Journal of Algebra, Number Theory: Advances and Applications
Volume 24, Number 2, 2021, Pages 155-195
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jantaa_7100122177

OCTAD ORBITS FOR CERTAIN SUBGROUPS OF M_{24}

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Abstract

Using Curtis's MOG [3], we display the orbits and orbit representatives for various subgroups of the Mathieu group M_{24} acting on the octads of the Steiner system $S(24, 8, 5)$. This information is deployed in [8] and [9] to study a graph associated with the largest simple Fischer group.

2020 Mathematics Subject Classification: 05E18, 20D08.
Keywords and phrases: Mathieu group, subgroups, octad orbits.
Received February 13, 2021; Revised May 24, 2021

1. Introduction and Notation

Since its discovery in the second half of the nineteenth century by Emil Mathieu [5, 6], M_{24}, the Mathieu group of degree 24, has been much studied. Being a highly transitive permutation group, and not an alternating or symmetric group, was the reason for the early interest. The many facetted combinatorial nature of this group began to emerge following the two papers of Witt [10, 11], and later elaborated by many authors (see Chapter 11 of [2]). In addition to its impact in the combinatorial arena, M_{24} is intimately connected, in one way or another, with many of the sporadic finite simple groups. Our interest here is prompted by this particular aspect of M_{24}. In [8], the first steps are taken in analyzing the point-line collinearity graph of the Fi_{24}' maximal 2-local geometry (Fi_{24}' being the largest Fischer simple group). Now, the residue of a point in this geometry is isomorphic to the M_{24} maximal 2-local geometry (see [7]). And the lines in a point residue correspond to the octads (or blocks) of the Steiner system $S(24, 8, 5)$ upon which M_{24} acts. As a consequence, the action of certain subgroups of M_{24} on the octads are of paramount importance to the arguments in [8].

The purpose of this paper is to establish a data bank of octad orbits (for various subgroups of M_{24}) for use in [8] as well as in [9], and to highlight their combinatorial significance via Curtis's MOG [3]. In addition to recording the size of each orbit we also exhibit an orbit representative in MOG format – these are a valuable visual aid in [8]. Some of these configurations may be of interest in their own right. Also, for related work, see Kilic [4].

We now establish some notation. Throughout this paper G denotes M_{24} which is assumed to act upon the 24-element set Ω. The Steiner system $S(24, 8, 5)$ on Ω which G leaves invariant is assumed to be as described in [3]. We use the same names for the elements of Ω as in [3]; the heavy bricks of the MOG being denoted by O_1, O_2, O_3. Thus

$$\Omega = \begin{array}{|cc|cc|cc|} \hline \infty & 14 & 17 & 11 & 22 & 19 \\ 0 & 8 & 4 & 13 & 1 & 9 \\ 3 & 20 & 16 & 7 & 12 & 5 \\ 15 & 18 & 10 & 2 & 21 & 6 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline O_1 & O_2 & O_3 \\ & & \\ & & \\ \hline \end{array}.$$

The subgroup of G whose octad orbits we scrutinize will always be denoted by $L -$ the structure of L being described using ATLAS [1] notation and conventions. We will use ~ to mean that two groups have the same shape. Let $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ be either subsets of Ω or partitions of a subset of Ω. Then we set

$$Stab_G\{\Lambda_1, \Lambda_2, \dots, \Lambda_n\} = Stab_G\Lambda_1 \cap Stab_G\Lambda_2 \cap \cdots \cap Stab_G\Lambda_n.$$

In fact, L is often of the form $Stab_G\{\Lambda_1, \dots, \Lambda_n\}$ for certain $\Lambda_1, \dots, \Lambda_n$. As a result the octad orbits are frequently parameterized by the type of intersection octads have with the sets/partitions $\Lambda_1, \dots, \Lambda_n$. By $\alpha_{d_1, d_2, \dots, d_n}$ we mean the set of all octads O of Ω for which $|O \cap \Lambda_i| = d_i$ (if Λ_i is a subset of Ω) or O cuts Λ_i in d_i (if Λ_i is a partition of some subset of Ω). In the latter case, if the sets of the partition Λ_i are $\Lambda_i^j (j = 1, \dots, m)$, we describe the way in which O cuts Λ_i by $e_1^{f_1} e_2^{f_2} \cdots (e_{1 \geq} e_{2 \geq} \cdots)$ meaning that f_1 of the sets Λ_i^j intersect O in a set of size e_1, f_2 of the sets Λ_i^j intersect O in a set of size e_2, and so on. We omit the term $e_k^{f_k}$ if $e_k = 0$, except when O cuts Λ_i in 0^{f_1}. On the rare occasion when the $d_1, \dots, d_n$ do not serve to distinguish an L-orbit, we employ superscripts. Two types of partition we encounter frequently are the trios and sextets of Ω. We recall that the standard trio and the standard sextet are, respectively,

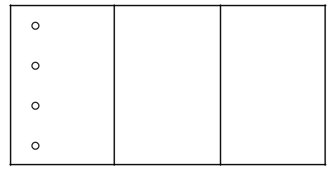
$$\mathscr{T}_0 = \left|\begin{array}{cc|cc|cc} \circ & \circ & + & + & - & - \\ \circ & \circ & + & + & - & - \\ \circ & \circ & + & + & - & - \\ \circ & \circ & + & + & - & - \end{array}\right| \text{ and } \mathscr{L}_0 = \left|\begin{array}{cc|cc|cc} \circ & * & + & \square & - & \times \\ \circ & * & + & \square & - & \times \\ \circ & * & + & \square & - & \times \\ \circ & * & + & \square & - & \times \end{array}\right|.$$

2. Octad Orbits

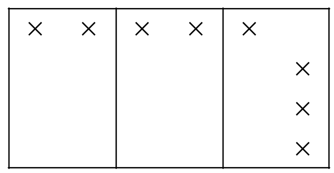
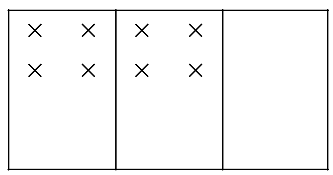
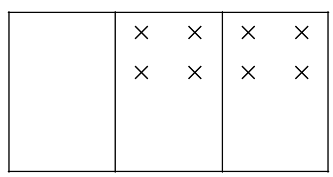
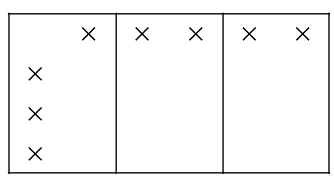
For each octad orbit below we indicate to which G_a-orbit $\Delta_j^i(a)$ in [8] the information will be applied.

2.1. $(\Delta_1(a))L = Stab_G\{\Lambda_1\}$, where $\Lambda_1 = O_1$. So $L \sim 2^4 : A_8$.

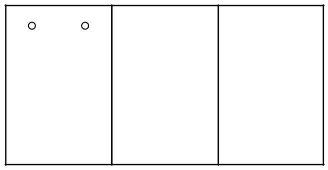
L-Orbit	Size	Representative
α_8	1	O_1
α_0	30	O_2
α_2	448	$\left|\begin{array}{cc\|cc\|cc} \times & \times & \times & \times & \times & \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \end{array}\right|$
α_4	280	$\left|\begin{array}{cc\|cc\|cc} \times & & \times & & & \\ \times & & \times & & & \\ \times & & \times & & & \\ \times & & \times & & & \end{array}\right|$

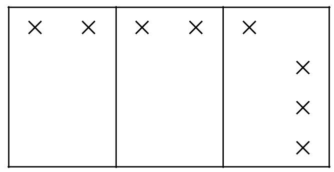
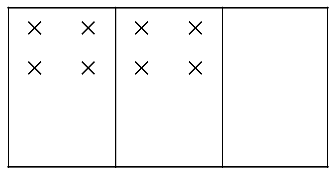
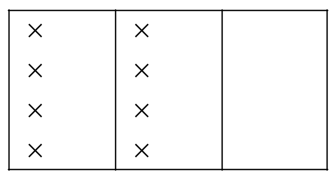
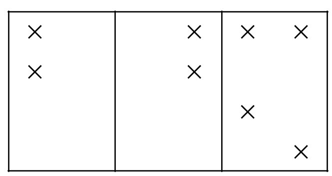
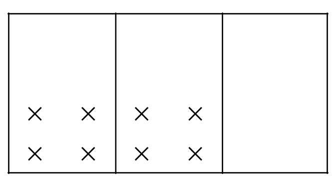
2.2. $(\Delta^1_2(a))L = Stab_G\{\Lambda_1, \Lambda_2\}$, where $\Lambda_1 =$ 

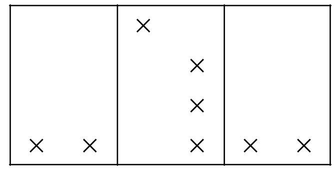
and $\Lambda_2 = \mathscr{L}_0$. So $L \sim 2^6 : (3 \times S_5)$.

