Journal of Mathematical Sciences: Advances and Applications
Volume 66, 2021, Pages 1-19
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122195

SCHUR-CONVEXITY OF A CLASS OF ELEMENTARY SYMMETRIC COMPOSITE FUNCTIONS

HUAN-NAN SHI[1], TAO ZHANG[2] and BO-YAN XI[3]

[1]Department of Basic Teaching
Teacher's College
Beijing Union University
Beijing 100011
P. R. China
e-mail: shihuannan2014@qq.com
sfthuannan@buu.edu.cn

[2]School of Mathematical Sciences
Inner Mongolia University
Hohhot 010021
P. R. China
e-mail: zhangtaocx@163.com

2020 Mathematics Subject Classification: 26E60, 26D16, 26A51.
Keywords and phrases: Schur-convesxity, Schur-geometric convexity, Schur-harmonic convexity, elementary symmetric function.
Received April 5, 2021

[3]College of Mathematics
Inner Mongolia University for Nationalities
Tongliao City
Inner Mongolia Autonomous Region, 028043
P. R. China
e-mail: baoyintu78@qq.com
baoyintu68@sohu.com
baoyintu78@imun.edu.cn

Abstract

In this paper, using the properties of Schur-convex function, Schur-geometrically convex function and Schur-harmonically convex function, we provide much simpler proofs of the Schur-convexity, Schur-geometric convexity on $(1, +\infty)^n$, and Schur-harmonic convexity on $(1, +\infty)^n$ for a composite function of the elementary symmetric functions.

1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \cdots, n\},$$

$$\mathbb{R}_+^n = \{\boldsymbol{x} \in \mathbb{R}^n : x_i > 0, i = 1, 2, \cdots, n\},$$

$$\mathbb{R}_-^n = \{\boldsymbol{x} \in \mathbb{R}^n : x_i < 0, i = 1, 2, \cdots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}_+^1$, respectively.

Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. The elementary symmetric functions are defined by

$$E_k(\boldsymbol{x}) = E_k(x_1, x_2, \cdots, x_n) := \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} x_{i_j}, \, k = 1, \cdots, n,$$

$E_0(\boldsymbol{x}) = 1$ and $E_k(\boldsymbol{x}) = 0$ for $k < 0$ or $k > n$.

In 2006, Hua Mei et al. [7] studied the Schur-convexity of the following composite function of $E_k(\boldsymbol{x})$:

$$F_k(\boldsymbol{x}) = E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right) := \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \left(\frac{1}{x_{i_j}} - x_{i_j}\right), \ k = 1, \cdots, n. \tag{1.1}$$

In 2012, using the Lemma 2.2 and Lemma 2.3 in second section, Shao [8] proved following the Theorem A and Theorem B, respectively.

Theorem A ([8])**.** (1) *The function* $E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically convex on* $(0, 1]^n$, $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically concave on* $(0, 1]^n$.

(2) *For* $2 \le k \le n-1$ *and* $a = \frac{\sqrt{n-1} - \sqrt{k-1}}{\sqrt{n-k}}$, $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically convex on* $(0, a]^n$, *and* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically concave on* $[a, 1]^n$.

Theorem B ([8])**.** *For* $k = 1, \cdots, n$, $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave on* $(0, 1]^n$.

The above results only relates to area $(0, 1]^n$, in this paper, we study Schur-convexity and Schur-harmonic convexity of $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ on $\mathbb{R}^n$, and Schur-geometric convexity of $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ on $\mathbb{R}^n_+$, we prove the following results:

Theorem 1.1. *Let* $n \ge 2$.

(1) *The function* $E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex on* $\mathbb{R}^n_+$, *and* $E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-concave on* $\mathbb{R}^n_-$.

(2) *The function* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-concave on* $(-\infty, -1]^n$.

If n *is odd numbers* (*or even numbers*), *then* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex* (*or Schur-concave*) *function on* $(-1, -\sqrt{\sqrt{5}-2}\,]^n$ *respectively.*

If n *is odd numbers* (*or even numbers*), *then* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-concave* (*or Schur-convex*) *function on* $[-\sqrt{\sqrt{5}-2}, 0]^n$ *respectively.*

The function $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex on* $(0, \sqrt{\sqrt{5}-2}\,]^n$, *and* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-concave on* $(\sqrt{\sqrt{5}-2}, 1]^n$.

