar{x}) = \frac{\beta\delta}{(\gamma+\delta)^2} = -a_2,$$

$$f_v(\bar{x}, \bar{x}, \bar{x}) = \alpha + \frac{\beta}{\gamma+\delta} = -a_1,$$

$$f_w(\bar{x}, \bar{x}, \bar{x}) = \frac{-\beta\delta}{(\gamma+\delta)^2} = -a_0.$$

The linearized equation of Equation (1) about $\bar{x}$ is

$$y_{n+1} + a_2 y_{n-l} + a_1 y_{n-2} + a_0 y_{n-4} = 0,$$

$$y_{n+1} + \frac{\beta\delta}{(\gamma+\delta)^2} y_{n-1} + \left(\alpha + \frac{\beta}{\gamma+\delta}\right) y_{n-2} - \frac{\beta\delta}{(\gamma+\delta)^2} y_{n-4} = 0. \tag{7}$$

Theorem 1. *Assume that*

$$\beta(\gamma + 3\delta) < (1-\alpha)(\gamma+\delta)^2 \ \textit{and} \ \alpha < 1.$$

Then the equilibrium point of Equation (1) *is locally asymptotically stable.*

Proof. It is follows by Theorem A that, Equation (7) is asymptotically stable if

$$\left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| + \left|\alpha + \frac{\beta}{\gamma+\delta}\right| + \left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| < 1,$$

and so

$$\beta\delta + \alpha(\gamma+\delta)^2 + \beta(\gamma+\delta) + \beta\delta < (\gamma+\delta)^2.$$

Therefore,

$$2\beta\delta + \beta(\gamma+\delta) < (\gamma+\delta)^2 - \alpha(\gamma+\delta)^2.$$

Then

$$2\beta\delta + \beta\gamma + \beta\delta < (1 - \alpha)(\gamma + \delta)^2.$$

$$\frac{\beta\gamma + 3\beta\delta}{(c + d)^2} < (1 - a).$$

$$\frac{\beta(\gamma + 3\delta)}{(\gamma + \delta)^2} < (1 - \alpha).$$

The proof is complete.

2.2. Global attractor of the equilibrium point of Equation (1)

The global attractively character of solutions of Equation (1) has been investigated as follows:

Theorem 2. *The equilibrium point* $\overline{x}$ *of Equation* (1) *is global attractor if* $\gamma(1 - \alpha) \neq \beta$.

Proof. Let p, q are real numbers and assume that $g : [p, q]^3 \longrightarrow [p, q]$ be a function defined by $g(u, v, w) = \alpha v + \frac{\beta uv}{\gamma u + \delta w}$, then we can easily see that the function $g(u, v, w)$ increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system

$$M = g(M, M, m) \text{ and } m = g(m, m, M).$$

Then from Equation (1), we see that

$$M = \alpha M + \frac{\beta M^2}{\gamma M + \delta m}, \quad m = \alpha m + \frac{\beta m^2}{\gamma m + \delta M},$$

or,

$$M(1 - \alpha) = \frac{\beta M^2}{\gamma M + \delta m}, \quad m(1 - \alpha) = \frac{\beta m^2}{\gamma m + \delta M},$$

then

$$\gamma(1-\alpha)M^2 + \delta(1-\alpha)Mm = \beta M^2, \; \gamma(1-\alpha)m^2 + \delta(1-\alpha)Mm = \beta m^2,$$

Subtracting we obtain

$$\gamma(1-\alpha)(M^2 - m^2) = \beta(M^2 - m^2), \quad \gamma(1-\alpha) \neq \beta.$$

Thus

$$M = m.$$

It follows by Theorem B that $\bar{x}$ is a global attractor of Equation (1) and then the proof is complete.

2.3. Boundedness of solutions of Equation (1)

The boundedness of solutions of Equation (1) has been examined as follows:

Theorem 3. *Every solution of Equation* (1) *is bounded if* $\left(\alpha + \frac{\beta}{\gamma}\right) < 1.$

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1} = \alpha x_{n-2} + \frac{\beta x_n x_{n-2}}{\gamma x_n + \delta x_{n-4}} \leq \alpha x_{n-2} + \frac{\beta x_n x_{n-2}}{\gamma x_n} = \left(\alpha + \frac{\beta}{\gamma}\right) x_{n-2}.$$

Then

$$x_{n+1} \leq x_{n-2} \text{ for all } n \geq 0.$$

Then the subsequences $\{x_{3n}\}_{n=0}^{\infty}, \{x_{3n+1}\}_{n=0}^{\infty}, \{x_{3n+2}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

2.4. Numerical examples

For emphasizing the results of this section, we take some numerical examples:

Example 1. In Figure 1 we consider $x_{-4} = 3$, $x_{-3} = 8$, $x_{-2} = 1.6$, $x_{-1} = 2$, $x_0 = 7$ and $\alpha = 0.8$, $\beta = 4$, $\gamma = 6$, $\delta = 7$.

