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SOME CLASSES OF QUANTALE MORPHISMS

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Abstract

This paper concerns some types of coherent quantale morphisms: Baer, minimalisant, quasi rigid, quasi r- and quasi $r*$-quantale morphisms. Firstly, we study how the reticulation functor $L(\cdot)$ preserves the properties that define these types of quantale morphisms. Secondly, we prove some characterization theorems for quasi rigid, quasi r- and quasi $r*$-quantale morphisms. These theorems extend some results existing in the literature of ring extensions and frame extensions.

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1. Introduction

The papers ([5], [22]) studied some important classes of ring morphisms and ring extensions. Among them we mention Baer, minimalisant, flat ring morphisms and rigid, r- and $r*$-ring extensions (resp., quasi rigid, quasi r- and quasi $r*$-ring extensions).

The quantales are multiplicative complete lattices that extend the lattices of ideals in (unital) commutative rings, as well as other lattices of congruences [23, 11, 21]. The reticulation of a quantale A is a bounded distributive lattice $L(A)$ whose prime spectrum is homeomorphic to the m-prime spectrum of A (cf. [13, 8]). In fact, the reticulation construction provides a covariant functor $L(\cdot)$ from the category of coherent quantales to the category of bounded distributive lattices (see [8]).

This paper concerns some types of coherent quantale morphisms: Baer, minimalisant, quasi rigid, quasi r- and quasi $r*$-quantale morphisms. These notions are abstractions of some remarkable types of morphisms and extensions studied in ring theory and frame theory (see [3-6, 22]). Firstly, we study how the reticulation functor $L(\cdot)$ preserves the properties that define these types of quantale morphisms. Secondly, we prove some characterization theorems for quasi rigid, quasi r- and quasi $r*$-quantale morphisms. These theorems extend some algebraic and topological results proved in [5] for ring extensions and in [3, 4] for frame extensions.

Now we shall present the structure of paper. Section 2 contains definitions and basic properties on quantales: arithmetical properties, radical and m-prime elements, Zariski and flat topologies on the m-prime spectrum (cf. [23, 11, 14]). Section 3 concerns some transfer properties of the reticulation regarding the m-prime elements and the annihilators. Section 4 deals with some elementary functorial properties of reticulation with emphasis on preservation of the annihilators. In Section 5, we find various descriptions of Baer and w-Baer quantale morphisms and we prove two preservation results related to these classes of quantale morphisms. In Section 6, we begin the study of m-quantale morphisms. Sections 7 and 8 concern the quasi r- and quasi $r*$-quantale morphisms respectively. We obtain algebraic and topological characterizations of these classes of quantale morphisms and some transfer results. Quasi rigid quantale morphisms are studied in Section 9.

2. Preliminaries on Quantales

This section contains some basic notions and results in quantale theory ([23, 11]). Let $(A, \bigvee, \wedge, \cdot, 0, 1)$ be a quantale and $K(A)$ be the set of its compact elements. A is said to be *integral* if $(A, \cdot, 1)$ is a monoid and *commutative*, if the multiplication $\cdot$ is commutative. A *frame* is a quantale in which the multiplication coincides with the meet [17]. The quantale A is *algebraic* if any $a \in A$ has the form $a = \bigvee X$ for some subset X of $K(A)$. An algebraic quantale A is *coherent* if $1 \in K(A)$ and $K(A)$ is closed under the multiplication. Throughout this paper, the quantales are assumed to be integral and commutative. Often we shall write ab instead of $a \cdot b$. We fix a quantale A.

