

## **FIXED POINT THEOREMS ON GENERALIZED RECTANGULAR METRIC SPACES**

**O. K. ADEWALE<sup>1</sup>, J. O. OLALERU<sup>2</sup>, H. OLAOLUWA<sup>3</sup>  
and H. AKEWE<sup>4</sup>**

Department of Mathematics

University of Lagos

Lagos

Nigeria

e-mail: [adewalekayode2@yahoo.com](mailto:adewalekayode2@yahoo.com)

[jolaleru@unilag.edu.ng](mailto:jolaleru@unilag.edu.ng)

[holaoluwa@unilag.edu.ng](mailto:holaoluwa@unilag.edu.ng)

[hakewe@unilag.edu.ng](mailto:hakewe@unilag.edu.ng)

### **Abstract**

In this paper, we introduce the notion of generalized rectangular metric spaces which extends rectangular metric spaces introduced by Branciari. Analogues of the some well-known fixed point theorems are proved in this space. With an example, it is shown that a generalized rectangular metric space is neither a  $G$ -metric space nor a rectangular metric space. Our results generalize many known results in fixed point theory.

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## 1. Introduction

The notion of a distance between two points as the case may be is as old as human civilization. Frechet [8] put this notion into an abstract form and came out with metric spaces in early 19th century. Metric space is a crucial device in functional analysis, non-linear analysis and topology. This aspect of functional analysis has attracted the attention of many mathematicians due to the development of fixed point theory in it.

Many authors made attempt to generalize the usual notion of metric space  $(X, d)$  to extend the known metric space theorems in a more general setting (see [1]-[12]). In attempt to generalize these usual metric spaces, 2-metric spaces,  $D$ -metric spaces,  $G$ -metric spaces, rectangular metric spaces, and  $\gamma$ -generalized quasi metric spaces were respectively, introduced by Gahler [9], Dhage [7], Mustafa and Sims [10], Branciari [6], and Adewale et al. [1].

Motivated by these generalizations and the fact that in real life, two points in a space may be distinct, we present the notion of generalized rectangular metric space which extends a rectangular metric space. We also use an example to show that the newly introduced generalized rectangular metric space is better than rectangular metric space and different from  $G$ -metric spaces. Some fixed point theorems are stated and proved in this new space.

## 2. Preliminary

Mustafa and Sims define  $G$ -metric spaces as follows:

**Definition 2.1** ([10]). For a non-empty set  $X$  and a function  $G : X^3 \rightarrow [0, \infty)$  satisfying the following properties:

- (i)  $G(x, y, z) = 0$  if and only if  $x = y = z$ ;
- (ii)  $G(x, x, y) > 0, \forall x, y \in X$ , with  $x \neq y$ ;
- (iii)  $G(x, x, y) < G(x, y, z), \forall x, y, z \in X$ , with  $z \neq y$ ;

$$(iv) \quad G(x, y, z) = G(y, z, x) = G(x, z, y) = \dots;$$

(v)  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ ,  $\forall a, x, y, z \in X$  (rectangle inequality).

The function  $G$  is called a  $G$ -metric and  $(X, G)$  is called a  $G$ -metric space.

Branciari also defined rectangular metric spaces as follows:

**Definition 2.2** ([6]). For a non-empty set  $X$  and a function  $d : X^2 \rightarrow [0, \infty)$  satisfying the following properties:

$$(i) \quad d(x, y) = 0 \text{ if and only if } x = y \text{ for all } x, y \in X;$$

$$(ii) \quad d(x, y) = d(y, x), \quad \forall x, y \in X;$$

(iii)  $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ ,  $\forall x, y \in X$  and all distinct points  $u, v \in X - \{x, y\}$ .

The function  $d$  is called a rectangular metric on  $X$  and  $(X, d)$  is called a rectangular metric space.

### 3. Main Results

We introduce the following:

**Definition 3.1.** For a non-empty set  $X$  and a function  $G : X^3 \rightarrow [0, \infty)$  satisfying the following properties:

$$(i) \quad G(x, y, z) = 0 \text{ if and only if } x = y = z.$$

$$(ii) \quad G(x, x, y) > 0, \quad \forall x, y \in X, \text{ with } x \neq y.$$

$$(iii) \quad G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$$

(iv)  $G(x, y, z) \leq G(x, a, a) + G(a, b, b) + G(b, y, y) + G(y, y, z)$ ,  $\forall x, y, z \in X$  and all distinct points  $a, b \in X - \{x, y, z\}$ .

The function  $G$  is called a generalized rectangular metric and  $(X, G)$  is called a generalized rectangular metric space.

**Remark 3.2.** If  $y = z$  and we set  $G(x, y, y) = d(x, y)$ , Definition 3.1 reduces to rectangular metric space in [6].

**Example 3.3.** Let  $X = R^+ \cup \{0\}$  and define  $G : X \times X \times X \rightarrow R^+ \cup \{0\} \cup \{0\}$  by

$$G(x, y, z) = \begin{cases} 0, & x = y = z, \\ 3, & \text{if } x \neq y \neq z \neq x, \\ 1, & \text{otherwise.} \end{cases}$$

Then  $(X, G)$  is a generalized rectangular metric space but neither a  $G$ -metric space nor a rectangular metric space because

- (i)  $G(1, 2, 3) > G(1, 2, 2) + G(2, 2, 3)$ ,
- (ii) the points  $a$  and  $b$  are distinct ( $a, b \notin \{x, y, z\}$ ).