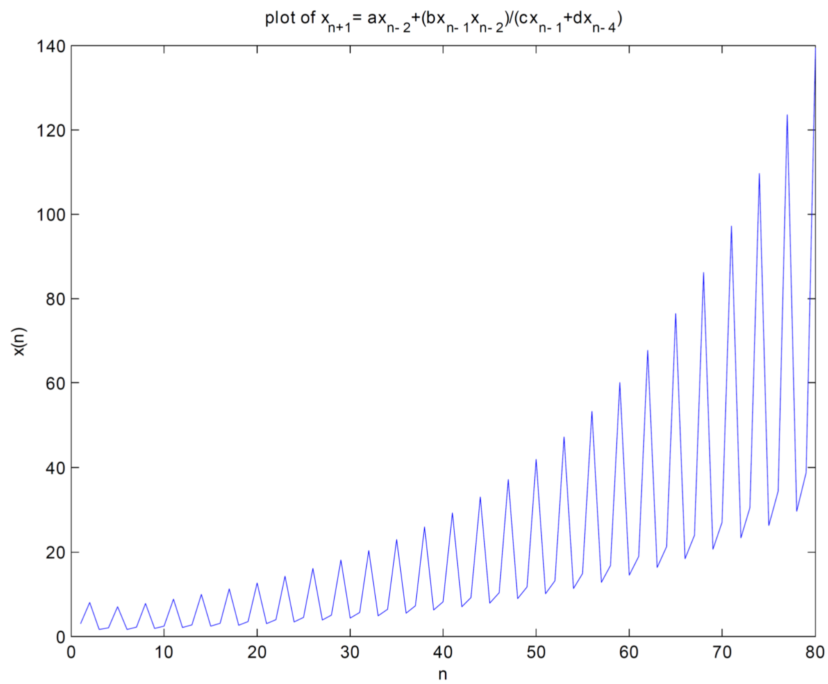

Figure 1. This figure shows the dynamics of the solution of Equation (1) wherever $x_{-4} = 3$, $x_{-3} = 8$, $x_{-2} = 1.6$, $x_{-1} = 2$, $x_0 = 7$ and $\alpha = 0.8$, $\beta = 4$, $\gamma = 6$, $\delta = 7$.

Example 2. This example shows the solutions when, $x_{-4} = 3$, $x_{-3} = 8$, $x_{-2} = 1.6$, $x_{-1} = 2$, $x_0 = 7$ and $\alpha = 0.5$, $\beta = 4$, $\gamma = 6$, $\delta = 7$, Figure 2.

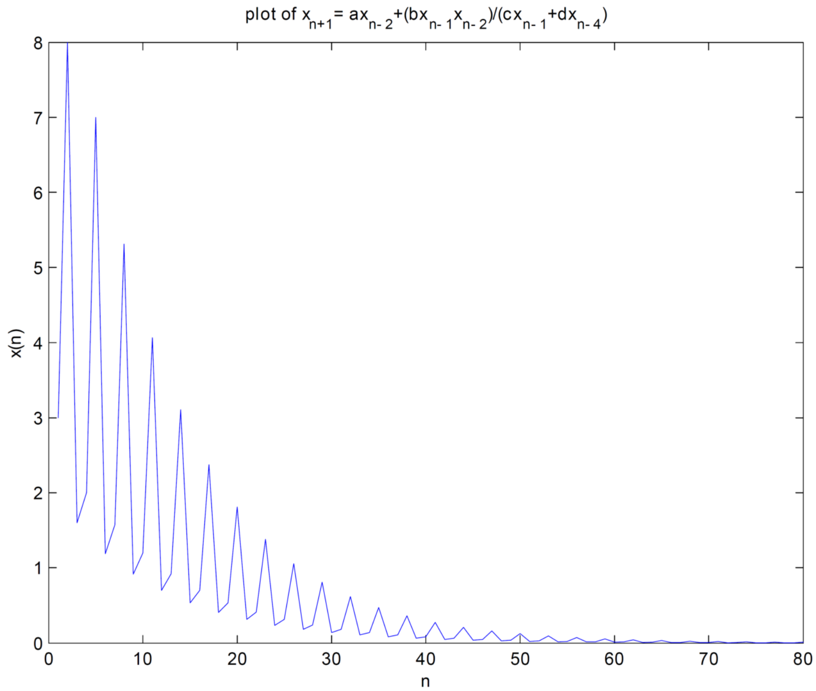

Figure 2. Plot of solution of Equation (1) when we set $x_{-4} = 3$, $x_{-3} = 8$, $x_{-2} = 1.6$, $x_{-1} = 2$, $x_0 = 7$ and $\alpha = 0.5$, $\beta = 4$, $\gamma = 6$, $\delta = 7$.

Example 3. See Figure 3 since $x_{-4} = 13$, $x_{-3} = 11$, $x_{-2} = 1.6$, $x_{-1} = 0.2$, $x_0 = 0.7$ and $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 0.6$, $\delta = 0.4$.

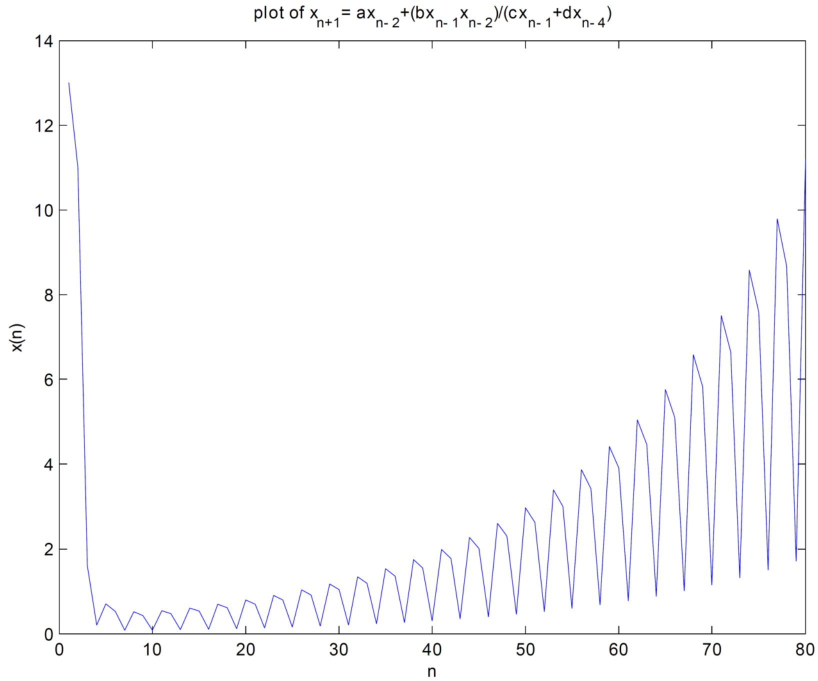

Figure 3. This figure offer the behaviour of solutions of Equation (1) when $x_{-4} = 13$, $x_{-3} = 11$, $x_{-2} = 1.6$, $x_{-1} = 0.2$, $x_0 = 0.7$ and $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 0.6$, $\delta = 0.4$.

Example 4. Figure 4 illustrates the case we choose $x_{-4} = 1.3$, $x_{-3} = 1.1$, $x_{-2} = 6$, $x_{-1} = 0.2$, $x_0 = 0.7$ and $\alpha = 0.3$, $\beta = 0.2$, $\gamma = 0.6$, $\delta = 4$.