L-Orbit	Size	Representative
$\alpha_{4,4^2}$	5	O_1
$\alpha_{0,4^2}$	10	O_3
$\alpha_{1,31^5}$	320	
$\alpha_{2,2^4}$	240	
$\alpha_{0,2^4}$	120	
$\alpha_{3,31^5}$	64	

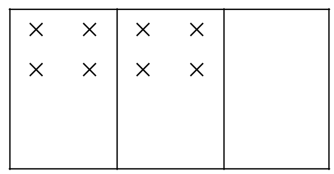
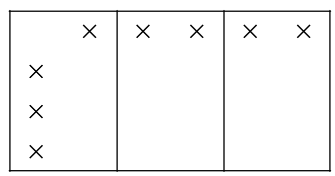
2.3. $(\Delta_2^2(a)$ and $\Delta_3^3(a))L = Stab_G\{\Lambda_1, \Lambda_2\}$, where $\Lambda_1 = O_1$ and

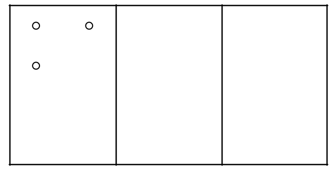
$\Lambda_2 =$  . So $L \sim 2^4 : S_6$.

L-Orbit	Size	Representative
$\alpha_{8,2}$	1	O_1
$\alpha_{2,2}$	16	
$\alpha_{4,2}$	60	
$\alpha_{4,1}$	160	
$\alpha_{2,1}$	192	
$\alpha_{4,0}$	60	

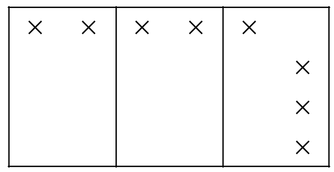
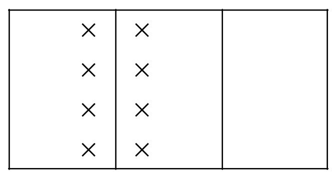
$\alpha_{2,0}$	240	
$\alpha_{0,0}$	30	O_3

2.4. $(\Delta_2^3(a))$ L is the derived subgroup of $Stab_G\{\Lambda_1\}$, where $\Lambda_1 = \mathscr{T}_0$. So $L \sim 2^6 : (L_3(2) \times 3)$.

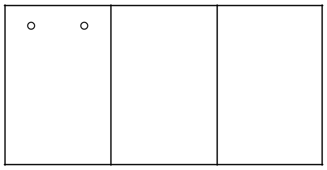
L-Orbit	Size	Representative
α_{80^2}	3	O_1
α_{4^2}	84	
α_{42^2}	672	

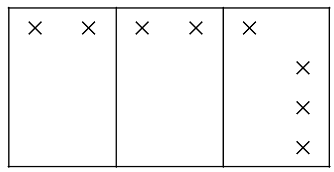
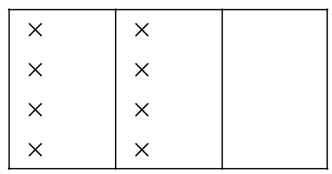
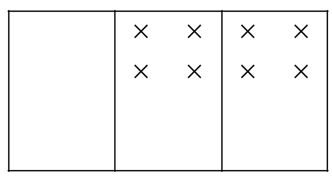
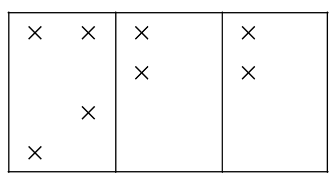
2.5. $(\Delta_3^1(a))L = Stab_G\{\Lambda_1\}$, where $\Lambda_1 =$  .

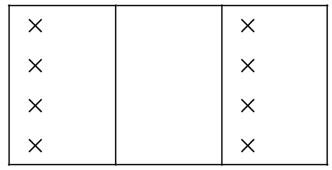
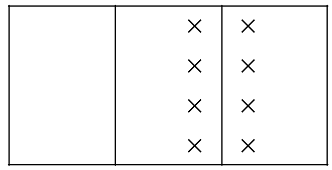
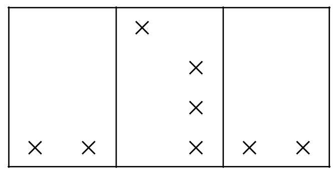
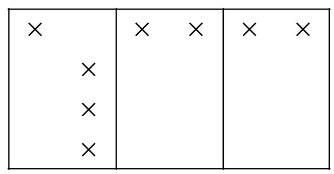
So $L \sim L_3(4) : S_3$.

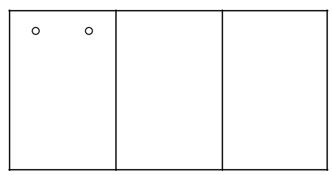
L-Orbit	Size	Representative
α_3	21	O_1
α_2	168	
α_0	210	O_3
α_1	360	

2.6. $(\Delta_3^2(a))L = Stab_G\{\Lambda_1, \Lambda_2\}$, where $\Lambda_1 = O_3$ and

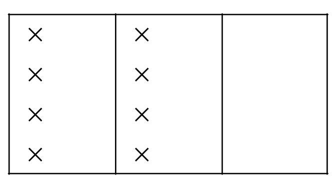
$\Lambda_2 =$  . So $L \sim 2^3 : (L_3(2) \times 2)$.

L-Orbit	Size	Representative
$\alpha_{8,0}$	1	O_3
$\alpha_{0,2}$	7	O_1
$\alpha_{0,0}$	7	O_2
$\alpha_{4,2}$	14	
$\alpha_{0,1}$	16	
$\alpha_{4,0}^{(1)}$	42	
$\alpha_{2,2}$	56	

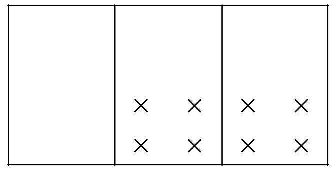
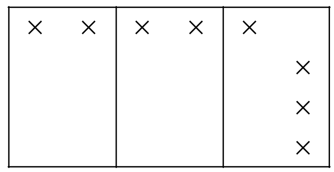
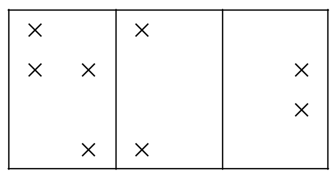
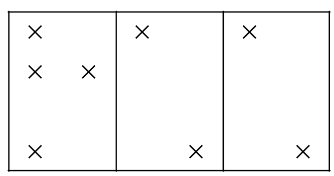
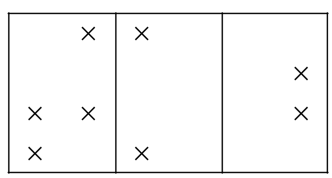
$\alpha_{4,1}$	112	
$\alpha_{4,0}^{(2)}$	112	
$\alpha_{2,0}$	168	
$\alpha_{2,1}$	224	

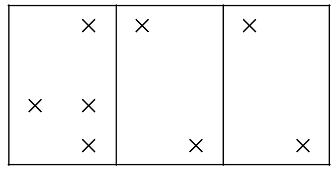
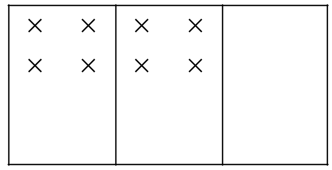
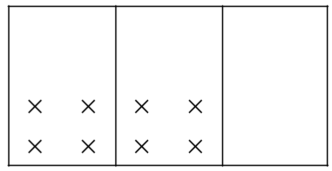
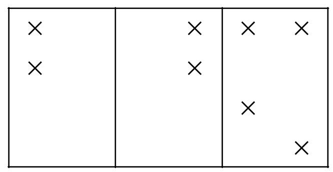
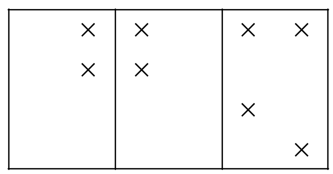
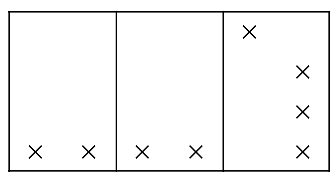
2.7. $(\Delta_3^4(\alpha))L = Stab_G\{\Lambda_1\}$, where $\Lambda_1 =$  .