Lemma 2.1 ([7]). *For all elements a, b, c of the quantale A the following hold*:

(1) *If $a \vee b = 1$, then $a \cdot b = a \wedge b$.*

(2) *If $a \vee b = 1$, then $a^n \vee b^n = 1$ for all integer numbers $n \geq 1$.*

(3) *If $a \vee b = a \vee c = 1$, then $a \vee (b \cdot c) = a \vee (b \wedge c) = 1$.*

On each quantale A one can consider a residuation operation $a \to b = \bigvee\{x \mid ax \leq b\}$ and a negation operation $a^{\perp} = a^{\perp_A}$, defined by $a^{\perp} = a \to 0 = \bigvee\{x \in A \mid ax = 0\}$. Thus for all $a, b, c \in A$ the following equivalence holds: $a \leq b \to c$ if and only if $ab \leq c$, so $(A, \vee, \wedge, \cdot, \to, 0, 1)$ becomes a (commutative) residuated lattice.

In this paper, we shall use without mention the basic arithmetical properties of a residuated lattice [12]. An element $p < 1$ of A is *m-prime* if for all $a, b \in A$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. If A is an algebraic quantale, then $p < 1$ is m-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. Let us introduce the following notations: $Spec(A)$ is the set of m-prime elements and $Max(A)$ is the set of maximal elements of A. If $1 \in K(A)$, then for any $a < 1$ there exists $m \in Max(A)$ such that $a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$.

The main example of quantale is the set $Id(R)$ of ideals of a (unital) commutative ring R and the main example of frame is the set $Id(L)$ of ideals of a bounded distributive lattice L. Thus the set $Spec(R)$ of prime ideals in R is the prime spectrum of the quantale $Id(R)$ and the set of prime ideals in L is the prime spectrum of the frame $Id(L)$.

Following [23], the *radical* $\rho(a) = \rho_A(a)$ of an element $a \in A$ is defined by $\rho_A(a) = \bigwedge\{p \in Spec(A) \mid a \leq p\}$; if $a = \rho(a)$ then a is a radical element. We shall denote by $R(A)$ the set of radical elements of A. The quantale A is said to be *semiprime* if $\rho(0) = 0$.

Lemma 2.2 ([23])**.** *For all elements* $a, b \in A$ *the following hold*:

(1) $a \leq \rho(a)$;

(2) $\rho(a \wedge b) = \rho(ab) = \rho(a) \wedge \rho(b)$;

(3) $\rho(a) = 1$ *iff* $a = 1$;

(4) $\rho(a \vee b) = \rho(\rho(a) \vee \rho(b))$;

(5) $\rho(\rho(a)) = \rho(a)$;

(6) $\rho(a) \vee \rho(b) = 1$ *iff* $a \vee b = 1$;

(7) $p(a^n) = \rho(a)$, *for all integer* $n \geq 1$.

Lemma 2.3 ([19])**.** *Let* A *be a coherent quantale and* $a \in A$. *Then*

(1) $\rho(a) = \bigvee\{c \in K(A) | c^k \leq a$ *for some integer* $k \geq 1\}$;

(2) *For any* $c \in K(A)$, $c \leq \rho(a)$ *iff* $c^k \leq a$ *for some* $k \geq 1$;

(3) A *is semiprime if and only if for any integer* $k \geq 1$, $c^k = 0$ *implies* $c = 0$.

Let A be a quantale such that $1 \in K(A)$. For any $a \in A$, denote $D_A(a) = D(a) = \{p \in Spec(A) | a \not\leq p\}$ and $V_A(a) = V(a) = \{p \in Spec(A) | a \leq p\}$. Then $Spec(A)$ is endowed with a topology whose closed sets are $(V(a))_{a \in A}$. If the quantale A is algebraic, then the family $(D(c))_{c \in K(A)}$ is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prime spectrum $Spec(R)$ of a commutative ring R [1]) and the Stone topology (defined on the prime spectrum $Spec_{Id}(L)$ of a bounded distributive lattice L [2]). Then this topology will be also called the Zariski topology of $Spec(A)$ and the corresponding topological space will be denoted by $Spec_Z(A)$. According to [14], $Spec_Z(A)$ is a *spectral space* in the sense of

[15]. The *flat topology* on $Spec(A)$ has as basis the family of the complements of compact open subsets of $Spec_Z(A)$ (cf. [9, 17]). Recall from [14] that the family $\{V(c) | c \in K(A)\}$ is a basis of open sets for the flat topology on $Spec(A)$. We shall denote by $Spec_F(A)$ this topological space.