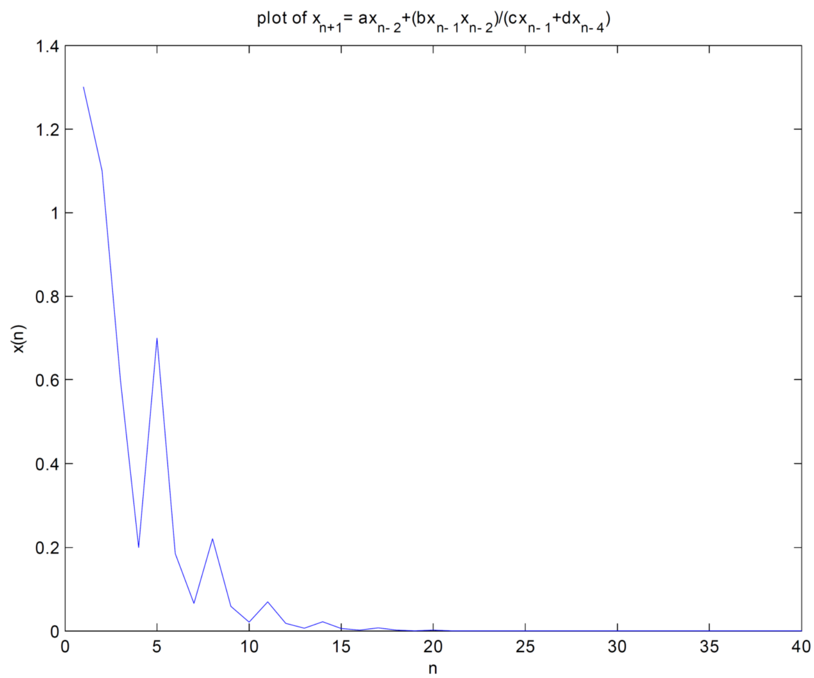

Figure 4. Draw the solution of Equation (1) while $x_{-4} = 1.3$, $x_{-3} = 1.1$, $x_{-2} = 6$, $x_{-1} = 0.2$, $x_0 = 0.7$ and $\alpha = 0.3$, $\beta = 0.2$, $\gamma = 0.6$, $\delta = 4$.

3. Special Cases of Equation (1)

In this section, we obtain the form of the solutions of Equation (1) since the constants α, β, γ, and δ are integers numbers.

3.1. First equation

The following special case of Equation (1) has been studied

$$x_{n+1} = x_{n-2} + \frac{x_{n-1}x_{n-2}}{x_{n-1} + x_{n-4}}, \tag{8}$$

where the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, and x_0 are arbitrary non zero real numbers.

Theorem 4. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Equation* (8). *Then for* $n = 0, 1, 2, \ldots$

$$x_{6n-4} = h\prod_{i=0}^{n-1}\left(\frac{f_{6i}p + f_{6i-1}h}{f_{6i-1}p + f_{6i-2}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right),$$

$$x_{6n-3} = k\prod_{i=0}^{n-1}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i}q + f_{6i-1}k}{f_{6i-1}q + f_{6i-2}k}\right),$$

$$x_{6n-2} = r\prod_{i=0}^{n-1}\left(\frac{f_{6i+2}p + f_{6i+1}h}{f_{6i+1}p + f_{6i}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right),$$

$$x_{6n-1} = p\prod_{i=0}^{n-1}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right),$$

$$x_{6n} = q\prod_{i=0}^{n-1}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right),$$

$$x_{6n+1} = r\left(\frac{2p + h}{p + h}\right)\prod_{i=0}^{n-1}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right),$$

where $x_{-4} = h$, $x_{-3} = k$, $x_{-2} = r$, $x_{-1} = p$, $x_0 = q$, $\{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, \ldots\}$, $f_0 = 0$, $f_1 = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 2$. That is,

$$x_{6n-9} = k\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i}q + f_{6i-1}k}{f_{6i-1}q + f_{6i-2}k}\right),$$

$$x_{6n-8} = r\prod_{i=0}^{n-2}\left(\frac{f_{6i+2}p + f_{6i+1}h}{f_{6i+1}p + f_{6i}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right),$$

$$x_{6n-7} = p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p+f_{6i+5}h}{f_{6i+5}p+f_{6i+4}h}\right)\left(\frac{f_{6i+2}q+f_{6i+1}k}{f_{6i+1}q+f_{6i}k}\right),$$

$$x_{6n-6} = q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p+f_{6i+3}h}{f_{6i+3}p+f_{6i+2}h}\right)\left(\frac{f_{6i+6}q+f_{6i+5}k}{f_{6i+5}q+f_{6i+4}k}\right),$$

$$x_{6n-5} = r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right).$$

Now, it follows from Equation (8) that

$$\begin{aligned}
x_{6n-4} &= x_{6n-7}+\frac{x_{6n-6}x_{6n-7}}{x_{6n-6}+x_{6n-9}}\\
&= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p+f_{6i+5}h}{f_{6i+5}p+f_{6i+4}h}\right)\left(\frac{f_{6i+2}q+f_{6i+1}k}{f_{6i+1}q+f_{6i}k}\right)\\
&\quad+\frac{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p+f_{6i+3}h}{f_{6i+3}p+f_{6i+2}h}\right)\left(\frac{f_{6i+6}q+f_{6i+5}k}{f_{6i+5}q+f_{6i+4}k}\right)}{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p+f_{6i+3}h}{f_{6i+3}p+f_{6i+2}h}\right)\left(\frac{f_{6i+6}q+f_{6i+5}k}{f_{6i+5}q+f_{6i+4}k}\right)}\\
&\qquad\times\frac{p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p+f_{6i+5}h}{f_{6i+5}p+f_{6i+4}h}\right)\left(\frac{f_{6i+2}q+f_{6i+1}k}{f_{6i+1}q+f_{6i}k}\right)}{+k\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p+f_{6i+3}h}{f_{6i+3}p+f_{6i+2}h}\right)\left(\frac{f_{6i}q+f_{6i-1}k}{f_{6i-1}q+f_{6i-2}k}\right)}\\
&= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p+f_{6i+5}h}{f_{6i+5}p+f_{6i+4}h}\right)\left(\frac{f_{6i+2}q+f_{6i+1}k}{f_{6i+1}q+f_{6i}k}\right)\\
&\quad+\frac{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}q+f_{6i+5}k}{f_{6i+5}q+f_{6i+4}k}\right)p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p+f_{6i+5}h}{f_{6i+5}p+f_{6i+4}h}\right)\left(\frac{f_{6i+2}q+f_{6i+1}k}{f_{6i+1}q+f_{6i}k}\right)}{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}q+f_{6i+5}k}{f_{6i+5}q+f_{6i+4}k}\right)+k\prod_{i=0}^{n-2}\left(\frac{f_{6i}q+f_{6i-1}k}{f_{6i-1}q+f_{6i-2}k}\right)}
\end{aligned}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)$$