So $L \sim M_{22} : 2$.

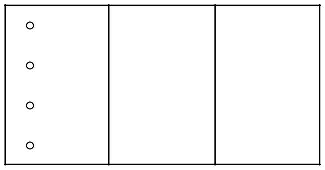
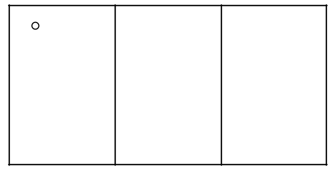
L-Orbit	Size	Representative
α_2	77	O_1
α_0	330	O_3
α_1	352	

2.8. $(\Delta_3^5(a))L \sim 2^4{:}A_5$, a subgroup of $Stab_G O_1$ fixing ∞ and 14 with L acting transitively on $O_1 \backslash \{\infty, 14\}$. (Note that L acts 2-transitively on $O_1 \backslash \{\infty, 14\}$ but not 3-transitively; suppose the L_{80}-orbits are $\{3, 18\}$ and $\{15, 20\}$.) Let $\Lambda_1 = O_1$, $\Lambda_2 = \{\infty\}$, and $\Lambda_3 = \{14\}$.

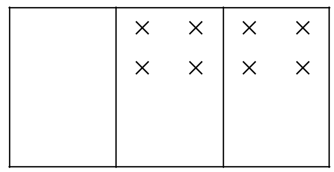
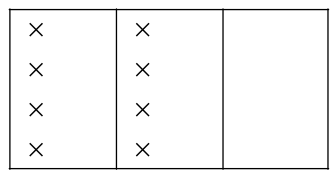
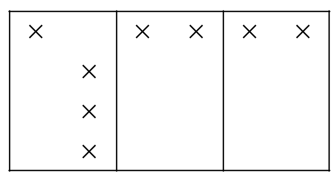
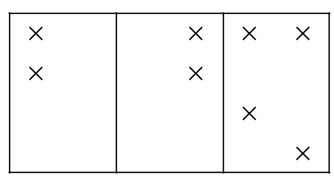
L-Orbit	Size	Representative
$\alpha_{8,1,1}$	1	O_1
$\alpha^{(1)}_{0,0,0}$	10	
$\alpha_{2,1,1}$	16	
$\alpha^{(2)}_{0,0,0}$	20	O_3
$\alpha^{(1)}_{4,1,0}$	40	
$\alpha^{(2)}_{4,1,0}$	40	
$\alpha^{(1)}_{4,0,1}$	40	

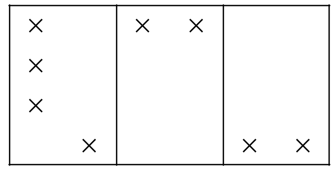
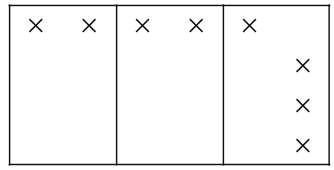
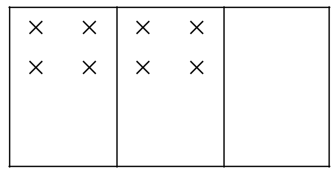
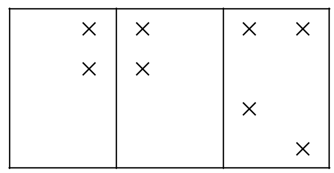
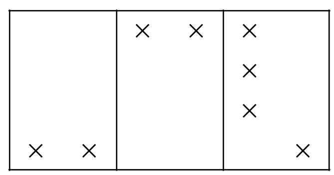
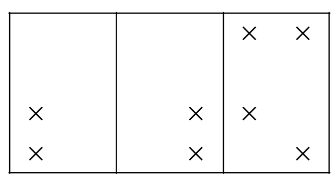
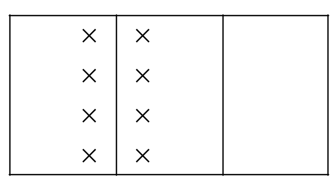
$\alpha^{(2)}_{4,0,1}$	40	
$\alpha_{4,1,1}$	60	
$\alpha_{4,0,0}$	60	
$\alpha_{2,1,0}$	96	
$\alpha_{2,0,1}$	96	
$\alpha_{2,0,0}$	240	

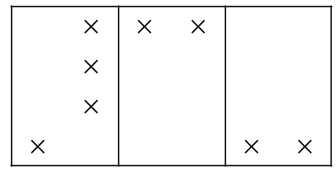
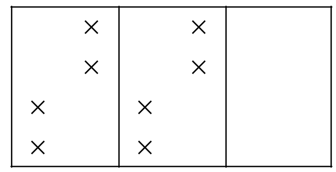
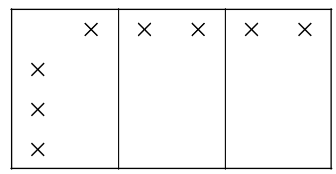
2.9. $(\Delta_3^6(a))L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_1 = O_1$,

$\Lambda_2 =$  and $\Lambda_3 =$ . So

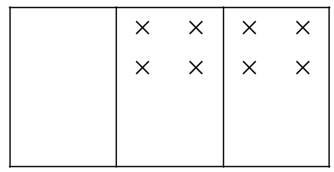
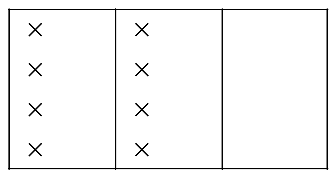
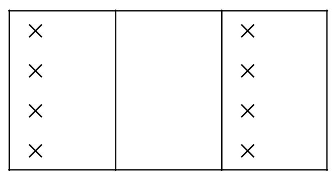
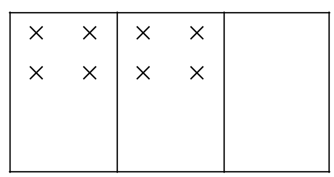
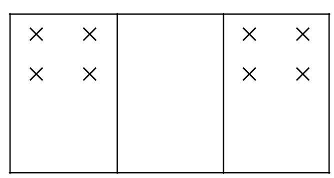
$L \sim 2^4{:}3.S_4$.

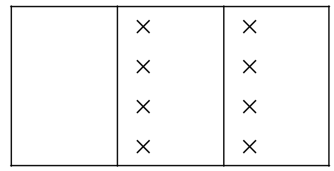
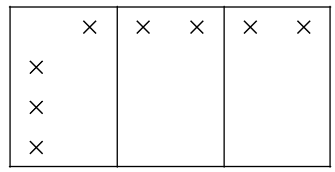
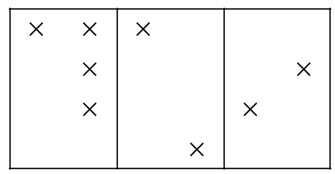
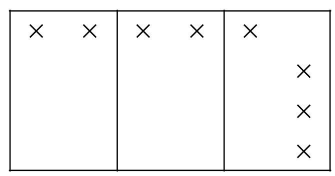
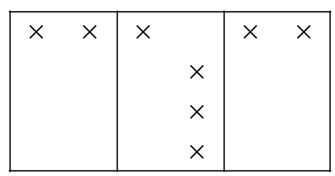
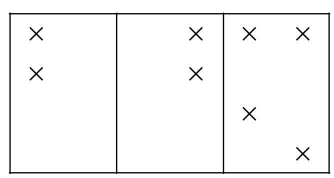
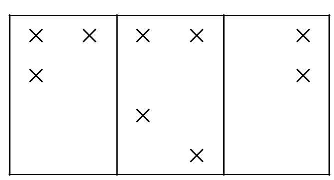
L-Orbit	Size	Representative
$\alpha_{8,4,1}$	1	O_1
$\alpha^{(1)}_{0,0,0}$	6	O_3
$\alpha^{(2)}_{0,0,0}$	24	
$\alpha_{4,4,1}$	4	
$\alpha_{4,1,1}$	16	
$\alpha_{2,2,1}$	48	

$\alpha_{4,3,1}$	48	
$\alpha_{2,1,1}$	64	
$\alpha_{4,2,1}$	72	
$\alpha_{2,0,0}$	96	
$\alpha_{2,1,0}$	192	
$\alpha_{2,2,0}$	48	
$\alpha_{4,0,0}$	4	

$\alpha_{4,1,0}$	48	
$\alpha_{4,2,0}$	72	
$\alpha_{4,3,0}$	16	

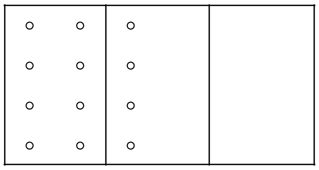
2.10. $(\Delta_3^7(a))\, L = Stab_G\{\Lambda_1,\, \Lambda_2,\, \Lambda_3\}$, where $\Lambda_1 = O_1$, $\Lambda_2 = O_2$ and Λ_3 is the partition of O_1 given by $\{\infty, 14\}, \{0, 8\}, \{3, 20\}, \{15, 18\}$. Note that L stabilizes $\mathscr{T}_0$. So $L \sim [2^6].S_4$.