**Definition 3.4.** Let  $(X, G)$  be a generalized rectangular metric space. For  $x \in X$ ,  $r > 0$ , the  $G$ -sphere with centre  $x$  and radius  $r$  is

$$S_G(x, r) = \{z \in X : G(x, z, z) < r\}.$$

**Definition 3.5.** Let  $(X, G)$  be a generalized rectangular metric space. The sequence  $\{x_n\} \subset X$  is  $G$ -convergent to  $z$  if it converges to  $z$  in the generalized rectangular metric spaces.

**Definition 3.6.** Let  $(X, G)$  and  $(\bar{X}, \bar{G})$  be two generalized rectangular metric spaces, a function  $T : X \rightarrow \bar{X}$  is  $G$ -continuous at a point  $x \in X$  if  $T^{-1}(S_{\bar{G}}(T(x), r)) \in \tau(G)$ , for all  $r > 0$ .  $T$  is  $G$ -continuous if it is  $G$ -continuous at all points of  $X$ .

The following lemmas will be used in this work.

**Lemma 3.7.** *Let  $(X, G)$  be a generalized rectangular metric space and  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  converges to  $x$  if and only if  $G(x_n, x, x) \rightarrow 0$  as  $n \rightarrow \infty$ .*

**Lemma 3.8.** *Let  $(X, G)$  be a generalized rectangular metric space and  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  is said to be a Cauchy sequence if and only if  $G(x_n, x_m, x_l) \rightarrow 0$  as  $n, m, l \rightarrow \infty$ .*

**Theorem 3.9.** *Let  $X$  be a complete generalized rectangular metric space and  $T : X \rightarrow X$  be a map for which there exist the real number,  $k$  satisfying  $0 \leq k < 1$  such that for each pair  $x, y, z \in X$ .*

$$G(Tx, Ty, Tz) \leq kG(x, y, z). \quad (1)$$

*Then  $T$  has a unique fixed point.*

**Proof.** Considering (1),

$$G(Tx, Ty, Ty) \leq kG(x, y, y). \quad (2)$$

Suppose  $T$  satisfies condition (2) and  $x_0 \in X$  be an arbitrary point and define the sequence  $x_n$  by  $x_n = T^n x_0$ , then

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \leq kG(x_{n-1}, x_n, x_n).$$

Setting  $g_n = G(x_n, x_{n+1}, x_{n+1})$ , we have

$$g_n \leq k g_{n-1}. \quad (3)$$

We deduce that

$$g_n \leq k g_{n-1},$$

$$g_n \leq k^n g_0, \quad \forall n \in N. \quad (4)$$

Suppose there exists  $n \in N$  such that  $x_0 = x_n$ .

$$G(x_0, Tx_0, Tx_0) = G(x_n, Tx_n, Tx_n),$$

$$G(x_0, x_1, x_1) = G(x_n, x_{n+1}, x_{n+1}),$$

$$g_0 = g_n,$$

$$g_0 \leq k^n g_0.$$

Contradiction since  $k < 1$ . Hence  $\forall n \in N, x_0 \neq x_n$ . Repeating this argument, we have that  $\forall n, m \in N$  with  $n \neq m, x_n \neq x_m$ . Then the terms of the sequence  $\{x_n\}$  are distinct.

By repeated use of (iv) in Definition 3.1 and all distinct points  $x_{n+1}, x_{n+2}, \dots, x_{m-1}$  with  $m > n$ , we have the following for all odd  $m - n$ :

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_m, x_m) \\ &\leq g_n + g_{n+1} + G(x_{n+2}, x_m, x_m) \\ &\leq g_n + g_{n+1} + g_{n+2} + g_{n+3} + G(x_{n+4}, x_m, x_m) \\ &\leq \sum_{i=n}^{n+3} g_i + G(x_{n+4}, x_m, x_m) \\ &\leq \sum_{i=n}^{m-1} g_i \leq \sum_{i=n}^{\infty} g_i. \end{aligned} \tag{5}$$

Similarly, if  $m - n \geq 4$  is even, we have

$$G(x_n, x_m, x_m) \leq \sum_{i=n}^{m-3} g_i + G(x_{m-2}, x_m, x_m). \tag{6}$$

From (4) and (5), we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq k^n g_0 + k^{n+1} g_0 + k^{n+2} g_0 + \dots + k^{m-2} g_0 + k^{m-1} g_0 \\
&\leq [k^n + k^{n+1} + k^{n+2} + k^{n+3} + \dots + k^{m-1}] g_0 \\
&\leq k^n [1 + k + k^2 + k^3 + k^4 + \dots + k^{m-n-1}] g_0 \\
&\leq k^n (1 - k)^{-1} g_0.
\end{aligned} \tag{7}$$

From (4) and (6), we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq k^n (1 - k)^{-1} g_0 + G(x_{m-2}, x_m, x_m) \\
&\leq k^n (1 - k)^{-1} g_0 + k^{m-2} G(x_0, x_2, x_2).
\end{aligned} \tag{8}$$

Taking the limit of  $G(x_n, x_m, x_m)$  as  $n, m \rightarrow \infty$ , we have

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x_m) = 0. \tag{9}$$

For  $n, m, l \in N$  with  $n > m > l$ ,

$$\begin{aligned}
G(x_n, x_m, x_l) &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) \\
&\quad + G(x_{n-2}, x_m, x_m) + G(x_m, x_m, x_l).
\end{aligned} \tag{10}$$

Taking the limit of  $G(x_n, x_m, x_l)$  as  $n, m, l \rightarrow \infty$ , we have

$$\lim_{n, m, l \rightarrow \infty} G(x_n, x_m, x_l) = 0. \tag{11}$$

So,  $\{x_n\}$  is a  $G$ -Cauchy sequence.