If n *is odd numbers* (*or even numbers*), *then* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex* (*or Schur-concave*) *function on* $(1, +\infty)^n$ *respectively.*

(3) *For* $2 \leq k \leq n-1$, *the* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-concave on* $(-\infty, -1]^n$.

If k *is odd numbers* (*or even numbers*), *then* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex* (*or Schur-concave*) *function on* $(-1, -\sqrt{\sqrt{5}-2}\,]^n$ *respectively.*

If k *is odd numbers* (*or even numbers*), *then* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-concave* (*or Schur-convex*) *function on* $[-\sqrt{\sqrt{5}-2}, 0]^n$ *respectively.*

The function $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex on* $[0, \sqrt{\sqrt{5}-2}\,]^n$.

If n *is odd numbers* (*or numbers even*), *then* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-convex* (*or Schur-concave*) *function on* $(1, +\infty)^n$ *respectively.*

Theorem 1.2. *For* $k = 1, 2, \cdots, n$ *with* $n \geq 2$.

(1) *The function* $E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically concave on* $[1, +\infty)^n$.

(2) *If* n *is odd numbers* (*or even numbers*), *then* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically concave* (*or Schur-geometrically convex*) *function on* $(1, +\infty)^n$ *respectively.*

(3) *For* $2 \leq k \leq n-1$, *and* $b = \frac{\sqrt{n-1} + \sqrt{k-1}}{\sqrt{n-k}}$, *if* k *is odd numbers* (*or even numbers*), *then* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically convex* (*or Schur-geometrically concave*) *function on* $[1, b]^n$ *respectively, and* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-geometrically concave* (*or Schur-geometrically convex*) *on* $[b, +\infty)^n$.

Theorem 1.3. *Let* $n \geq 2$.

(1) *The function* $E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically convex on* $\mathbb{R}_+^n$, *and the* $E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave on* $\mathbb{R}_-^n$.

(2) *The function* $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically convex on* $(-\infty, -\sqrt{2+\sqrt{5}}\,]^n$.

The function $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave on* $[-\sqrt{2+\sqrt{5}}, -1]^n$.

If n is odd numbers (or even numbers), then $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically convex (or Schur-harmonically concave) function on* $[-1, 0)^n$ *respectively.*

The $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave on* $(0, 1]^n$.

If n is odd numbers (or even numbers), then $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically convex (or Schur-harmonically concave) function on* $[1, \sqrt{2 + \sqrt{5}}\,]^n$ *respectively.*

If n is odd numbers (or even numbers), then $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave (or convex) function on* $[\sqrt{2 + \sqrt{5}}, +\infty)^n$ *respectively.*

(3) *If* $2 \le k \le n - 1$, *then the* $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically convex on* $(-\infty, -\sqrt{2 + \sqrt{5}}\,]^n$.

If n is odd numbers (or even numbers), then $E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically convex (or Schur-harmonically concave) function on* $[-1, 0)^n$ *respectively.*

The function $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave on* $(0, 1]^n$.

If n is odd numbers (or even numbers), then $E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right)$ *is Schur-harmonically concave (or Schur-harmonically convex) function on* $[\sqrt{2 + \sqrt{5}}, +\infty)^n$ *respectively.*

2. Definitions and Lemmas

For convenience, we recall some definitions as follows.

Definition 2.1. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

(1) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$.

(2) Let $\Omega \subseteq \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2.2 ([6, 10]). Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for all $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(2) $\Omega \subseteq \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \leq \alpha \leq 1$, implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \cdots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(3) Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric and convex set, $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be Schur-concave on Ω if and only if $-\varphi$ is a Schur-convex function on Ω.

Lemma 2.1 ([6, 10]) (*Schur-convexity decision theorem*). *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric and convex set with nonempty interior Ω°. The function $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and continuously differentiable on Ω°. Then φ is a Schur-convex (or Schur-concave, respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \ \ (\text{or} \leq 0, \text{ respectively}) \tag{2.1}$$

holds for any $\boldsymbol{x} \in \Omega^\circ$.

The first systematical study of the functions preserving the ordering of majorization was made by Issai Schur in 1923. In Schur's honor, such functions are said to be "Schur-convex". It has many important applications in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields. See [6].

Definition 2.3 ([11]). Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \Omega \subseteq \mathbb{R}_+^n$.