L-Orbit	Size	Representative
$\alpha_{8,0,2^4}$	1	O_1
$\alpha_{0,8,0^4}$	1	O_2
$\alpha_{0,0,0^4}$	1	O_3
$\alpha^{(1)}_{0,4,0^4}$	12	
$\alpha_{4,4,1^4}$	16	
$\alpha_{4,0,1^4}$	16	
$\alpha_{4,4,2^2}$	12	
$\alpha_{4,0,2^2}$	12	

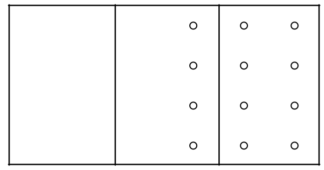
$\alpha^{(2)}_{0,4,0^4}$	16	
$\alpha_{4,2,1^4}$	32	
$\alpha_{4,2,21^2}$	192	
$\alpha_{2,2,2}$	32	
$\alpha_{2,4,2}$	32	
$\alpha_{2,2,1^2}$	192	
$\alpha_{2,4,1^2}$	192	

Remark. $\alpha^{(1)}_{0,4,0^4} \cup \alpha^{(2)}_{0,4,0^4} \cup \alpha_{4,4,1^4} \cup \alpha_{4,0,1^4} \cup \alpha_{4,4,2^2} \cup \alpha_{4,0,2^2} = \alpha_{4^2}$ (of (2.4)) and $\alpha_{4,2,1^4} \cup \alpha_{4,2,21^2} \cup \alpha_{2,2,2} \cup \alpha_{2,4,2} \cup \alpha_{2,2,1^2} \cup \alpha_{2,4,1^2} = \alpha_{4,2^2}$ (of (2.4)).

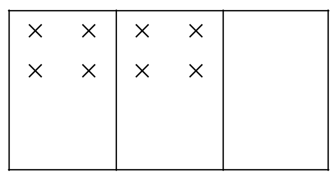
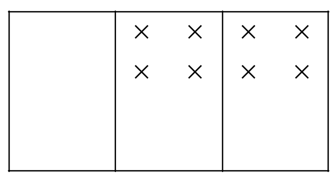
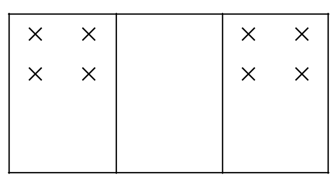
2.11. $(\Delta^8_3(a))\, L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_1 = \mathscr{L}_0$, $\Lambda_2 =$

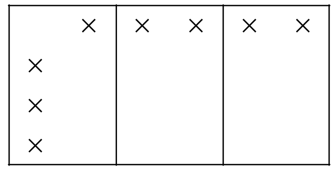
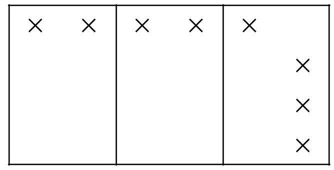


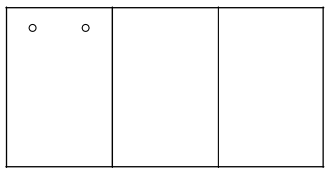
and $\Lambda_3 =$

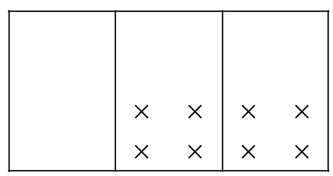
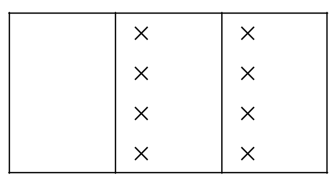


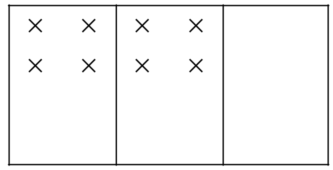
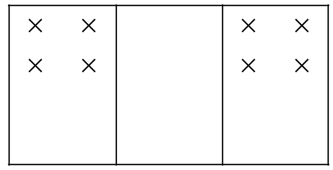
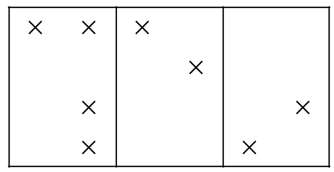
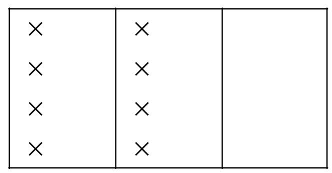
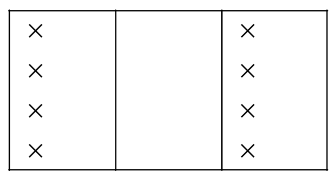
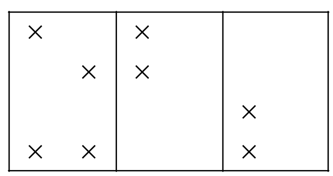
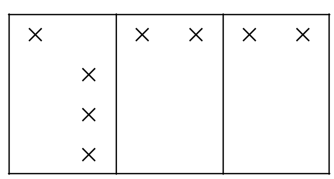
. So $L \sim 2^6{:}3.3^2{:}4$.

L-Orbit	Size	Representative
$\alpha_{4^2,8,0}$	3	O_1
$\alpha_{4^2,0,8}$	3	O_3
$\alpha_{4^2,4,4}$	9	O_2
$\alpha_{2^4,6,2}$	72	
$\alpha_{2^4,2,6}$	72	
$\alpha_{2^4,4,4}$	216	

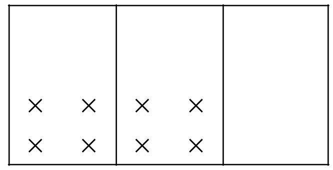
$\alpha_{31^5,5,3}$	192	
$\alpha_{31^5,3,5}$	192	

2.12. $(\Delta_3^9(a))$ $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$, where $\Lambda_1 = O_1$, $\Lambda_2 = O_2$, $\Lambda_3 = O_3$ and $\Lambda_4 =$ . So $L \sim 2^4{:}S_4$.

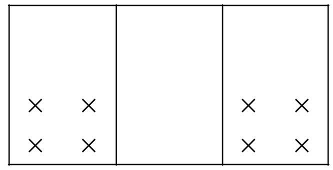
L-Orbit	Size	Representative
$\alpha_{8,0,0,2}$	1	O_1
$\alpha_{0,8,0,0}$	1	O_2
$\alpha_{0,0,8,0}$	1	O_3
$\alpha^{(1)}_{0,4,4,0}$	12	
$\alpha^{(2)}_{0,4,4,0}$	16	

$\alpha_{4,4,0,2}$	6	
$\alpha_{4,0,4,2}$	6	
$\alpha_{4,2,2,2}$	48	
$\alpha_{4,4,0,1}$	16	
$\alpha_{4,0,4,1}$	16	
$\alpha^{(1)}_{4,2,2,1}$	96	
$\alpha^{(2)}_{4,2,2,1}$	32	