By completeness of  $(X, G)$ , there exist  $u \in X$  such that  $x_n$  is  $G$ -convergent to  $u$ .

Suppose  $Tu \neq u$

$$G(x_n, Tu, Tu) \leq kG(x_{n-1}, u, u). \tag{12}$$

Taking the limit  $n \rightarrow \infty$  and using the fact that  $T$  is  $G$ -continuous in its variables, we get

$$G(u, Tu, Tu) \leq kG(u, u, u). \quad (13)$$

Hence,

$$G(u, Tu, Tu) \leq 0. \quad (14)$$

This is a contradiction. So,  $Tu = u$ . To show the uniqueness, suppose  $v \neq u$  is such that  $Tv = v$ , then

$$G(Tu, Tv, Tv) \leq kG(u, v, v). \quad (15)$$

Since  $Tu = u$  and  $Tv = v$ , we have

$$G(u, v, v) \leq 0, \quad (16)$$

which implies that  $v = u$ .

**Remark 3.10.** Let  $(X, G)$  be a rectangular  $G$ -metric space and  $d : X \times X \rightarrow [0, \infty)$  be a function defined by  $G(x, y, y) = d(x, y)$ , then Theorem 2.9 reduces to Banach contraction principle in rectangular-metric space (an analogue of Banach contraction principle in metric space).

**Theorem 3.11.** *Let  $X$  be a complete generalized rectangular metric space and  $T : X \rightarrow X$  be a map for which there exist the real number  $b$  satisfying  $0 \leq b < \frac{1}{3}$  such that for each pair  $x, y, z \in X$ .*

$$G(Tx, Ty, Tz) \leq b[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]. \quad (17)$$

*Then  $T$  has a unique fixed point.*

**Proof.** Considering (17),

$$G(Tx, Ty, Ty) \leq b[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]. \quad (18)$$



Suppose  $T$  satisfies condition (18) and  $x_0 \in X$  be an arbitrary point and define the sequence  $x_n$  by  $x_n = T^n x_0$ , then we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq b[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + G(x_n, x_{n+1}, x_{n+1})]. \end{aligned}$$

We deduce that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{b}{1-2b} G(x_{n-1}, x_n, x_n).$$

$$\text{Let } p = \frac{b}{1-2b} < 1$$

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq pG(x_{n-1}, x_n, x_n) \\ &\leq p^2 G(x_{n-2}, x_{n-1}, x_{n-1}) \end{aligned}$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq p^3 G(x_{n-3}, x_{n-2}, x_{n-2})$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq p^n G(x_0, x_1, x_1)$$

$$g_n \leq p^n g_0. \tag{19}$$

Suppose there exists  $n \in N$  such that  $x_0 = x_n$ .

$$G(x_0, Tx_0, Tx_0) = G(x_n, Tx_n, Tx_n),$$

$$G(x_0, x_1, x_1) = G(x_n, x_{n+1}, x_{n+1}),$$

$$g_0 = g_n,$$

$$g_0 \leq k^n g_0.$$

Contradiction since  $k < 1$ . Hence  $\forall n \in N, x_0 \neq x_n$ . Repeating this argument, we have that  $\forall n, m \in N$  with  $n \neq m, x_n \neq x_m$ . Then the terms of the sequence  $\{x_n\}$  are distinct.

By repeated use of (iv) in Definition 3.1 and all distinct points  $x_{n+1}, x_{n+2}, \dots, x_{m-1}$ , we have the following for all odd  $m - n$ :

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + G(x_{n+2}, x_m, x_m) \\
&\leq g_n + g_{n+1} + G(x_{n+2}, x_m, x_m) \\
&\leq g_n + g_{n+1} + g_{n+2} + g_{n+3} + G(x_{n+4}, x_m, x_m) \\
&\leq \sum_{i=n}^{n+3} g_i + G(x_{n+4}, x_m, x_m) \\
&\leq \sum_{i=n}^{m-1} g_i \leq \sum_{i=n}^{\infty} g_i. \tag{20}
\end{aligned}$$

Similarly, if  $m - n \geq 4$  is even, we have

$$G(x_n, x_m, x_m) \leq \sum_{i=n}^{m-3} g_i + G(x_{m-2}, x_m, x_m). \tag{21}$$

From (19) and (20), we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq p^n g_0 + p^{n+1} g_0 + p^{n+2} g_0 + \dots + p^{m-2} g_0 + p^{m-1} g_0 \\
&\leq [p^n + p^{n+1} + p^{n+2} + p^{n+3} + \dots + p^{m-1}] g_0 \\
&\leq p^n [1 + p + p^2 + p^3 + p^4 + \dots + p^{m-n-1}] g_0 \\
&\leq p^n (1 - p)^{-1} g_0. \tag{22}
\end{aligned}$$

From (19) and (21), we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq p^n (1 - p)^{-1} g_0 + G(x_{m-2}, x_m, x_m) \\
&\leq p^n (1 - p)^{-1} g_0 + p^{m-2} G(x_0, x_2, x_2). \tag{23}
\end{aligned}$$

Taking the limit of  $G(x_n, x_m, x_m)$  as  $n, m \rightarrow \infty$ , we have

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x_m) = 0. \quad (24)$$

For  $n, m, l \in N$  with  $n > m > l$ ,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) \\ &\quad + G(x_{n-2}, x_m, x_m) + G(x_m, x_m, x_l). \end{aligned} \quad (25)$$

Taking the limit of  $G(x_n, x_m, x_l)$  as  $n, m, l \rightarrow \infty$ , we have

$$\lim_{n, m, l \rightarrow \infty} G(x_n, x_m, x_l) = 0. \quad (26)$$

So,  $x_n$  is a  $G$ -Cauchy sequence.