(1) ([11, p. 64]) A set Ω is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \cdots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(2) ([11, p. 107]) Let Ω is a geometrically convex set. The function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω, for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, if $(\ln x_1, \ln x_2, \cdots, \ln x_n) \prec (\ln y_1, \ln y_2, \cdots, \ln y_n)$ implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is a Schur-geometrically convex function on Ω.

Lemma 2.2 ([11, p. 108]) (*Schur-geometrically convexity decision theorem*). *Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric and geometrically convex set with a nonempty interior Ω°. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial\varphi}{\partial x_1} - x_2 \frac{\partial\varphi}{\partial x_2}\right) \ge 0 \ (\textit{or} \le 0, \textit{ respectively}) \tag{2.2}$$

holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^\circ$, then φ is a Schur-geometrically convex (or Schur-geometrically concave, respectively) function.

The Schur-geometric convexity was proposed by Zhang [11] in 2004, and was investigated by Chu et al. [1], Guan [5], Sun et al. [9], and so on. We also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu ([4], [3], [2]) introduced the notion of Schur-harmonically convex function.

Definition 2.4 ([4, 3, 2])**.** Let $\Omega \subseteq \mathbb{R}_+^n$ or $\Omega \subseteq \mathbb{R}_-^n$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \Omega$.

(1) A set Ω is said to be harmonically convex if $\left(\frac{2x_1y_1}{x_1 + y_1}, \frac{2x_2y_2}{x_2 + y_2}, \cdots, \frac{2x_ny_n}{x_n + y_n}\right) \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.

(2) Let Ω is a harmonically convex set, a function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur-harmonically convex on Ω, for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, if $\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) \prec \left(\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_n}\right)$ implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. A function φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is a Schur-harmonically convex function on Ω.

Lemma 2.3 ([4, 3, 2]) (*Schur-harmonically convexity decision theorem*)**.** *Let* $\Omega \subseteq \mathbb{R}_+^n$ *or* $\Omega \subseteq \mathbb{R}_-^n$, *be a symmetric and harmonically convex set with inner points and let* $\varphi : \Omega \to \mathbb{R}$ *be a continuous symmetric function which is differentiable on* Ω°. *Then* φ *is Schur-harmonically convex* (*or Schur-harmonically concave, respectively*) *on* Ω *if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \ge 0 \ (or \le 0, \ respectively), \ \boldsymbol{x} \in \Omega^\circ. \tag{2.3}$$

Remark 1. We extend the definition and determination theorem of Schur-harmonically convex function established by Chu as follows:

(1) $\Omega \subseteq \mathbb{R}_+^n$ is extended to $\Omega \subseteq \mathbb{R}_+^n$ or $\Omega \subseteq \mathbb{R}_-^n$.

(2) The function $\varphi : \Omega \to \mathbb{R}$ must not be a positive function.

Lemma 2.4 ([6, 10])**.** *The function $E_k(\boldsymbol{x})$ is increasing and Schur-concave on $\mathbb{R}_+^n$. If $k > 1$, $E_k(\boldsymbol{x})$ is strictly Schur-concave on $\mathbb{R}_+^n$.*

Lemma 2.5. *If k is even numbers (or odd numbers, respectively), then $E_k(\boldsymbol{x})$ is decreasing and Schur-concave (or increasing and Schur-convex, respectively) on $\mathbb{R}_-^n$.*

Proof. Notice that $E_k(-\boldsymbol{x}) = (-1)^k E_k(\boldsymbol{x})$ for all $x \in \mathbb{R}_+^n$. Using the Lemma 2.4, it is easy to the desired result. Lemma 2.5 is proved. □

It is easy to see that

Lemma 2.6. *Let function $g(x) = \dfrac{x(1-x^2)}{1+x^2}$ for all $x \in \mathbb{R}$. Then*

(1) *the function g is decreasing on* $(-\infty, -\sqrt{\sqrt{5}-2}]$;

(2) *the function g is increasing on* $[-\sqrt{\sqrt{5}-2}, -\sqrt{\sqrt{5}-2}]$;

(3) *the function g is decreasing on* $[\sqrt{\sqrt{5}-2}, +\infty)$.