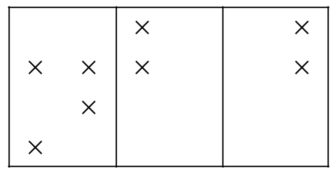
$\alpha_{4,4,0,0}$ 6



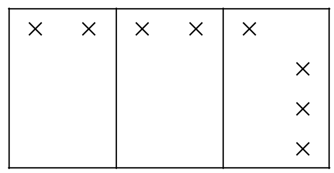
$\alpha_{4,0,4,0}$ 6



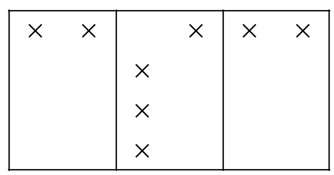
$\alpha_{4,2,2,0}$ 48



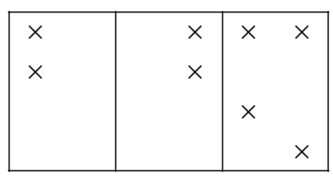
$\alpha_{2,2,4,2}$ 8



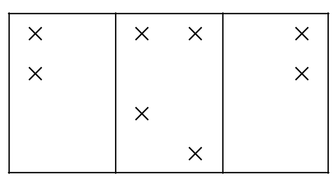
$\alpha_{2,4,2,2}$ 8

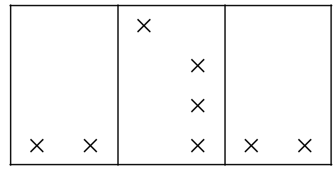
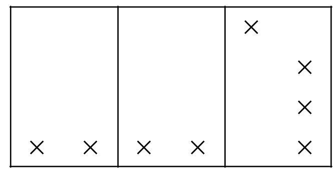
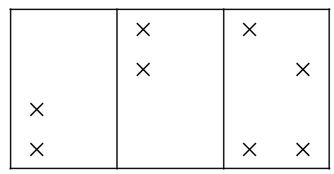
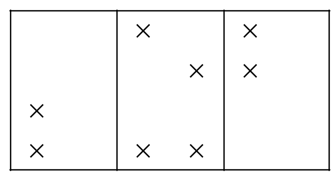


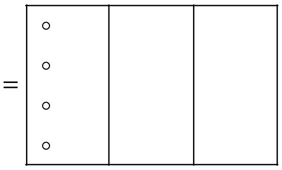
$\alpha_{2,2,4,1}$ 96

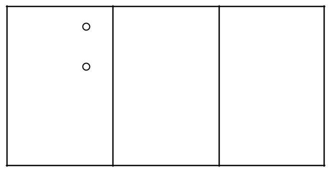
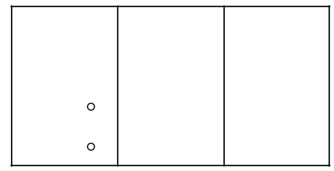


$\alpha_{2,4,2,1}$ 96



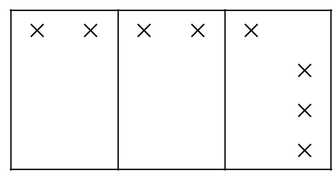
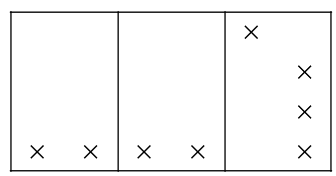
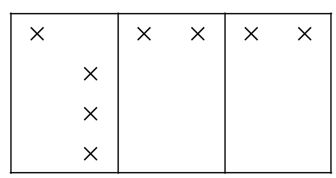
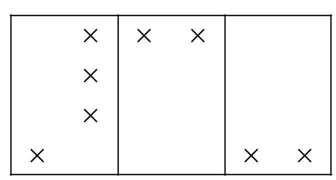
$\alpha^{(1)}_{2,4,2,0}$	24	
$\alpha^{(1)}_{2,2,4,0}$	24	
$\alpha^{(2)}_{2,2,4,0}$	96	
$\alpha^{(2)}_{2,4,2,0}$	96	

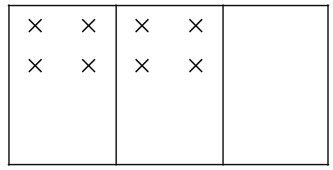
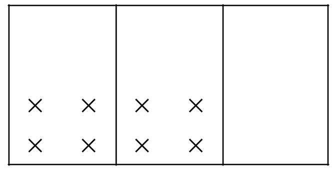
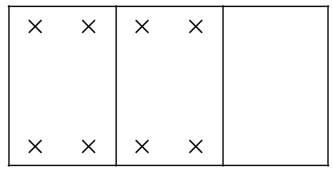
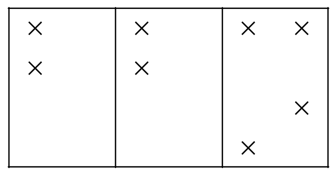
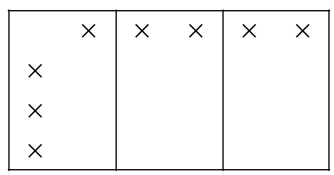
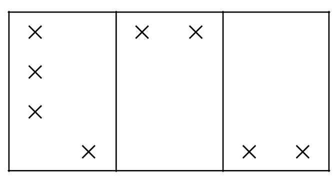
2.13. $(\Delta_3^{10}(a))\, L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_1 =$ ,

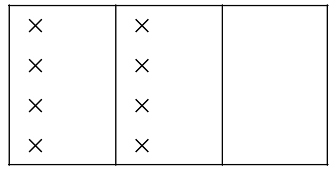
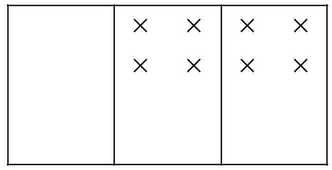
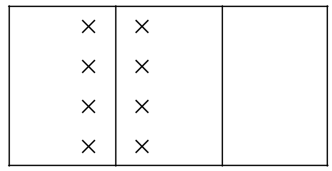
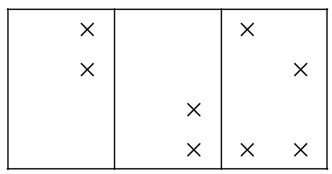
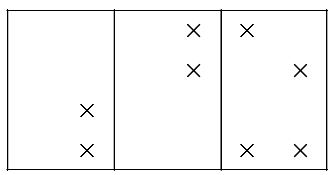
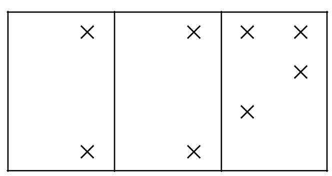
$\Lambda_2 =$  and $\Lambda_3 =$ . So

$L \sim [2^5]{:}S_4$.

(Note that $L \leq Stab_G O_1$.)

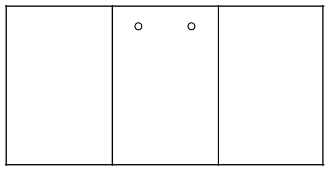
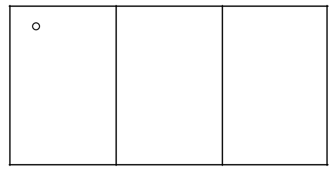
L-Orbit	Size	Representative
$\alpha_{1,1,0}$	128	
$\alpha_{1,0,1}$	128	
$\alpha_{1,1,2}$	32	
$\alpha_{1,2,1}$	32	

$\alpha_{2,2,0}$	24	
$\alpha_{2,0,2}$	24	
$\alpha_{2,1,1}$	96	
$\alpha_{2,0,0}$	96	
$\alpha_{3,1,0}$	32	
$\alpha_{3,0,1}$	32	
$\alpha_{4,2,2}$	1	O_1

$\alpha_{4,0,0}$	4	
$\alpha^{(2)}_{0,0,0}$	34	
$\alpha_{0,2,2}$	4	
$\alpha^{(1)}_{0,0,0}$	6	O_3
$\alpha_{0,2,0}$	16	
$\alpha_{0,0,2}$	16	
$\alpha_{0,1,1}$	64	

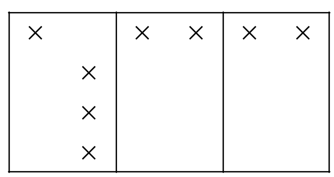
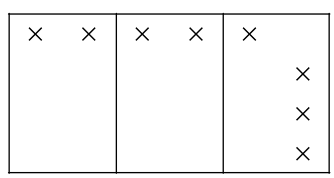
Remark. $\alpha_{1,1,0} \cup \alpha_{1,1,2} \cup \alpha_{1,2,1}$, $\alpha_{2,2,0} \cup \alpha_{2,0,2} \cup \alpha_{2,1,1} \cup \alpha_{2,0,0}$, $\alpha_{3,1,0} \cup \alpha_{3,0,1}$, $\alpha_{4,2,2} \cup \alpha_{4,0,0}$, $\alpha^{(1)}_{0,0,0} \cup \alpha_{0,2,2}$, $\alpha^{(2)}_{0,0,0} \cup \alpha_{0,2,0} \cup \alpha_{0,0,2} \cup \alpha_{0,1,1}$ equal, respectively, $\alpha_{1,31^5}$, $\alpha_{2,2^4}$, $\alpha_{3,31^5}$, $\alpha_{4,4^2}$, $\alpha_{0,4^2}$ and $\alpha_{0,2^4}$ of (2.2).