By completeness of  $(X, G)$ , there exist  $u \in X$  such that  $x_n$  is  $G$ -convergent to  $u$ .

Suppose  $Tu \neq u$

$$\begin{aligned} G(x_n, Tu, Tu) &\leq b[G(x_{n-1}, x_n, x_n) + G(u, Tu, Tu) + G(u, Tu, Tu)] \\ &\leq b[G(x_{n-1}, x_n, x_n) + 2G(u, Tu, Tu)]. \end{aligned} \quad (27)$$

Taking the limit as  $n \rightarrow \infty$  and using the fact that  $T$  is  $G$ -continuous in its variables, we get

$$G(u, Tu, Tu) \leq 2bG(u, Tu, Tu). \quad (28)$$

Hence,

$$G(u, Tu, Tu) \leq 0. \quad (29)$$

This is a contradiction. So,  $Tu = u$ . To show the uniqueness, suppose  $v \neq u$  is such that  $Tv = v$ , then

$$G(Tu, Tv, Tv) \leq b[G(u, Tu, Tu) + G(v, Tv, Tv) + G(v, Tv, Tv)]. \quad (30)$$

Since  $Tu = u$  and  $Tv = v$ , we have

$$G(u, v, v) \leq 0, \quad (31)$$

which implies that  $v = u$ .

**Remark 3.12.** Let  $(X, G)$  be a rectangular  $G$ -metric space and  $d : X \times X \rightarrow [0, \infty)$  be a function defined by  $G(x, y, y) = d(x, y)$ , then Theorem 3.11 reduces to Kannan's fixed point theorem in rectangular-metric space (an analogue of Kannan's fixed point theorem in metric space).

**Theorem 3.13.** Let  $X$  be a complete generalized rectangular metric space and  $T : X \rightarrow X$  be a map for which there exists real numbers  $a, b, c$  satisfying  $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$  with  $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$  and

$$\phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{t}{3}, & \text{if } t \neq 0, \end{cases}$$

such that for each pair  $x, y, z \in X$ .

$$G(Tx, Ty, Tz) \leq \phi(\delta G(x, y, z) + 2\delta G(x, Tx, Tx)). \quad (32)$$

Then  $T$  has a unique fixed point.

**Proof.** Considering (32),

$$G(Tx, Ty, Ty) \leq \phi(\delta G(x, y, y) + 2\delta G(x, Tx, Tx)) \quad (33)$$

Suppose  $T$  satisfies condition (33) and  $x_0 \in X$  be an arbitrary point and define the sequence  $x_n$  by  $x_n = T^n x_0$ , then

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \phi(\delta G(x_{n-1}, x_n, x_n) + 2\delta G(x_{n-1}, x_n, x_n)) \\ &\leq \phi(3\delta G(x_{n-1}, x_n, x_n)) \\ &\leq \delta G(x_{n-1}, x_n, x_n). \end{aligned}$$

Setting  $g_n = G(x_n, x_{n+1}, x_{n+1})$ , then

$$g_n \leq \delta g_{n-1}. \quad (34)$$

Deducing

$$\begin{aligned} g_n &\leq \delta g_{n-1}, \\ g_n &\leq \delta^n g_0, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (35)$$

Suppose there exists  $n \in \mathbb{N}$  such that  $x_0 = x_n$ .

$$\begin{aligned} G(x_0, Tx_0, Tx_0) &= G(x_n, Tx_n, Tx_n), \\ G(x_0, x_1, x_1) &= G(x_n, x_{n+1}, x_{n+1}), \\ g_0 &= g_n, \\ g_0 &\leq \delta^n g_0. \end{aligned}$$

Contradiction since  $\delta < 1$ . Hence  $\forall n \in \mathbb{N} \cup \{0\}$ ,  $x_0 \neq x_n$ . Repeating this argument,  $\forall n, m \in \mathbb{N} \cup \{0\}$  with  $n \neq m$ ,  $x_n \neq x_m$ . Then the terms of the sequence  $\{x_n\}$  are distinct.

By repeated use of (iv) in Definition 3.1 and all distinct points  $x_{n+1}, x_{n+2}, \dots, x_{m-1}$  with  $m > n$ , we have the following for all odd  $m - n$ :

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_m, x_m) \\ &\leq g_n + g_{n+1} + G(x_{n+2}, x_m, x_m) \\ &\leq g_n + g_{n+1} + g_{n+2} + g_{n+3} + G(x_{n+4}, x_m, x_m) \\ &\leq \sum_{i=n}^{n+3} g_i + G(x_{n+4}, x_m, x_m) \\ &\leq \sum_{i=n}^{m-1} g_i \leq \sum_{i=n}^{\infty} g_i. \end{aligned} \quad (36)$$

Similarly, if  $m - n \geq 4$  is even, we have

$$G(x_n, x_m, x_m) \leq \sum_{i=n}^{m-3} g_i + G(x_{m-2}, x_m, x_m). \quad (37)$$

From (34) and (36), the following are obtained.