Lemma 2.7. *Let function $g(x) = \dfrac{1-x^2}{x(1+x^2)}$ for all $x \in \mathbb{R}\setminus\{0\}$. Then*

(1) *the function g is increasing on* $(-\infty, -\sqrt{2+\sqrt{5}}]$ *and g is decreasing on* $[-\sqrt{2+\sqrt{5}}, 0)$;

(2) *the function g is decreasing on* $(0, \sqrt{2+\sqrt{5}}]$ *and g is increasing on* $[\sqrt{2+\sqrt{5}}, +\infty)$.

It is easy to prove that the following lemma holds.

Lemma 2.8. *Let function* $h(x) = \frac{1}{x} - x$ *for all* $x \in \mathbb{R} \backslash \{0\}$. *Then*

(1) *the function* $h(x) > 0$ *for* $x \in (-\infty, -1) \cup (0, 1)$;

(2) *the function* $h(x) < 0$ *for* $x \in (-1, 0) \cup (1, +\infty)$.

3. Proof of Theorems

Proof of Theorem 1.1. For $k = 1, \cdots, n, n \geq 2,$ write

$$\Delta_k(\boldsymbol{x}) = (x_1 - x_2)\left(\frac{\partial F_k(\boldsymbol{x})}{\partial x_1} - \frac{\partial F_k(\boldsymbol{x})}{\partial x_2}\right). \tag{3.1}$$

The proof is divided into three cases.

(1) If $k = 1$ and $\boldsymbol{x} \in \mathbb{R}_+^n$ (or $\boldsymbol{x} \in \mathbb{R}_-^n$, respectively), we have

$$\Delta_1(\boldsymbol{x}) = \frac{(x_1 - x_2)^2 (x_1 + x_2)}{x_1^2 x_2^2} \geq 0 \text{ (or } \leq 0, \text{ respectively).}$$

By Lemma 2.1, it follows that Theorem 1.1(1) is holds.

(2) If $k = n$ by Lemma 2.6 and Lemma 2.8, and notice that $\sqrt{\sqrt{5}-2} < 1$, we have

$$\Delta_n(\boldsymbol{x}) = \frac{(x_1 - x_2)(1 + x_1^2)(1 + x_2^2)}{x_1^2 x_2^2} \left[\frac{x_1(1 - x_1^2)}{1 + x_1^2} - \frac{x_2(1 - x_2^2)}{1 + x_2^2}\right] F_{n-2}(\tilde{\boldsymbol{x}})$$

$$= \begin{cases} \leq 0, & \boldsymbol{x} \in (-\infty, -1]^n, \\ \geq 0, & \boldsymbol{x} \in (-1, -\sqrt{\sqrt{5}-2}\,]^n, \quad n \text{ is odd number}, \\ \leq 0, & \boldsymbol{x} \in (-1, -\sqrt{\sqrt{5}-2}\,]^n, \quad n \text{ is even number}, \\ \leq 0, & \boldsymbol{x} \in [-\sqrt{\sqrt{5}-2},\, 0]^n, \quad n \text{ is odd number}, \\ \geq 0, & \boldsymbol{x} \in [-\sqrt{\sqrt{5}-2},\, 0]^n, \quad n \text{ is even number}, \\ \geq 0, & \boldsymbol{x} \in (0, -\sqrt{\sqrt{5}-2}\,]^n, \\ \leq 0, & \boldsymbol{x} \in (\sqrt{\sqrt{5}-2},\, 1]^n, \\ \geq 0, & \boldsymbol{x} \in (1, +\infty)^n, \quad n \text{ is odd number}, \\ \leq 0, & \boldsymbol{x} \in (1, +\infty)^n, \quad n \text{ is even number}, \end{cases}$$

where $\tilde{\boldsymbol{x}} = (x_3, \cdots, x_n)$.

By Lemma 2.1, it follows that Theorem 1.1 (2) is holds.

(3) If $2 \le k \le n-1$ and $\boldsymbol{x} \in (0, \sqrt{\sqrt{5}-2}\,]^n$, by Lemma 2.6 and Lemma 2.8, and notice that $\sqrt{\sqrt{5}-2} < 1$, we have

$$\Delta_n(\boldsymbol{x}) = \frac{(x_1 - x_2)(1 + x_1^2)(1 + x_2^2)}{x_1^2 x_2^2}\left[\frac{x_1(1 - x_1^2)}{1 + x_1^2} - \frac{x_2(1 - x_2^2)}{1 + x_2^2}\right] F_{k-2}(\widetilde{\boldsymbol{x}})$$