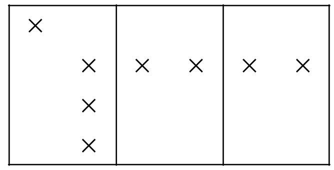
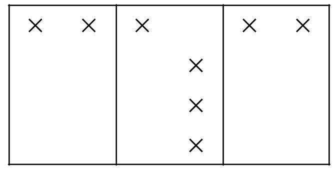
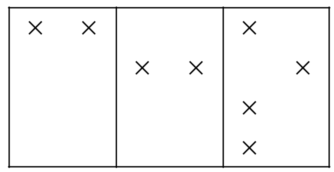
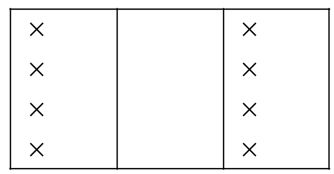
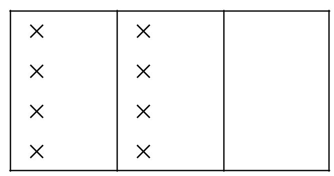
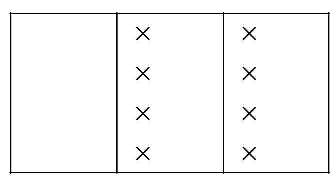
2.14. $(\Delta^1_4(a))\, L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_1 = O_1$,

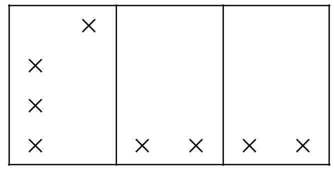
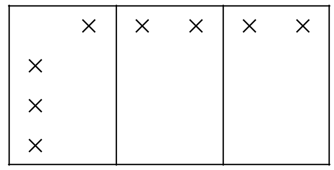
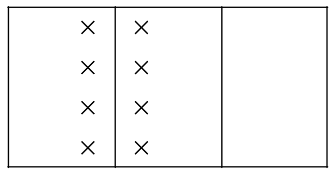
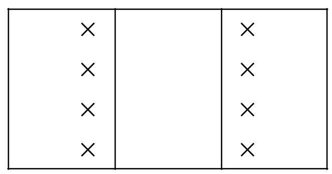
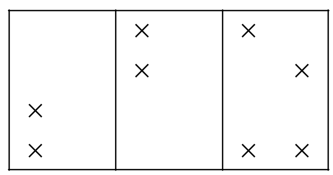
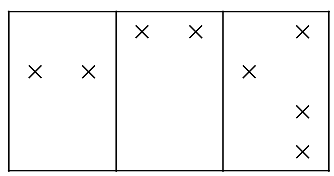
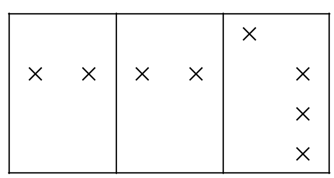
$\Lambda_2 =$ , and $\Lambda_3 =$ .

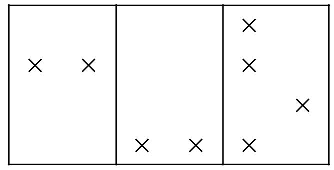
So $L \sim L_3(2) : 2$.

The seven octads in $\alpha_{4,2,1}$ (which all contain Λ_3) intersect $O_1 \setminus \Lambda_3$ in seven 3-element subsets which together may be regarded as the lines of a projective plane on $O_1 \setminus \Lambda_3$. Denoting this collection of 3-element subsets of $O_1 \setminus \Lambda_3$ by $\mathscr{L}$, we let $\alpha^{\mathscr{L}}_{4,0,1}$ consist of all octads in $\alpha_{4,0,1}$ which intersect $O_1 \setminus \Lambda_3$ in a 3-element subset in $\mathscr{L}$. Set $\alpha^{\mathscr{L}^c}_{4,0,1} = \alpha_{4,0,1} \setminus \alpha^{\mathscr{L}^c}_{4,0,1}$.

L-Orbit	Size	Representative
$\alpha_{8,0,1}$	1	O_1
$\alpha_{4,2,1}$	7	
$\alpha_{2,2,1}$	14	

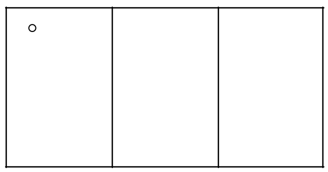
$\alpha^{\mathscr{L}}_{4,0,1}$	21	
$\alpha_{2,1,1}$	56	
$\alpha_{2,0,1}$	42	
$\alpha^{\mathscr{L}^c}_{4,0,1}$	56	
$\alpha_{4,1,1}$	56	
$\alpha_{0,0,0}$	7	O_3
$\alpha_{0,2,0}$	7	O_2
$\alpha_{0,1,0}$	16	

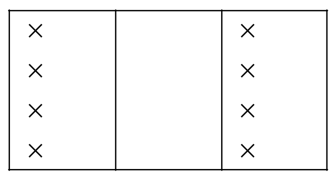
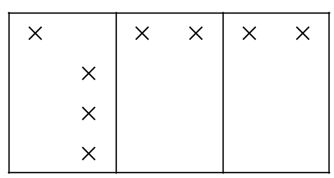
$\alpha_{4,0,0}^{(1)}$	21	
$\alpha_{4,2,0}$	7	
$\alpha_{4,1,0}$	56	
$\alpha_{4,0,0}^{(2)}$	56	
$\alpha_{2,1,0}$	168	
$\alpha_{2,2,0}$	42	
$\alpha_{2,0,0}^{(1)}$	42	

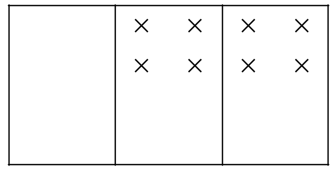
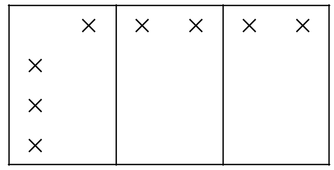
$\alpha^{(2)}_{2,0,0}$	84	

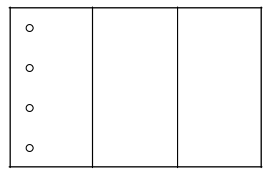
Remark. Putting $K = Stab_L\{0, 8\} (\in Syl_2 L)$ we have that the orbits of K upon O_2 are Λ_2, $\{4, 13\}$ and $\{16, 7, 10, 2\}$. Hence we see that the representatives given above for $\alpha^{(1)}_{2,0,0}$ and $\alpha^{(2)}_{2,0,0}$ cannot be in the same L-orbit.

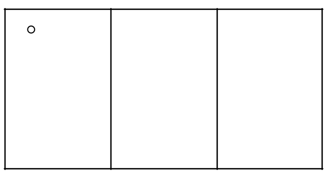
2.15. $(\Delta^2_4(a))\, L = Stab_G\{\Lambda_1, \Lambda_2\}$, where $\Lambda_1 = O_3$ and

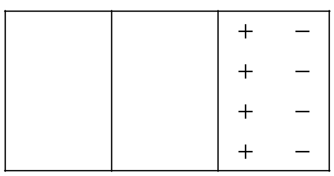
$\Lambda_2 =$ . So $L \cong A_8$.