$$+ \frac{pq\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{\prod_{i=0}^{n-2}\left(\frac{f_{6i+5}q + f_{6i+4}k}{f_{6i+6}q + f_{6i+5}k}\right)\left[q\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right) + k\prod_{i=0}^{n-2}\left(\frac{f_{6i}q + f_{6i-1}k}{f_{6i-1}q + f_{6i-2}k}\right)\right]}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)$$

$$+ \frac{pq\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{\left[q + k\prod_{i=0}^{n-2}\left(\frac{f_{6i}q + f_{6i-1}k}{f_{6i+6}q + f_{6i+5}k}\right)\left(\frac{f_{6i+5}q + f_{6i+4}k}{f_{6i-1}q + f_{6i-2}k}\right)\right]}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)$$

$$+ \frac{pq\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{\left[q + q\left(\frac{f_{6n-7}q + f_{6n-8}k}{f_{6n-6}q + f_{6n-7}k}\right)\right]}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)$$

$$+ \frac{p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{\left[1 + \left(\frac{f_{6n-7}q + f_{6n-8}k}{f_{6n-6}q + f_{6n-7}k}\right)\right]}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)$$

$$+ \frac{p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{\left(\frac{f_{6n-6}q + f_{6n-7}k + f_{6n-7}q + f_{6n-8}k}{f_{6n-6}q + f_{6n-7}k}\right)}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)$$

$$+ \frac{p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{\left(\frac{f_{6n-5}q + f_{6n-6}k}{f_{6n-6}q + f_{6n-7}k}\right)}$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)\left[1 + \frac{f_{6n-6}q + f_{6n-7}k}{f_{6n-5}q + f_{6n-6}k}\right]$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)\left[\frac{f_{6n-5}q + f_{6n-6}k + f_{6n-6}q + f_{6n-7}k}{f_{6n-5}q + f_{6n-6}k}\right]$$

$$= p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)\left[\frac{f_{6n-4}q + f_{6n-5}k}{f_{6n-5}q + f_{6n-6}k}\right].$$

Therefore

$$x_{6n-4} = h\prod_{i=0}^{n-1}\left(\frac{f_{6i}p + f_{6i-1}h}{f_{6i-1}p + f_{6i-2}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right).$$

Also, from Equation (8), we see that

$$x_{6n-3} = x_{6n-6} + \frac{x_{6n-5}x_{6n-6}}{x_{6n-5} + x_{6n-8}}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)}{r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)}$$

$$\times \frac{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{+ r\prod_{i=0}^{n-2}\left(\frac{f_{6i+2}p + f_{6i+1}h}{f_{6i+1}p + f_{6i}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)},$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{\left(\frac{2p+h}{p+h}\right)q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{\prod_{i=0}^{n-2}\left(\frac{f_{6i+7}p + f_{6i+6}h}{f_{6i+8}p + f_{6i+7}h}\right)\left[\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right) + \prod_{i=0}^{n-2}\left(\frac{f_{6i+2}p + f_{6i+1}h}{f_{6i+1}p + f_{6i}h}\right)\right]}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{\left(\frac{2p+h}{p+h}\right) q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{\left(\frac{2p+h}{p+h}\right) + \prod_{i=0}^{n-2}\left(\frac{f_{6i+2}p + f_{6i+1}h}{f_{6i+8}p + f_{6i+7}h}\right)\left(\frac{f_{6i+7}p + f_{6i+6}h}{f_{6i+1}p + f_{6i}h}\right)}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{\left(\frac{2p+h}{p+h}\right) q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{\left(\frac{2p+h}{p+h}\right) + \left(\frac{2p+h}{p+h}\right)\left(\frac{f_{6n-5}p + f_{6n-6}h}{f_{6n-4}p + f_{6n-5}h}\right)}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{\left[1 + \frac{f_{6n-5}p + f_{6n-6}h}{f_{6n-4}p + f_{6n-5}h}\right]}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{\left[\frac{f_{6n-4}p + f_{6n-5}h + f_{6n-5}p + f_{6n-6}h}{f_{6n-4}p + f_{6n-5}h}\right]}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)$$

$$+ \frac{q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)}{\left[\frac{f_{6n-3}p + f_{6n-4}h}{f_{6n-4}p + f_{6n-5}h}\right]}$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)\left[1 + \left(\frac{f_{6n-4}p + f_{6n-5}h}{f_{6n-3}p + f_{6n-4}h}\right)\right]$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)\left[\frac{f_{6n-3}p + f_{6n-4}h + f_{6n-4}p + f_{6n-5}h}{f_{6n-3}p + f_{6n-4}h}\right]$$

$$= q\prod_{i=0}^{n-2}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i+6}q + f_{6i+5}k}{f_{6i+5}q + f_{6i+4}k}\right)\left(\frac{f_{6n-2}p + f_{6n-3}h}{f_{6n-3}p + f_{6n-4}h}\right).$$

Therefore

$$x_{6n-3} = k\prod_{i=0}^{n-1}\left(\frac{f_{6i+4}p + f_{6i+3}h}{f_{6i+3}p + f_{6i+2}h}\right)\left(\frac{f_{6i}q + f_{6i-1}k}{f_{6i-1}q + f_{6i-2}k}\right).$$