$$+ \frac{(x_1 - x_2)^2(x_1 + x_2)}{x_1^2 x_2^2} F_{k-1}(\widetilde{\boldsymbol{x}})$$

$$= \begin{cases} \le 0, & \boldsymbol{x} \in (-\infty, -1]^n, \\ \ge 0, & \boldsymbol{x} \in (-1, -\sqrt{\sqrt{5}-2}\,]^n, \quad n \text{ is odd number}, \\ \le 0, & \boldsymbol{x} \in (-1, -\sqrt{\sqrt{5}-2}\,]^n, \quad n \text{ is even number}, \\ \le 0, & \boldsymbol{x} \in [-\sqrt{\sqrt{5}-2}, 0]^n, \quad n \text{ is odd number}, \\ \ge 0, & \boldsymbol{x} \in [-\sqrt{\sqrt{5}-2}, 0]^n, \quad n \text{ is even number}, \\ \ge 0, & \boldsymbol{x} \in (0, \sqrt{\sqrt{5}-2}\,]^n, \\ \ge 0, & \boldsymbol{x} \in (1, +\infty)^n, \quad n \text{ is odd number}, \\ \le 0, & \boldsymbol{x} \in (1, +\infty)^n, \quad n \text{ is even number}. \end{cases}$$

By the Lemma 2.1, it follows that Theorem 1.1(3) is holds.

The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. For $\boldsymbol{x}, \boldsymbol{y} \in [1, +\infty)^n$, if

$$(\ln x_1, \ln x_2, \cdots, \ln x_n) \prec (\ln y_1, \ln y_2, \cdots, \ln y_n)$$

implies

$$\left(\ln\frac{1}{x_1}, \ln\frac{1}{x_2}, \cdots, \ln\frac{1}{x_n}\right) \prec \left(\ln\frac{1}{y_1}, \ln\frac{1}{y_2}, \cdots, \ln\frac{1}{y_n}\right) \text{ for } \frac{1}{\boldsymbol{x}}, \frac{1}{\boldsymbol{y}} \in (0, 1]^n.$$

Notice that $\frac{1}{b} = \frac{\sqrt{n-1} - \sqrt{k-1}}{\sqrt{n-k}} = a$, by Theorem A, this shows that

(1) If $k = 1$, we have

$$-E_1\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right) = E_1\left(\frac{1}{\frac{1}{\boldsymbol{x}}} - \frac{1}{\boldsymbol{x}}\right) \leq E_1\left(\frac{1}{\frac{1}{\boldsymbol{y}}} - \frac{1}{\boldsymbol{y}}\right) = -E_k\left(\frac{1}{\boldsymbol{y}} - \boldsymbol{y}\right)$$

and if $k = n$, then

$$(-1)^n E_n\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right) = E_n\left(\frac{1}{\frac{1}{\boldsymbol{x}}} - \frac{1}{\boldsymbol{x}}\right) \geq E_n\left(\frac{1}{\frac{1}{\boldsymbol{y}}} - \frac{1}{\boldsymbol{y}}\right) = (-1)^n E_n\left(\frac{1}{\boldsymbol{y}} - \boldsymbol{y}\right).$$

(2) If $2 \leq k \leq n-1$ and $\boldsymbol{x}, \boldsymbol{y} \in [1, b]^n$, then

$$(-1)^k E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right) = E_k\left(\frac{1}{\frac{1}{\boldsymbol{x}}} - \frac{1}{\boldsymbol{x}}\right) \geq E_k\left(\frac{1}{\frac{1}{\boldsymbol{y}}} - \frac{1}{\boldsymbol{y}}\right) = (-1)^k E_k\left(\frac{1}{\boldsymbol{y}} - \boldsymbol{y}\right) \quad (3.2)$$

and if $2 \leq k \leq n-1$ and $\boldsymbol{x}, \boldsymbol{y} \in [b, +\infty)^n$, the above inequalities (3.2) is reversed.

By the Definition 2.4(2), from (1) and (2), it follows that Theorem 1.2 is holds.

□

Proof of Theorem 1.3. Let $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}_+^n$ and $k = 1, \cdots, n$, $n \geq 2$. Put

$$\Lambda_k(\boldsymbol{x}) = (x_1 - x_2)\left(x_1^2 \frac{\partial F_k(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial F_k(\boldsymbol{x})}{\partial x_2}\right). \quad (3.3)$$

The proof is divided into three cases.