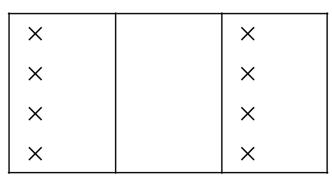
L-Orbit	Size	Representative
$\alpha_{0,1}$	15	O_1
$\alpha_{4,1}$	70	
$\alpha_{2,1}$	168	
$\alpha_{0,0}$	15	O_2

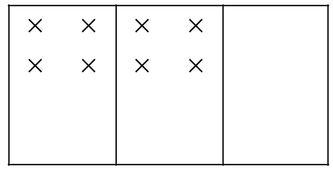
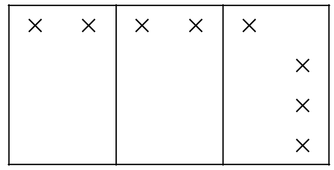
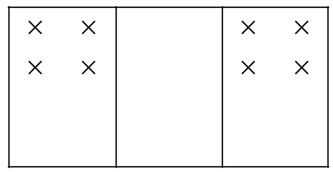
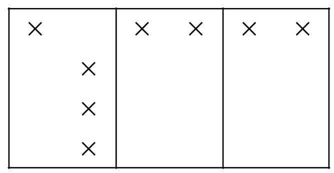
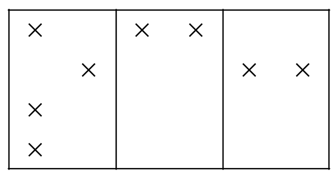
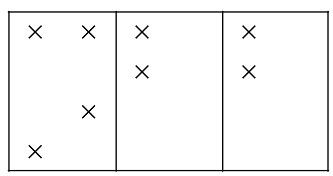
$\alpha_{4,0}$	210	
$\alpha_{2,0}$	280	
$\alpha_{8,0}$	1	O_3

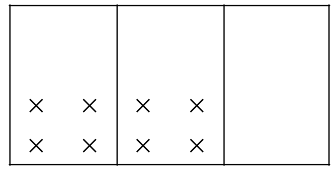
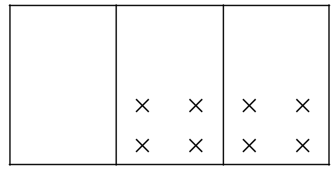
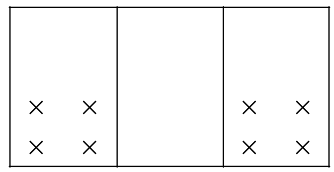
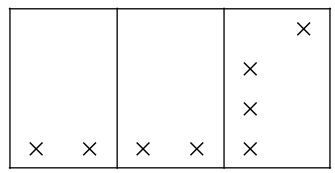
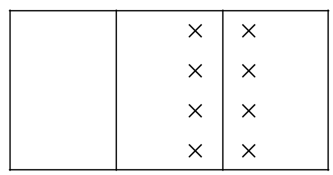
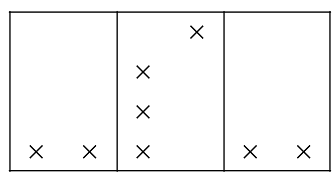
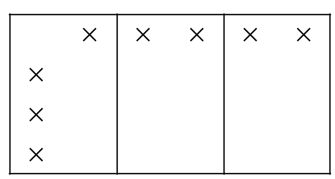
2.16. $(\Delta_4^3(a))$ $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where Λ_1 = ,

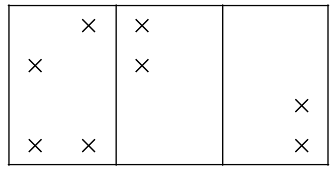
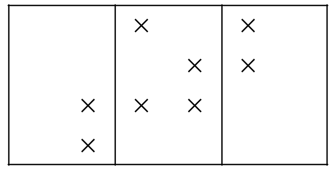
Λ_2 =  and Λ_3 is the 4^2 partition of O_3 given

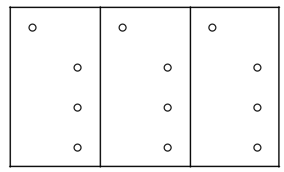
by . So $L \sim 2^6{:}3^2$.

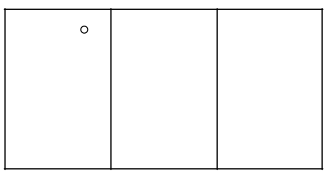
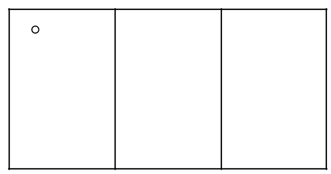
L-Orbit	Size	Representative
$\alpha_{4,1,4}$	2	
$\alpha_{4,1,0^2}$	3	O_1

$\alpha_{2,1,0^2}$	12	
$\alpha_{1,1,31}$	32	
$\alpha_{2,1,2^2}$	36	
$\alpha_{1,1,1^2}$	48	
$\alpha_{3,1,1^2}$	48	
$\alpha_{2,1.2}$	72	
$\alpha_{0,0,4^2}$	1	O_3
$\alpha_{0,0,0^2}$	3	O_2

$\alpha_{2,0,0^2}$	12	
$\alpha_{0,0,2^2}$	72	
$\alpha_{2,0,2^2}$	36	
$\alpha_{1,0,31}$	96	
$\alpha_{0,0,4}$	6	
$\alpha_{1,0,1^2}$	144	
$\alpha_{3,0,1^2}$	16	

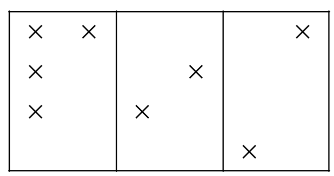
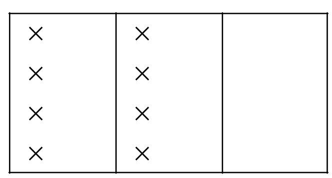
$\alpha_{2,0,2}$	72	
$\alpha_{0,0,2}$	48	

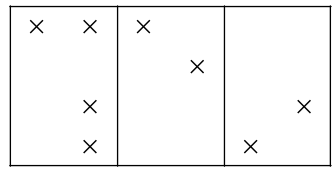
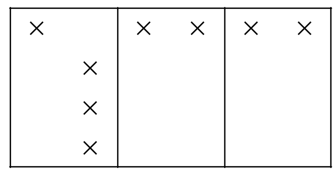
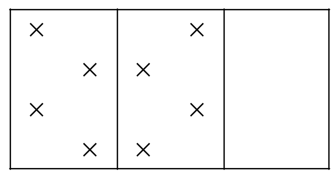
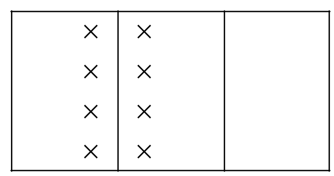
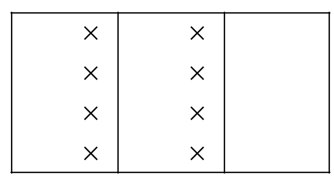
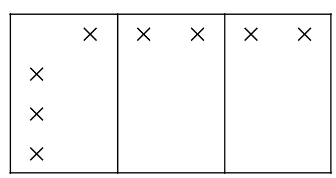
2.17. $(\Delta_4^4(a))\, L = Stab_G\{\Lambda_1,\ \Lambda_2,\ \Lambda_3\}$, where $\Lambda_1 =$ ,

$\Lambda_2 =$ , and $\Lambda_3 =$ .

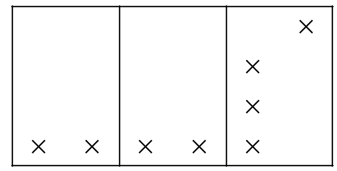
So $L \cong L_2(11)$.

(Note that Λ_1 is a dodecad of Ω.)