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \delta^n g_0 + \delta^{n+1} g_0 + \delta^{n+2} g_0 + \dots + \delta^{m-2} g_0 + \delta^{m-1} g_0 \\ &\leq [\delta^n + \delta^{n+1} + \delta^{n+2} + \delta^{n+3} + \dots + \delta^{m-1}] g_0 \\ &\leq \delta^n [1 + \delta + \delta^2 + \delta^3 + \delta^4 + \dots + \delta^{m-n-1}] g_0 \\ &\leq \delta^n (1 - \delta)^{-1} g_0. \end{aligned} \quad (38)$$

From (34) and (37), we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \delta^n (1 - \delta)^{-1} g_0 + G(x_{m-2}, x_m, x_m) \\ &\leq \delta^n (1 - \delta)^{-1} g_0 + \delta^{m-2} G(x_0, x_2, x_2). \end{aligned} \quad (39)$$

Taking the limit of  $G(x_n, x_m, x_m)$  as  $n, m \rightarrow \infty$ ,

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x_m) = 0. \quad (40)$$

For  $n, m, l \in \mathbb{N} \cup \{0\}$  with  $n > m > l$ ,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) \\ &\quad + G(x_{n-2}, x_m, x_m) + G(x_m, x_m, x_l). \end{aligned} \quad (41)$$

Taking the limit of  $G(x_n, x_m, x_l)$  as  $n, m, l \rightarrow \infty$ ,

$$\lim_{n, m, l \rightarrow \infty} G(x_n, x_m, x_l) = 0. \quad (42)$$

So,  $\{x_n\}$  is a  $G$ -Cauchy sequence.

By completeness of  $(X, G)$ , there exist  $u \in X$  such that  $x_n$  is  $G$ -convergent to  $u$ .

Suppose  $Tu \neq u$

$$G(x_n, Tu, Tu) \leq \phi(\delta G(x_{n-1}, u, u) + 2\delta G(x_{n-1}, x_n, x_n)). \quad (43)$$

Taking the limit as  $n \rightarrow \infty$  and using the fact that  $T$  is  $G$ -continuous in its variables,

$$G(u, Tu, Tu) \leq \phi(\delta G(u, u, u) + 2\delta G(u, u, u)). \quad (44)$$

Hence,

$$G(u, Tu, Tu) \leq 0. \quad (45)$$

This is a contradiction. So,  $Tu = u$ . To show the uniqueness, suppose  $v \neq u$  is such that  $Tv = v$ , then

$$G(Tu, Tv, Tv) \leq \phi(\delta G(u, v, v) + 2\delta G(u, Tu, Tu)). \quad (46)$$

Since  $Tu = u$  and  $Tv = v$ , then

$$G(u, v, v) \leq 0, \quad (47)$$

which implies that  $v = u$ .

**Remark 3.14.** Let  $(X, G)$  be a rectangular  $G$ -metric space and  $d : X \times X \rightarrow [0, \infty)$  be a function defined by  $G(x, y, y) = d(x, y)$  with  $\phi(t) = t$ , then Theorem 3.13 reduces to Zamfirescu's fixed point theorem in rectangular-metric space (an analogue of Zamfirescu's fixed point theorem in metric space).

**Theorem 3.15.** Let  $X$  be a complete generalized rectangular metric space and  $T : X \rightarrow X$  be a map for which there exists real numbers  $a, b, c$  satisfying  $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$  with  $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$  such that for each pair  $x, y, z \in X$ .

$$G(Tx, Ty, Tz) \leq \delta G(x, y, z) + \nu(2\delta G(Tx, y, z)), \quad (48)$$

where  $\delta \in [0, 1)$  and function  $\mathfrak{p} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\mathfrak{p}(t) = t$  a monotone increasing sequence. Then  $T$  has a unique fixed point.

**Proof.** Considering (48),

$$G(Tx, Ty, Ty) \leq \delta G(x, y, y) + \mathfrak{p}(2\delta G(Tx, y, y)). \quad (49)$$

Suppose  $T$  satisfies condition (49) and  $x_0 \in X$  be an arbitrary point and define the sequence  $x_n$  by  $x_n = T^n x_0$ , then

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \delta G(x_{n-1}, x_n, x_n) + \mathfrak{p}(2\delta G(x_n, x_n, x_n)) \\ &\leq \delta G(x_{n-1}, x_n, x_n). \end{aligned}$$

Setting  $g_n = G(x_n, x_{n+1}, x_{n+1})$ , then

$$g_n \leq \delta g_{n-1}. \quad (50)$$

Deducing

$$\begin{aligned} g_n &\leq \delta g_{n-1}, \\ g_n &\leq \delta^n g_0, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (51)$$

Suppose there exists  $n \in \mathbb{N}$  such that  $x_0 = x_n$ .

$$\begin{aligned} G(x_0, Tx_0, Tx_0) &= G(x_n, Tx_n, Tx_n), \\ G(x_0, x_1, x_1) &= G(x_n, x_{n+1}, x_{n+1}), \\ g_0 &= g_n, \\ g_0 &\leq \delta^n g_0. \end{aligned}$$

Contradiction since  $\delta < 1$ . Hence  $\forall n \in \mathbb{N} \cup \{0\}$ ,  $x_0 \neq x_n$ . Repeating this argument,  $\forall n, m \in \mathbb{N} \cup \{0\}$  with  $n \neq m$ ,  $x_n \neq x_m$ . Then the terms of the sequence  $\{x_n\}$  are distinct.