(1) If $k = 1$, and $\boldsymbol{x} \in \mathbb{R}_+^n$ (or $\boldsymbol{x} \in \mathbb{R}_-^n$, respectively), we have

$$\Delta_1(\boldsymbol{x}) = -(x_1 - x_2)^2(x_1 + x_2) \leq 0 \text{ (or } \leq 0, \text{ respectively).}$$

By Lemma 2.3, it follows that Theorem 1.3(1) is holds.

(2) If $k = n$, by Lemma 2.7 and Lemma 2.8, and notice that $\sqrt{\sqrt{5}+2} > 2$, we get

$$\Lambda_n(\boldsymbol{x}) = (x_1 - x_2)(1 + x_1^2)(1 + x_2^2)\left[\frac{1 - x_1^2}{x_1(1 + x_1^2)} - \frac{1 - x_2^2}{x_2(1 + x_1^2)}\right]F_{n-2}(\widetilde{\boldsymbol{x}})$$

$$= \begin{cases} \geq 0, & \boldsymbol{x} \in (-\infty, -\sqrt{2+\sqrt{5}}\,]^n, \\ \leq 0, & \boldsymbol{x} \in [-\sqrt{2+\sqrt{5}}, -1]^n, \\ \geq 0, & \boldsymbol{x} \in [-1, 0)^n, \quad n \text{ is odd number}, \\ \leq 0, & \boldsymbol{x} \in [-1, 0)^n, \quad n \text{ is even number}, \\ \leq 0, & x \in [0, 1]^n, \\ \geq 0, & \boldsymbol{x} \in [1, \sqrt{2+\sqrt{5}}\,]^n, \quad n \text{ is odd number}, \\ \leq 0, & \boldsymbol{x} \in [1, \sqrt{2+\sqrt{5}}\,]^n, \quad n \text{ is even number}, \\ \leq 0, & \boldsymbol{x} \in [\sqrt{2+\sqrt{5}}, +\infty)^n, \quad n \text{ is odd number}, \\ \geq 0, & \boldsymbol{x} \in [\sqrt{2+\sqrt{5}}, +\infty)^n, \quad n \text{ is even number}. \end{cases}$$

By Lemma 2.3, it follows that Theorem 1.3(2) is holds.

(3) If $2 \le k \le n-1$, by Lemma 2.7 and Lemma 2.8, and notice that $\sqrt{\sqrt{5}+2} > 2,$, we get

$$\Lambda_k(\boldsymbol{x}) = (x_1 - x_2)(1 + x_1^2)(1 + x_2^2)\left[\frac{1 - x_1^2}{x_1(1 + x_1^2)} - \frac{1 - x_2^2}{x_2(1 + x_2^2)}\right]$$

$$\times [F_{k-2}(\tilde{\boldsymbol{x}})] - (x_1 - x_2)^2(x_1 + x_2)[F_{k-1}(\tilde{\boldsymbol{x}})]$$

$$= \begin{cases} \ge 0, & \boldsymbol{x} \in (-\infty, -\sqrt{2+\sqrt{5}}\,]^n, \\ \ge 0, & \boldsymbol{x} \in [-1, 0)^n, \quad n \text{ is odd number}, \\ \le 0, & \boldsymbol{x} \in [-1, 0)^n, \quad n \text{ is even number}, \\ \le 0, & x \in [0, 1]^n, \\ \le 0, & \boldsymbol{x} \in [\sqrt{2+\sqrt{5}}, +\infty)^n, \quad n \text{ is odd number}, \\ \ge 0, & \boldsymbol{x} \in [\sqrt{2+\sqrt{5}}, +\infty)^n, \quad n \text{ is even number}. \end{cases}$$

By Lemma 2.3, it follows that Theorem 1.3(1) is holds.

The proof of Theorem 1.3 is completed. □

4. Applications

Define

$$A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ for } x \in \mathbb{R}^n,\ G_n(\boldsymbol{x}) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \text{ for } \boldsymbol{x} \in \mathbb{R}_+^n, \tag{4.1}$$

and

$$H_n(\boldsymbol{x}) = n\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^{-n} \text{ for } \boldsymbol{x} \in \mathbb{R}_+^n \cup \mathbb{R}_-^n. \tag{4.2}$$

These means $A_n(\boldsymbol{x}), G_n(\boldsymbol{x})$, and $H_n(\boldsymbol{x})$ are respectively called the arithmetic, geometric, and harmonic means of numbers $x_1, x_2, \cdots, x_n$.