Also, from Equation (8), we see that

$$x_{6n-2} = x_{6n-5} + \frac{x_{6n-4}x_{6n-5}}{x_{6n-4} + x_{6n-7}}$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)$$

$$+ \frac{h\prod_{i=0}^{n-1}\left(\frac{f_{6i}p + f_{6i-1}h}{f_{6i-1}p + f_{6i-2}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}{h\prod_{i=0}^{n-1}\left(\frac{f_{6i}p + f_{6i-1}h}{f_{6i-1}p + f_{6i-2}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}$$

$$\times \frac{r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)}{+\, p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\left(\frac{f_{6i+2}q + f_{6i+1}k}{f_{6i+1}q + f_{6i}k}\right)}$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)$$

$$+ \frac{hr\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p + f_{6i+7}h}{f_{6i+7}p + f_{6i+6}h}\right)\left(\frac{f_{6i+4}q + f_{6i+3}k}{f_{6i+3}q + f_{6i+2}k}\right)}{\left(\frac{f_{6n}q + f_{6n-1}k}{f_{6n+1}q + f_{6n}k}\right)\prod_{i=0}^{n-1}\left(\frac{f_{6i-1}p + f_{6i-2}h}{f_{6i}p + f_{6i-1}h}\right)\left[h\left(\frac{f_{6n+1}q + f_{6n}k}{f_{6n}q + f_{6n-1}k}\right)\prod_{i=0}^{n-1}\left(\frac{f_{6i}p + f_{6i-1}h}{f_{6i-1}p + f_{6i-2}h}\right) + p\prod_{i=0}^{n-2}\left(\frac{f_{6i+6}p + f_{6i+5}h}{f_{6i+5}p + f_{6i+4}h}\right)\right]}$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)$$

$$+\frac{h\left[r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)\right]}{h+p\left(\frac{f_{6n}q+f_{6n-1}k}{f_{6n+1}q+f_{6n}k}\right)\left(\frac{\prod_{i=0}^{n-1} f_{6i-1}p+f_{6i-2}h}{\prod_{i=0}^{n-2} f_{6i+5}p+f_{6i+4}h}\right)\left(\frac{\prod_{i=0}^{n-2} f_{6i+6}p+f_{6i+5}h}{\prod_{i=0}^{n-1} f_{6i}p+f_{6i-1}h}\right)}$$

$$x_{6n-2} = r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)$$

$$+\frac{h\left[r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)\right]}{h+p\left(\frac{f_{6n}q+f_{6n-1}k}{f_{6n+1}q+f_{6n}k}\right)\frac{h}{p}}$$

$$x_{6n-2} = r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)$$

$$+\frac{h\left[r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)\right]}{h\left(1+\frac{f_{6n}q+f_{6n-1}k}{f_{6n+1}q+f_{6n}k}\right)}$$

$$x_{6n-2} = r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)$$

$$+\frac{\left[r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)\right]}{\left[\frac{f_{6n+1}q+f_{6n}k+f_{6n}q+f_{6n-1}k}{f_{6n+1}q+f_{6n}k}\right]}$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)$$

$$+\frac{r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)}{\frac{f_{6n+2}q+f_{6n+1}k}{f_{6n+1}q+f_{6n}k}}$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)\left[1+\frac{f_{6n+1}q+f_{6n}k}{f_{6n+2}q+f_{6n+1}k}\right]$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)$$

$$\times\left[\frac{f_{6n+2}q+f_{6n+1}k+f_{6n+1}q+f_{6n}k}{f_{6n+2}q+f_{6n+1}k}\right]$$

$$= r\left(\frac{2p+h}{p+h}\right)\prod_{i=0}^{n-2}\left(\frac{f_{6i+8}p+f_{6i+7}h}{f_{6i+7}p+f_{6i+6}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right)\left[\frac{f_{6n+3}q+f_{6n+2}k}{f_{6n+2}q+f_{6n+1}k}\right].$$

Therefore

$$x_{6n-2} = r\prod_{i=0}^{n-1}\left(\frac{f_{6i+2}p+f_{6i+1}h}{f_{6i+1}p+f_{6i}h}\right)\left(\frac{f_{6i+4}q+f_{6i+3}k}{f_{6i+3}q+f_{6i+2}k}\right).$$

Also, other formulas can be proved similarly. Hence the proof is completed.

Example 5. To illustrate this case, we choose $x_{-4} = 1.23$, $x_{-3} = 1.1$, $x_{-2} = 5$, $x_{-1} = 1.52$, and $x_0 = 0.9$. We notice in the Figure 5.

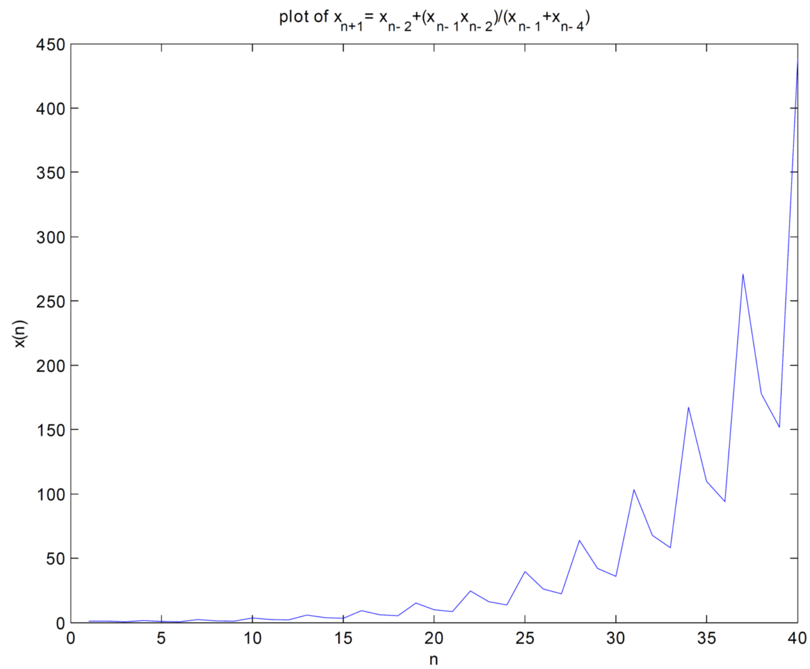

Figure 5. This figure presents the dynamics for solutions of (8) while $x_{-4} = 1.23$, $x_{-3} = 1.1$, $x_{-2} = 5$, $x_{-1} = 1.52$, and $x_0 = 0.9$.