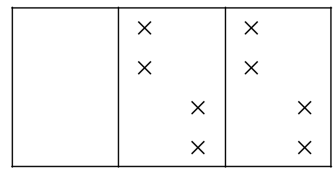
L-Orbit	Size	Representative
$\alpha_{2,1,1}$	11	
$\alpha_{2,0,1}$	11	

$\alpha_{6,1,1}$	11	
$\alpha_{4,1,1}$	55	O_1
$\alpha_{6,0,1}$	55	
$\alpha_{4,0,1}$	110	
$\alpha_{4,1,0}$	110	
$\alpha_{6,1,0}$	11	
$\alpha_{2,1,0}$	55	
$\alpha^{(1)}_{4,0,0}$	55	O_3

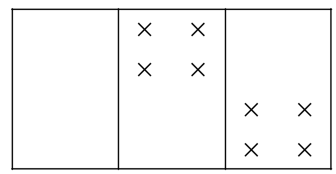
$\alpha_{2,0,0}$ 55



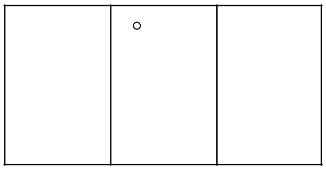
$\alpha_{6,0,0}$ 55



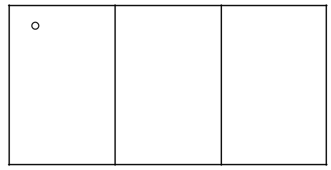
$\alpha^{(2)}_{4,0,0}$ 165



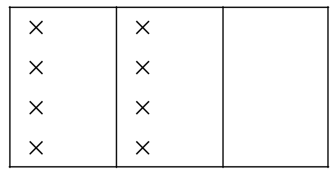
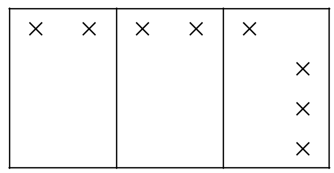
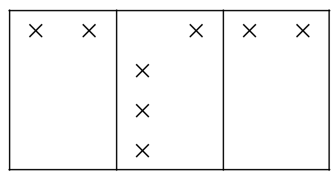
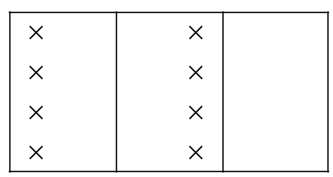
2.18. $(\Delta_4^5(a))\, L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_1, O_1, \Lambda_2 =$

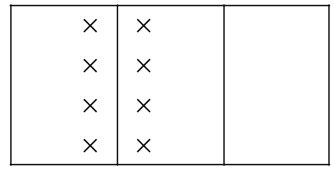
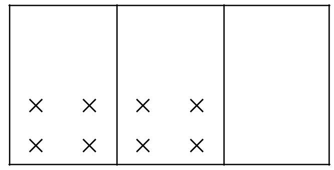
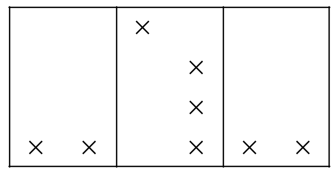
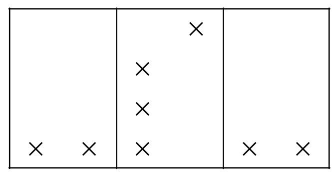


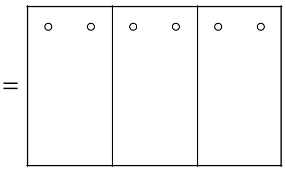
, and $\Lambda_3 =$

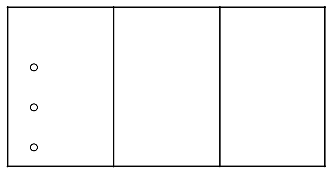
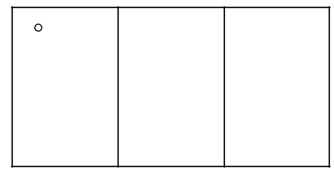


. So $L \cong A_7$.

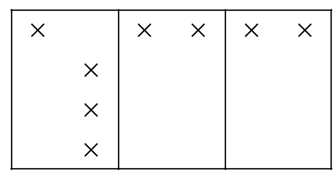
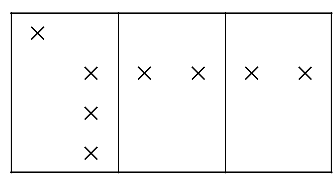
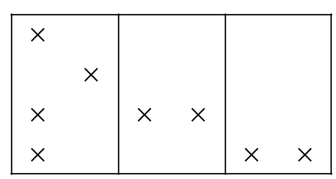
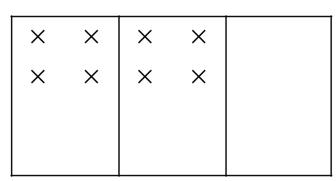
L-Orbit	Size	Representative
$\alpha_{8,0,1}$	1	O_1
$\alpha_{4,1,1}$	35	
$\alpha_{2,1,1}$	42	
$\alpha_{2,0,1}$	70	
$\alpha_{4,0,1}$	105	
$\alpha_{0,1,0}$	15	O_2
$\alpha_{0,0,0}$	15	O_3

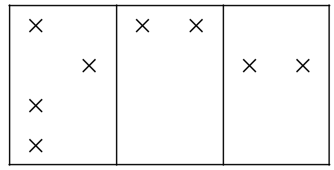
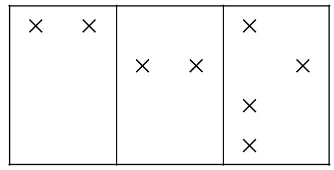
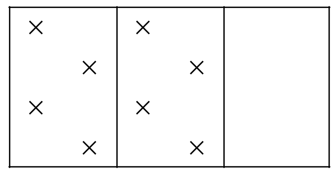
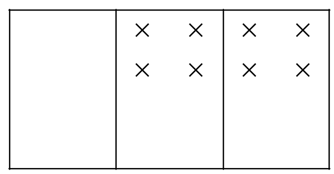
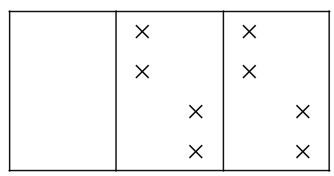
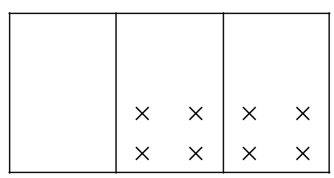
$\alpha_{4,1,0}$	35	
$\alpha_{4,0,0}$	105	
$\alpha_{2,1,0}$	126	
$\alpha_{2,0,0}$	210	

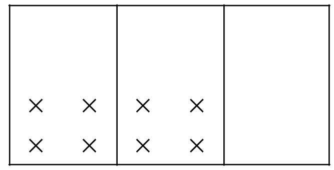
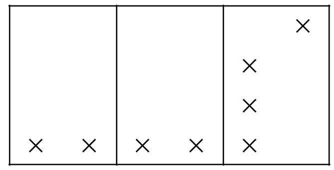
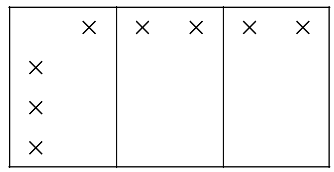
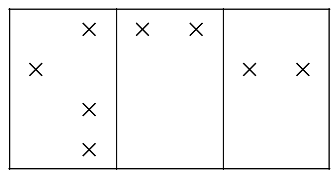
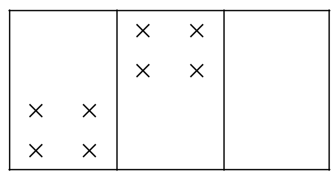
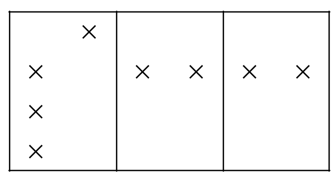
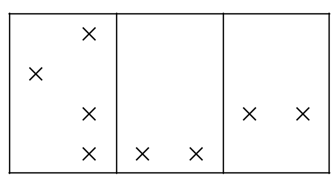
2.19. $(\Delta_4^6(a))L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where $\Lambda_1 =$ ,

$\Lambda_2 =$ , and $\Lambda_3 =$ . So

$L \sim (3 \times A_5):2.$

L-Orbit	Size	Representative
$\alpha_{4,0,1}$	5	
$\alpha_{1,3,1}$	5	O_1
$\alpha_{0,0,1}$	15	
$\alpha_{0,2,1}$	18	
$\alpha_{3,1,1}$	30	

$\alpha_{2,2,1}$	30	
$\alpha_{2,0,1}$	60	
$\alpha_{1,1,1}$	90	
$\alpha^{(1)}_{2,0,0}$	10	O_3
$\alpha_{4,0,0}$	15	
$\alpha^{(2)}_{2,0,0}$	90	
$\alpha_{0,0,0}$	15	

$\alpha_{0,2,0}$	30	
$\alpha^{(1)}_{1,1,0}$	60	
$\alpha_{5,3,0}$	1	
$\alpha_{3,1,0}$	90	
$\alpha_{2,2,0}$	90	
$\alpha_{1,3,0}$	15	
$\alpha^{(2)}_{1,1,0}$	90	

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