Theorem 4.1. *Let* $n \geq 2$. *If* $\boldsymbol{x} \in (0, \sqrt{\sqrt{5}-2}\,]^n$ *or* $\boldsymbol{x} \in (-1, -\sqrt{\sqrt{5}-2}\,]^n \cup (1, +\infty)^n$ $(2 \leq k \leq n$ *and* k *is odd number*) *or* $\boldsymbol{x} \in [-\sqrt{\sqrt{5}-2}, 0]^n$ $(2 \leq k \leq n$ *and* k *is even number*), *or* $\boldsymbol{x} \in \mathbb{R}_+^n$ $(k = 1)$, *then*

$$\binom{n}{k}\left(\frac{1}{A_n(\boldsymbol{x})} - A_n(\boldsymbol{x})\right)^k \leq E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right). \tag{4.3}$$

If $x \in (-\infty, -1]^n$ *or* $\boldsymbol{x} \in (-1, -\sqrt{\sqrt{5}-2}\,]^n \cup (1, +\infty)^n$ $(2 \leq k \leq n$ *and* k *is even number*) *or* $\boldsymbol{x} \in [-\sqrt{\sqrt{5}-2}, 0]^n$ $(2 \leq k \leq n$ *and* k *is odd number*), *or* $\boldsymbol{x} \in \mathbb{R}_+^n$ $(k = 1)$, *then the inequalities* (4.3) *is reversed.*

Proof. For $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, we have

$$(A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \cdots, A_n(\boldsymbol{x})) \prec (x_1, x_2, \cdots, x_n) = \boldsymbol{x}. \tag{4.4}$$

By Theorem 1.1 and Definition 2.2(3), the inequalities (4.3) holds. The proof is complete. □

Theorem 4.2. *Let* $n \geq 2$, *and let* $a = \dfrac{\sqrt{n-1} - \sqrt{k-1}}{\sqrt{n-k}}$ *and* $b = \dfrac{\sqrt{n-1} + \sqrt{k-1}}{\sqrt{n-k}}$.

If $\boldsymbol{x} \in (0, 1]^n (k = 1)$ *or* $\boldsymbol{x} \in (0, a]^n$ $(2 \leq k \leq n-1)$, *or* $\boldsymbol{x} \in (1, +\infty)^n$ $(k = n$ *and* k *is even number*), *or* $\boldsymbol{x} \in (1, b]^n$ $(2 \leq k \leq n-1$ *and* k *is odd number*), *or* $\boldsymbol{x} \in [b, +\infty)^n$ $(2 \leq k \leq n-1$ *and* k *is even number*), *then*

$$\binom{n}{k}\left(\frac{1}{G_n(\boldsymbol{x})} - G_n(\boldsymbol{x})\right)^k \leq E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right). \tag{4.5}$$

If $\boldsymbol{x} \in (0, 1]^n$ $(k = n)$ *or and* $\boldsymbol{x} \in (a, 1]^n$ $(2 \leq k \leq n-1)$, *or* $\boldsymbol{x} \in (1, +\infty]^n$ $(k = 1)$, *or* $\boldsymbol{x} \in (1, +\infty]^n (k = n$ *and* k *is odd number*), *or* $\boldsymbol{x} \in (1, b]^n$ $(2 \leq k \leq n-1$ *and* k *is even number*), *or* $\boldsymbol{x} \in [b, +\infty)^n$ $(2 \leq k \leq n-1$ *and* k *is odd number*), *then the inequalities* (4.5) *is reversed.*

Proof. For all $\boldsymbol{x} \in \mathbb{R}_+^n$, we have

$$(\ln G_n(\boldsymbol{x}), \ln G_n(\boldsymbol{x}), \cdots, \ln G_n(\boldsymbol{x})) \prec (\ln x_1, \ln x_2, \cdots, \ln x_n).$$

Form Definition 2.3(2), Theorem A and Theorem 1.2, we obtain the inequalities (4.5). Theorem 4.2 is proved. □