Example 6. In Figure 6, we choose $x_{-4} = 6.23$, $x_{-3} = 11.1$, $x_{-2} = 0.5$, $x_{-1} = 1.52$, and $x_0 = -9.9$.

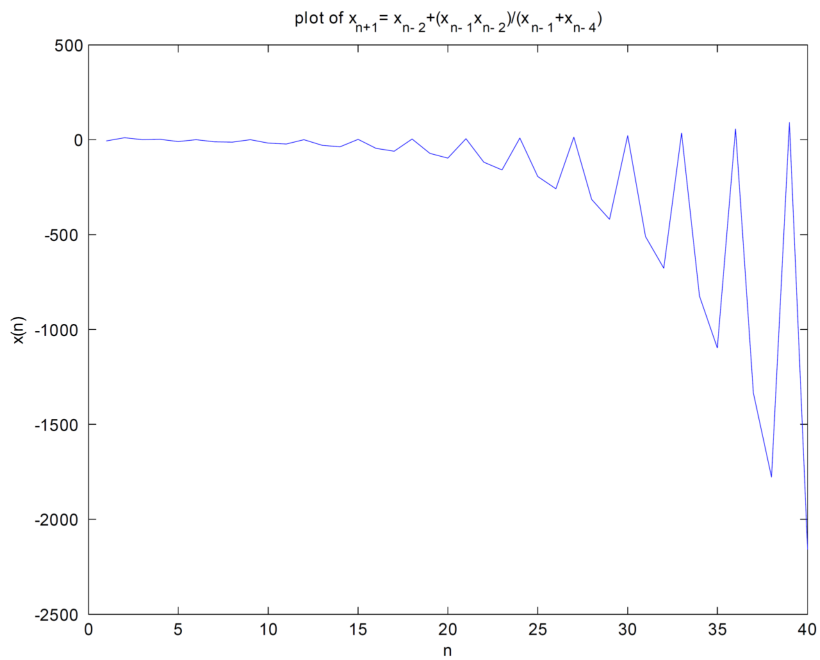

Figure 6. Sketch the behaviour of Equation (8) since $x_{-4} = 6.23$, $x_{-3} = 11.1$, $x_{-2} = 0.5$, $x_{-1} = 1.52$, and $x_0 = -9.9$.

3.2. Second equation

A specific form of the solutions of the difference equation has been provided

$$x_{n+1} = x_{n-2} + \frac{x_{n-1}x_{n-2}}{x_{n-1} - x_{n-4}}, \tag{9}$$

where the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, and x_0 are arbitrary non zero real numbers with $x_{-4} \neq x_{-1}$.

Theorem 5. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Equation* (9). *Then for* $n = 0, 1, 2, \ldots$

$$x_{6n-4} = h\prod_{i=0}^{n-1}\left(\frac{f_{3i+1}p - f_{3i-1}h}{f_{3i-1}p - f_{3i-3}h}\right)\left(\frac{f_{3i+2}q - f_{3i}k}{f_{3i}q - f_{3i-2}k}\right),$$

$$x_{6n-3} = k\prod_{i=0}^{n-1}\left(\frac{f_{3i+3}p - f_{3i+1}h}{f_{3i+1}p - f_{3i-1}h}\right)\left(\frac{f_{3i+1}q - f_{3i-1}k}{f_{3i-1}q - f_{3i-3}k}\right),$$

$$x_{6n-2} = r\prod_{i=0}^{n-1}\left(\frac{f_{3i+2}p - f_{3i}h}{f_{3i}p - f_{3i-2}h}\right)\left(\frac{f_{3i+3}q - f_{3i+1}k}{f_{3i+1}q - f_{3i-1}k}\right),$$

$$x_{6n-1} = p\prod_{i=0}^{n-1}\left(\frac{f_{3i+4}p - f_{3i+2}h}{f_{3i+2}p - f_{3i}h}\right)\left(\frac{f_{3i+2}q - f_{3i}k}{f_{3i}q - f_{3i-2}k}\right),$$

$$x_{6n} = q\prod_{i=0}^{n-1}\left(\frac{f_{3i+3}p - f_{3i+1}h}{f_{3i+1}p - f_{3i-1}h}\right)\left(\frac{f_{3i+4}q - f_{3i+2}k}{f_{3i+2}q - f_{3i}k}\right),$$

$$x_{6n+1} = r\left(\frac{2p - h}{p - h}\right)\prod_{i=0}^{n-1}\left(\frac{f_{3i+5}p - f_{3i+3}h}{f_{3i+3}p - f_{3i+1}h}\right)\left(\frac{f_{3i+3}q - f_{3i+1}k}{f_{3i+1}q - f_{3i-1}k}\right),$$

where $x_{-4} = h$, $x_{-3} = k$, $x_{-2} = r$, $x_{-1} = p$, $x_0 = q$, $\{f_m\}_{m=-1}^{\infty} = \{-1, 1, 0, 1, 1, 2, 3, 5, 8, \ldots\}$.

Proof. The proof is similar to the proof of Theorem 4 and therefore it will be omitted.

Example 7. Assume that $x_{-4} = 13$, $x_{-3} = 7$, $x_{-2} = 5$, $x_{-1} = 12$, and $x_0 = 9$. See Figure 7.