By repeated use of (iv) in Definition 3.1 and all distinct points  $x_{n+1}, x_{n+2}, \dots, x_{m-1}$  with  $m > n$ , we have the following for all odd  $m - n$ :

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + G(x_{n+2}, x_m, x_m) \\
&\leq g_n + g_{n+1} + G(x_{n+2}, x_m, x_m) \\
&\leq g_n + g_{n+1} + g_{n+2} + g_{n+3} + G(x_{n+4}, x_m, x_m) \\
&\leq \sum_{i=n}^{n+3} g_i + G(x_{n+4}, x_m, x_m) \\
&\leq \sum_{i=n}^{m-1} g_i \leq \sum_{i=n}^{\infty} g_i.
\end{aligned} \tag{52}$$

Similarly, if  $m - n \geq 4$  is even, we have

$$G(x_n, x_m, x_m) \leq \sum_{i=n}^{m-3} g_i + G(x_{m-2}, x_m, x_m). \tag{53}$$

From (50) and (52), the following are obtained:

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq \delta^n g_0 + \delta^{n+1} g_0 + \delta^{n+2} g_0 + \dots + \delta^{m-2} g_0 + \delta^{m-1} g_0 \\
&\leq [\delta^n + \delta^{n+1} + \delta^{n+2} + \delta^{n+3} + \dots + \delta^{m-1}] g_0 \\
&\leq \delta^n [1 + \delta + \delta^2 + \delta^3 + \delta^4 + \dots + \delta^{m-n-1}] g_0 \\
&\leq \delta^n (1 - \delta)^{-1} g_0.
\end{aligned} \tag{54}$$

From (50) and (53), we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq \delta^n (1 - \delta)^{-1} g_0 + G(x_{m-2}, x_m, x_m) \\
&\leq \delta^n (1 - \delta)^{-1} g_0 + \delta^{m-2} G(x_0, x_2, x_2).
\end{aligned} \tag{55}$$

Taking the limit of  $G(x_n, x_m, x_m)$  as  $n, m \rightarrow \infty$ ,

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x_m) = 0. \quad (56)$$

For  $n, m, l \in \mathbb{N} \cup \{0\}$  with  $n > m > l$ ,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) \\ &\quad + G(x_{n-2}, x_m, x_m) + G(x_m, x_m, x_l). \end{aligned} \quad (57)$$

Taking the limit of  $G(x_n, x_m, x_l)$  as  $n, m, l \rightarrow \infty$ ,

$$\lim_{n, m, l \rightarrow \infty} G(x_n, x_m, x_l) = 0. \quad (58)$$

So,  $\{x_n\}$  is a  $G$ -Cauchy sequence.

By completeness of  $(X, G)$ , there exist  $u \in X$  such that  $x_n$  is  $G$ -convergent to  $u$ .

Suppose  $Tu \neq u$

$$G(x_n, Tu, Tu) \leq \delta G(x_{n-1}, u, u) + \mathfrak{p}(2\delta G(x_n, u, u)). \quad (59)$$

Taking the limit as  $n \rightarrow \infty$  and using the fact that  $T$  is  $G$ -continuous in its variables,

$$G(u, Tu, Tu) \leq \delta G(u, u, u) + \mathfrak{p}(2\delta G(u, u, u)). \quad (60)$$

Hence,

$$G(u, Tu, Tu) \leq 0. \quad (61)$$

This is a contradiction. So,  $Tu = u$ . To show the uniqueness, suppose  $v \neq u$  is such that  $Tv = v$ , then

$$G(Tu, Tv, Tv) \leq \delta G(u, v, v) + \mathfrak{p}(2\delta G(Tu, v, v)). \quad (62)$$

Since  $Tu = u$  and  $Tv = v$ , then

$$G(u, v, v) \leq 0, \quad (63)$$

which implies that  $v = u$ .

**Remark 3.16.** Let  $(X, G)$  be a rectangular  $G$ -metric space and  $d : X \times X \rightarrow [0, \infty)$  be a function defined by  $G(x, y, y) = d(x, y)$  with  $\varphi(2\delta G(Tx, y, z)) = 0$ , then Theorem 3.15 reduces to Banach contraction principle in rectangular-metric space (an analogue of Banach contraction principle in metric space).