Theorem 4.3. *Let* $n \geq 2$. *If* $\boldsymbol{x} \in \mathbb{R}_+^n (k = 1)$, *or* $\boldsymbol{x} \in (-\infty, -\sqrt{2+\sqrt{5}}\,]^n$ $(2 \leq k \leq n)$, *or* $\boldsymbol{x} \in (-1, 0]^n$ $(2 \leq k \leq n$ *and* k *is odd number*), *or* $\boldsymbol{x} \in (1, \sqrt{2+\sqrt{5}}\,]^n (k = n$ *and* k *is odd number*), *or* $\boldsymbol{x} \in (\sqrt{2+\sqrt{5}}, +\infty)^n$ $(2 \leq k \leq n$ *and* k *is even number*), *then*

$$\binom{n}{k}\left(\frac{1}{H_n(\boldsymbol{x})} - H_n(\boldsymbol{x})\right)^k \leq E_k\left(\frac{1}{\boldsymbol{x}} - \boldsymbol{x}\right). \tag{4.6}$$

If $\boldsymbol{x} \in \mathbb{R}_-^n (k = 1)$, *or* $\boldsymbol{x} \in [-\sqrt{2+\sqrt{5}}, -1]^n (k = n)$, *or* $\boldsymbol{x} \in (-1, 0]^n (2 \leq k \leq n$ *and* k *is even number*), *or* $\boldsymbol{x} \in (0, 1]^n$ $(2 \leq k \leq n)$ *or* $\boldsymbol{x} \in (1, \sqrt{2+\sqrt{5}}\,]^n$ $(k = n$ *and* k *is even number*), *or* $\boldsymbol{x} \in (\sqrt{2+\sqrt{5}}, +\infty)^n$ $(2 \leq k \leq n$ *and* k *is odd number*), *then the inequalities* (4.6) *is reversed.*

Proof. For $\boldsymbol{x} \in \mathbb{R}_+^n \cup \mathbb{R}_-^n$, we obtain

$$\left(\frac{1}{H_n(\boldsymbol{x})}, \frac{1}{H_n(\boldsymbol{x})}, \cdots, \frac{1}{H_n(\boldsymbol{x})}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right).$$

Using Definition 2.4(2), Theorem 1.3, the inequalities (4.6) holds. The proof is complete. □

Acknowledgements

This research was supported partially by the Inner Mongolia Normal University High-level Talent Research Startup Funding (2019YJRC010).

References

[1] Y.-M. Chu, X.-M. Zhang and G.-D. Wang, The Schur geometrical convexity of the extended mean values, Journal of Convex Analysis 15(4) (2008), 707-718.

[2] Y.-M. Chu and Y.-P. Lv, The Schur harmonic convexity of the Hamy symmetric function and its applications, Journal of Inequalities and Applications (2009); Article ID 838529, 10 pages.

DOI: https://doi.org/10.1155/2009/838529

[3] Y. M. Chu and T. C. Sun, The Schur harmonic convexity for a class of symmetric functions, Acta Mathematica Scientia 30(5) (2010), 1501-1506.

DOI: https://doi.org/10.1016/S0252-9602(10)60142-7

[4] Y.-M. Chu, G.-D.Wang and X.-H. Zhang, The Schur multiplicative and harmonic convexities of the complete symmetric function, Mathematische Nachrichten 284(5-6) (2011), 653-663.

DOI: https://doi.org/10.1002/mana.200810197

[5] K.-Z. Guan, Schur-convexity of the complete symmetric function, Mathematical Inequalities & Applications 9(4) (2006), 567-576.

DOI: https://doi.org/10.7153/mia-09-52

[6] A. W. Marshall, I. Olkin and B. C. Arnold, Inequalities: Theory of Majorization and its Application, Second Edition, Springer, New York, 2011.

DOI: https://doi.org/10.1007/978-0-387-68276-1

[7] H. Mei, C.-L. Bai and H. Man, Extension of an Inequality Guess, Journal of Inner Mongolia University for Nationalities 21(2) (2006), 127-129. (in Chinese)

[8] Z.-H. Shao, The Schur-geometrical convexity and Schur-harmonic convexity for a class of symmetric functions, Mathematics in Practice and Theory 42(16) (2012), 199-206. (in Chinese)

[9] T.-C. Sun, Y.-P. Lv and Y.-M. Chu, Schur multiplicative and harmonic convexities of generalized Heronian mean in n variables and their applications, International Journal of Pure and Applied Mathematics 55(1) (2009), 25-33.

[10] B.-Y. Wang, Foundations of Majorization Inequalities, Beijing Normal University Press, Beijing, 1990. (in Chinese)

[11] X.-M. Zhang, Geometrically Convex Functions, An'hui University Press, Hefei, 2004. (in Chinese)

■