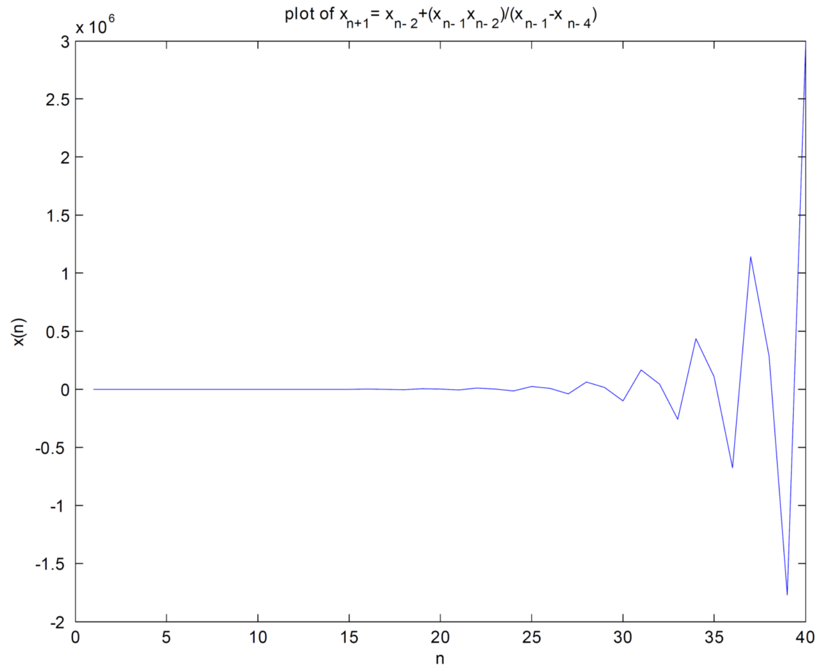

Figure 7. This figure shows the dynamics of the solution of Equation (9) wherever $x_{-4} = 13$, $x_{-3} = 7$, $x_{-2} = 5$, $x_{-1} = 12$, and $x_0 = 9$.

3.3. Third equation

The solution of the following special case of Equation (1) has been obtained

$$x_{n+1} = x_{n-2} - \frac{x_{n-1}x_{n-2}}{x_{n-1} + x_{n-4}}, \tag{10}$$

where the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, and x_0 are arbitrary non zero real numbers:

Theorem 6. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Equation* (10). *Then for* $n = 0, 1, 2, \ldots$

$$x_{6n+1} = r\prod_{i=0}^{n-1}\left(\frac{f_{3i+2}p + f_{3i+3}h}{f_{3i+3}p + f_{3i+4}h}\right)\left(\frac{f_{3i}q + f_{3i+1}k}{f_{3i+1}q + f_{3i+2}k}\right),$$

$$x_{6n-4} = h\prod_{i=0}^{n-1}\left(\frac{f_{3i-2}p + f_{3i-1}h}{f_{3i-1}p + f_{3i}h}\right)\left(\frac{f_{3i-1}q + f_{3i}k}{f_{3i}q + f_{3i+1}k}\right),$$

$$x_{6n-3} = k\prod_{i=0}^{n-1}\left(\frac{f_{3i}p + f_{3i+1}h}{f_{3i+1}p + f_{3i+2}h}\right)\left(\frac{f_{3i-2}q + f_{3i-1}k}{f_{3i-1}q + f_{3i}k}\right),$$

$$x_{6n-2} = r\prod_{i=0}^{n-1}\left(\frac{f_{3i-1}p + f_{3i}h}{f_{3i}p + f_{3i+1}h}\right)\left(\frac{f_{3i}q + f_{3i+1}k}{f_{3i+1}q + f_{3i+2}k}\right),$$

$$x_{6n-1} = p\prod_{i=0}^{n-1}\left(\frac{f_{3i+1}p + f_{3i+2}h}{f_{3i+2}p + f_{3i+3}h}\right)\left(\frac{f_{3i-1}q - f_{3i}k}{f_{3i}q - f_{3i+1}k}\right),$$

$$x_{6n} = q\prod_{i=0}^{n-1}\left(\frac{f_{3i}p + f_{3i+1}h}{f_{3i+1}p + f_{3i+2}h}\right)\left(\frac{f_{3i+1}q + f_{3i+2}k}{f_{3i+2}q + f_{3i+3}k}\right).$$

Proof. The proof is similar to the proof of Theorem 4 and therefore it will be omitted.

Example 8. For $x_{-4} = 13$, $x_{-3} = 7$, $x_{-2} = 5$, $x_{-1} = 12$, and $x_0 = 9$. See Figure 8.

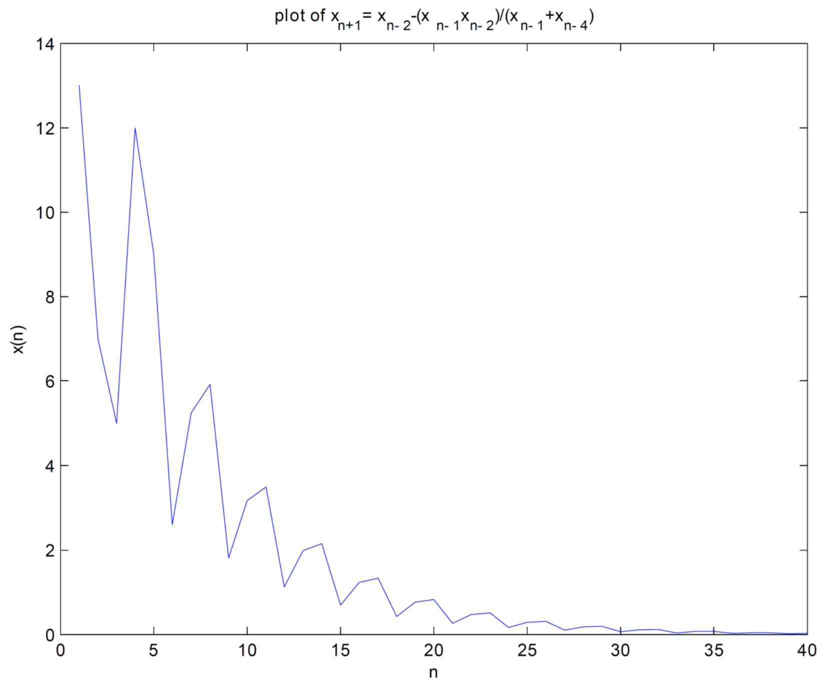

Figure 8. Plot of solution of Equation (10) when we set $x_{-4} = 13$, $x_{-3} = 7$, $x_{-2} = 5$, $x_{-1} = 12$, and $x_0 = 9$.

3.4. Fourth equation

The following special case of Equation (1) has been obtained is follows:

$$x_{n+1} = x_{n-2} - \frac{x_{n-1}x_{n-2}}{x_{n-1} - x_{n-4}}, \tag{11}$$

where the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, and x_0 are arbitrary non zero real numbers with $x_{-4} \neq x_{-1}$ and $x_{-3} \neq x_0$.