**Example 3.17.** Let  $X = \mathbb{R}$  endowed with a generalized rectangular metric  $G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$  for all  $x, y, z \in X$ . Define the mapping  $T : X \rightarrow X$  by

$$T(x) = \begin{cases} 2, & \text{if } x \geq 1, \\ 1, & \text{if } x \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

It is observed that Banach contraction principle in  $G$ -metric space introduced by Mustafa and Sims [10] cannot be applied in this case because

$$\begin{aligned} G(T(-1), T(1), T(3)) &= G(0, 2, 2) \\ &= \max\{|0 - 2|, |0 - 2|, |2 - 2|\} \\ &= \max\{2, 2, 0\} \\ &= 2 \\ &> kG(-1, 1, 3) \\ &> k \max\{|-1 - 1|, |-1 - 3|, |1 - 3|\} \\ &> k \max\{2, 4, 2\} \\ &> 4k = 4 \times 0.25 = 1, 0.25 \in [0, 1). \end{aligned}$$

With the newly introduced contractive mapping, the above case can be taken care off. As a matter of fact, for all  $x, y, z \in X$ ,

$$G(Tx, Ty, Tz) \leq \delta G(x, y, z) + \mathfrak{v}(2\delta G(Tx, y, z)).$$

Since  $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ ,  $0 \leq G(Tx, Ty, Tz) \leq 2$  and  $\mathfrak{v}(t) = t$ .

**Theorem 3.18.** *Let  $X$  be a complete generalized rectangular metric space and  $T : X \rightarrow X$  be a map for which there exists real numbers  $a, b, c$  satisfying  $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$  with  $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$  such that for each pair  $x, y, z \in X$ .*

$$G(Tx, Ty, Tz) \leq \phi(\delta G(x, y, z)) + \mathfrak{v}(2\delta G(x, Tx, Tx)), \quad (64)$$

where  $\delta \in [0, 1)$  and functions  $\phi, \mathfrak{v} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\mathfrak{v}(t) = \frac{t}{4}$  and  $\phi(t) = \frac{t}{2}$  a monotone increasing sequences. Then  $T$  has a unique fixed point.

**Proof.** Considering (64),

$$G(Tx, Ty, Ty) \leq \phi(\delta G(x, y, y)) + \mathfrak{v}(2\delta G(x, Tx, Tx)). \quad (65)$$

Suppose  $T$  satisfies condition (65) and  $x_0 \in X$  be an arbitrary point and define the sequence  $x_n$  by  $x_n = T^n x_0$ , then

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \phi(\delta G(x_{n-1}, x_n, x_n)) + \mathfrak{v}(2\delta G(x_{n-1}, x_n, x_n)) \\ &\leq \delta G(x_{n-1}, x_n, x_n). \end{aligned}$$

Setting  $g_n = G(x_n, x_{n+1}, x_{n+1})$ , then

$$g_n \leq \delta g_{n-1}. \quad (66)$$

Deducing

$$\begin{aligned} g_n &\leq \delta g_{n-1}, \\ g_n &\leq \delta^n g_0, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{67}$$

Suppose there exists  $n \in \mathbb{N}$  such that  $x_0 = x_n$ .

$$\begin{aligned} G(x_0, Tx_0, Tx_0) &= G(x_n, Tx_n, Tx_n), \\ G(x_0, x_1, x_1) &= G(x_n, x_{n+1}, x_{n+1}), \\ g_0 &= g_n, \\ g_0 &\leq \delta^n g_0. \end{aligned}$$

Contradiction since  $\delta < 1$ ). Hence  $\forall n \in \mathbb{N} \cup \{0\}$ ,  $x_0 \neq x_n$ . Repeating this argument,  $\forall n, m \in \mathbb{N} \cup \{0\}$  with  $n \neq m$ ,  $x_n \neq x_m$ . Then the terms of the sequence  $\{x_n\}$  are distinct.

By repeated use of (iv) in Definition 3.1 and all distinct points  $x_{n+1}, x_{n+2}, \dots, x_{m-1}$  with  $m > n$ , we have the following for all odd  $m - n$ :

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_m, x_m) \\ &\leq g_n + g_{n+1} + G(x_{n+2}, x_m, x_m) \\ &\leq g_n + g_{n+1} + g_{n+2} + g_{n+3} + G(x_{n+4}, x_m, x_m) \\ &\leq \sum_{i=n}^{n+3} g_i + G(x_{n+4}, x_m, x_m) \\ &\leq \sum_{i=n}^{m-1} g_i \leq \sum_{i=n}^{\infty} g_i. \end{aligned} \tag{68}$$

Similarly, if  $m - n \geq 4$  is even, we have

$$G(x_n, x_m, x_m) \leq \sum_{i=n}^{m-3} g_i + G(x_{m-2}, x_m, x_m). \quad (69)$$

From (66) and (68), the following are obtained:

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \delta^n g_0 + \delta^{n+1} g_0 + \delta^{n+2} g_0 + \dots + \delta^{m-2} g_0 + \delta^{m-1} g_0 \\ &\leq [\delta^n + \delta^{n+1} + \delta^{n+2} + \delta^{n+3} + \dots + \delta^{m-1}] g_0 \\ &\leq \delta^n [1 + \delta + \delta^2 + \delta^3 + \delta^4 + \dots + \delta^{m-n-1}] g_0 \\ &\leq \delta^n (1 - \delta)^{-1} g_0. \end{aligned} \quad (70)$$

From (66) and (69), we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \delta^n (1 - \delta)^{-1} g_0 + G(x_{m-2}, x_m, x_m) \\ &\leq \delta^n (1 - \delta)^{-1} g_0 + \delta^{m-2} G(x_0, x_2, x_2). \end{aligned} \quad (71)$$

Taking the limit of  $G(x_n, x_m, x_m)$  as  $n, m \rightarrow \infty$ ,

$$\lim_{n, m \rightarrow \infty} G(x_n, x_m, x_m) = 0. \quad (72)$$

For  $n, m, l \in \mathbb{N} \cup \{0\}$  with  $n > m > l$ ,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) \\ &\quad + G(x_{n-2}, x_m, x_m) + G(x_m, x_m, x_l). \end{aligned} \quad (73)$$

Taking the limit of  $G(x_n, x_m, x_l)$  as  $n, m, l \rightarrow \infty$ ,

$$\lim_{n, m, l \rightarrow \infty} G(x_n, x_m, x_l) = 0. \quad (74)$$

So,  $\{x_n\}$  is a  $G$ -Cauchy sequence.

By completeness of  $(X, G)$ , there exist  $u \in X$  such that  $x_n$  is  $G$ -convergent to  $u$ .

Suppose  $Tu \neq u$

$$G(x_n, Tu, Tu) \leq \phi(\delta G(x_{n-1}, u, u)) + \psi(2\delta G(x_{n-1}, x_n, x_n)). \quad (75)$$

Taking the limit as  $n \rightarrow \infty$  and using the fact that  $T$  is  $G$ -continuous in its variables,

$$G(u, Tu, Tu) \leq \phi(\delta G(u, u, u)) + \psi(2\delta G(u, u, u)). \quad (76)$$

Hence,

$$G(u, Tu, Tu) \leq 0. \quad (77)$$

This is a contradiction. So,  $Tu = u$ . To show the uniqueness, suppose  $v \neq u$  is such that  $Tv = v$ , then

$$G(Tu, Tv, Tv) \leq \phi(\delta G(u, v, v)) + \psi(2\delta G(u, Tu, Tu)). \quad (78)$$

Since  $Tu = u$  and  $Tv = v$ , then

$$G(u, v, v) \leq 0, \quad (79)$$

which implies that  $v = u$ .

**Remark 3.19.** Let  $(X, G)$  be a rectangular  $G$ -metric space and  $d : X \times X \rightarrow [0, \infty)$  be a function defined by  $G(x, y, y) = d(x, y)$  with  $\phi(t) = t$  and  $\psi(2\delta G(x, Tx, Tx)) = 0$ , then Theorem 3.18 reduces to Banach contraction principle in rectangular-metric space.

Clearly, Remark 3.14 can be applied in the case Remark 3.16.

**Example 3.20.** Let  $X = \mathbb{R}$  endowed with the a generalized rectangular metric  $G(x, y, z) = |x - y| + |x - z| + |y - z|$  for all  $x, y, z \in X$ . Define the mapping  $T : X \rightarrow X$  by

$$T(x) = \begin{cases} \frac{4}{x}, & \text{if } x > 2, \\ 2x, & \text{if } x \in (0, 2], \\ 0, & \text{otherwise.} \end{cases}$$

At first, it is observed that Banach contraction principle in  $G$ -metric space introduced by Mustafa and Sims [10] cannot be applied in this case too because

$$\begin{aligned} G(T3, T1, T4) &= G\left(\frac{4}{3}, 2, 1\right) \\ &= \left|\frac{4}{3} - 2\right| + \left|\frac{4}{3} - 1\right| + |2 - 1| \\ &= 2 \\ &> kG(3, 1, 4) \\ &> k(|3 - 1| + |3 - 4| + |1 - 4|) \\ &> 6k = 6 \times \frac{1}{8} = \frac{3}{4}, \frac{1}{8} \in [0, 1). \end{aligned}$$

But Theorem 3.15 can be applied in this case because

$$\begin{aligned} G(T3, T1, T4) &= G\left(\frac{4}{3}, 2, 1\right) \\ &= \left|\frac{4}{3} - 2\right| + \left|\frac{4}{3} - 1\right| + |2 - 1| \\ &= 2 \\ &\leq \delta G(3, 1, 4) + 2\delta G\left(\frac{4}{3}, 1, 4\right) \\ &\leq \delta[G(3, 1, 4) + 2G\left(\frac{4}{3}, 1, 4\right)] \\ &\leq \delta[(|3 - 1| + |3 - 4| + |1 - 4|) \\ &\quad + 2(|\frac{4}{3} - 1| + |\frac{4}{3} - 4| + |1 - 4|)] \\ &\leq \frac{56}{3} \times \delta. \end{aligned}$$



Since  $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ . Theorem 3.18 can be applied in this case too because

$$\begin{aligned}
 G(T3, T1, T4) &= G(\frac{4}{3}, 2, 1) \\
 &= |\frac{4}{3} - 2| + |\frac{4}{3} - 1| + |2 - 1| \\
 &= 2 \\
 &\leq 0.5 \times \delta G(3, 1, 4) + 0.5 \times \delta G(3, \frac{4}{3}, \frac{4}{3}) \\
 &\leq 0.5 \times \delta [G(3, 1, 4) + G(3, \frac{4}{3}, \frac{4}{3})] \\
 &\leq 0.5 \times \delta [(|3 - 1| + |3 - 4| + |1 - 4|) \\
 &\quad + |3 - \frac{4}{3}| + |3 - \frac{4}{3}| + |\frac{4}{3} - \frac{4}{3}|] \\
 &\leq \frac{79}{18} \times \delta.
 \end{aligned}$$

Since  $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ .

#### 4. Concluding Remarks

In this paper, we present the notion of generalized rectangular metric space which extends a rectangular metric space. We also use an example to show that the newly introduced generalized rectangular metric space is better than rectangular metric space and different from  $G$ -metric spaces. Some fixed point theorems are stated and proved in this new space.

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