Theorem 7. *Let* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Equation* (11). *Then every solution of Equation* (11) *is unbounded. Moreover* $\{x_n\}_{n=-3}^{\infty}$ *takes the form*

$$x_{6n-4} = p\left(\frac{k}{k-q}\right)^{n}\left(\frac{p}{h}\right)^{n-1}, \quad x_{6n-3} = q\left(\frac{p-h}{p}\right)^{n}\left(\frac{q}{k}\right)^{n-1},$$

$$x_{6n-2} = r\left(\frac{h}{h-p}\right)^{n}\left(\frac{q-k}{q}\right)^{n}, \quad x_{6n-1} = p\left(\frac{k}{k-q}\right)^{n}\left(\frac{p}{h}\right)^{n},$$

$$x_{6n} = q\left(\frac{p-h}{p}\right)^{n}\left(\frac{q}{k}\right)^{n}, \quad x_{6n+1} = r\left(\frac{h}{h-p}\right)^{n+1}\left(\frac{q-k}{q}\right)^{n}.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{6n-10} = p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-2}, \quad x_{6n-9} = q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-2},$$

$$x_{6n-8} = r\left(\frac{h}{h-p}\right)^{n-1}\left(\frac{q-k}{q}\right)^{n-1}, \quad x_{6n-7} = p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1},$$

$$x_{6n-6} = q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1}, \quad x_{6n-5} = r\left(\frac{h}{h-p}\right)^{n}\left(\frac{q-k}{q}\right)^{n-1}.$$

Now, it follows from Equation (11) that

$$x_{6n-4} = x_{6n-7} - \frac{x_{6n-6}x_{6n-7}}{x_{6n-6} - x_{6n-9}}$$

$$= p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1} - \frac{q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1} p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1}}{q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1} - q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-2}}$$

$$= p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1} - \frac{\left(\frac{q}{k}\right)\left[p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1}\right]}{\left(\frac{q}{k}\right) - 1}$$

$$= p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1} - \frac{\left(\frac{q}{k}\right)\left[p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1}\right]}{\frac{q-k}{k}}$$

$$= p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1}\left[1-\frac{q}{q-k}\right]$$

$$= p\left(\frac{k}{k-q}\right)^{n-1}\left(\frac{p}{h}\right)^{n-1}\left[\frac{k}{k-q}\right].$$

Therefore

$$x_{6n-4} = p\left(\frac{k}{k-q}\right)^{n}\left(\frac{p}{h}\right)^{n-1}.$$

Also, from Equation (11), we see that

$$x_{6n-3} = x_{6n-6} - \frac{x_{6n-5}x_{6n-6}}{x_{6n-5} - x_{6n-8}}$$

$$= q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1} - \frac{r\left(\frac{h}{h-p}\right)^{n}\left(\frac{q-k}{q}\right)^{n-1} q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1}}{r\left(\frac{h}{h-p}\right)^{n}\left(\frac{q-k}{q}\right)^{n-1} - r\left(\frac{h}{h-p}\right)^{n-1}\left(\frac{q-k}{q}\right)^{n-1}}$$

$$= q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1} - \frac{\left(\frac{h}{h-p}\right)q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1}}{\frac{h}{h-p}-1}$$

$$= q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1} - \frac{\left(\frac{h}{h-p}\right)q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1}}{\frac{p}{h-p}}$$

$$= q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1}\left(1-\frac{h}{p}\right)$$

$$= q\left(\frac{p-h}{p}\right)^{n-1}\left(\frac{q}{k}\right)^{n-1}\left(\frac{p-h}{p}\right).$$

Therefore

$$x_{6n-3} = q\left(\frac{p-h}{p}\right)^{n}\left(\frac{q}{k}\right)^{n-1}.$$

Also, we can prove the other relations by similar way, the proof is completed.

Example 9. Figure 9 shows the solution when $x_{-4} = 0.13$, $x_{-3} = 0.1$, $x_{-2} = 0.15$, $x_{-1} = 0.2$, and $x_0 = 0.29$.

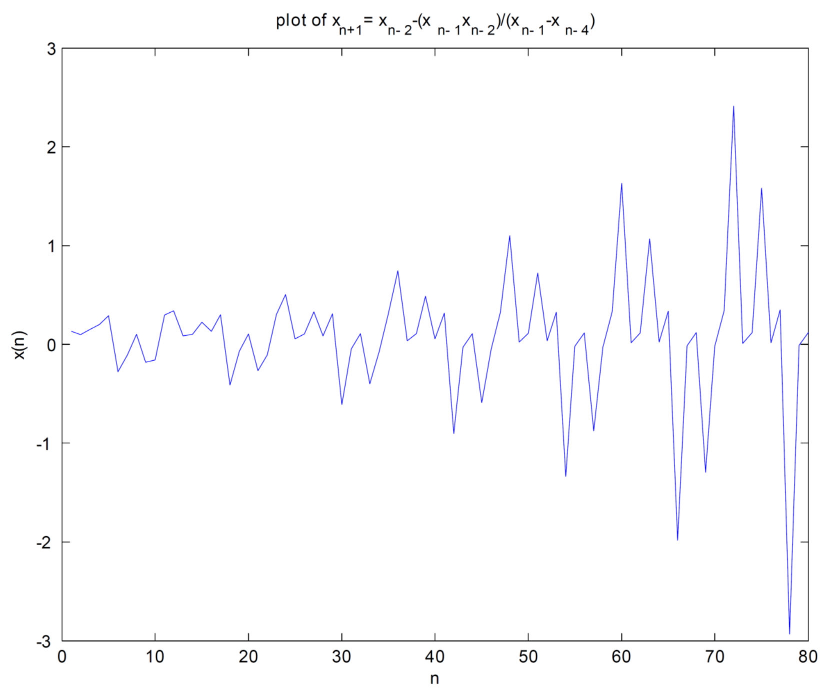

Figure 9. This figure shows the behaviour of solutions of Equation (11) when $x_{-4} = 0.13$, $x_{-3} = 0.1$, $x_{-2} = 0.15$, $x_{-1} = 0.2$, and $x_0 = 0.29$.

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