Let L be a bounded distributive lattice. For any $x \in L$, denote $D_{Id}(x) = \{P \in Spec_{Id}(L) | x \notin P\}$ and $V_{Id}(x) = \{P \in Spec_{Id,Z}(L) | x \in P\}$. The family $(D_{Id}(x))_{x \in L}$ is a basis of open sets for the Stone topology on $Spec_{Id}(L)$; this topological space will be denoted by $Spec_{Id,Z}(L)$. We will denote by $Spec_{Id,F}(L)$ the prime spectrum $Spec_{Id}(L)$ endowed with the flat topology; the family $(V_{Id}(x))_{x \in L}$ is a basis of open sets for the at topology.

An element e of the quantale A is a complemented element if there exists $f \in A$ such that $e \vee f = 1$ and $e \wedge f = 0$. The Boolean center of the quantale A is the set $B(A)$ of complemented elements of A (cf. [7, 16]).

The following lemma collects some elementary properties of the elements of $B(A)$.

Lemma 2.4 ([7, 16]). *Let A be a quantale and $a, b \in A$, $e \in B(A)$. Then the following properties hold*:

(1) $a \in B(A)$ *iff* $a \vee a^{\perp} = 1$;

(2) $a \wedge e = ae$;

(3) $e \to a = e^{\perp} \vee a$;

(4) *If* $a \vee b = 1$ *and* $ab = 0$, *then* $a, b \in B(A)$;

(5) $(a \wedge b) \vee e = (a \vee e) \wedge (b \wedge e)$.

Proposition 2.5. *If* $a \in A$, *then* $a^{\perp} = \bigwedge(V(a^{\perp}) \bigcap Min(A))$.

3. Reticulation of a Coherent Quantale

The reticulation $L(R)$ of a commutative ring R was studied by many authors, but the main references on this topic remain [24, 17]. The *reticulation* $L(A)$ of a quantale A (introduced in [13] as a generalization of the reticulation of a commutative ring) is a bounded distributive lattice whose prime spectrum $Spec_{Id}(L(A))$ is homeomorphic to the prime spectrum $Spec(A)$ of the quantale A. In this section, we shall recall from [8, 13] the axiomatic definition of the reticulation of the coherent quantale and some of its basic properties. Let A be a coherent quantale and $K(A)$ the set of its compact elements.

Definition 3.1 ([8]). A reticulation of the quantale A is a bounded distributive lattice L together with a surjective function $\lambda : K(A) \to L$ such that for all $a, b \in K(A)$ the following properties hold:

(1) $\lambda(a \vee b) \leq \lambda(a) \vee \lambda(b)$;

(2) $\lambda(ab) = \lambda(a) \vee \lambda(b)$;

(3) $\lambda(a) \leq \lambda(b)$ iff $a^n \leq b$, for some integer $n \geq 1$.

In ([8, 13]), there were proven the existence and the uniqueness of the reticulation for each coherent quantale A; this unique reticulation will be denoted by $(L(A), \lambda_A : K(A) \to L(A))$ or shortly $L(A)$.

We remark that the reticulation $L(R)$ of a commutative ring R is isomorphic to the reticulation $L(Id(R))$ of the quantale $Id(R)$.

Lemma 3.2 ([8]). *For all elements* $a, b \in K(A)$ *the following properties hold*:

(1) $a \leq b$ *implies* $\lambda_A(a) \leq \lambda_A(b)$;

(2) $\lambda_A(a \vee b) = \lambda_A(a) \vee \lambda_A(b)$;

(3) $\lambda_A(a) = 1$ *iff* $a = 1$;

(4) $\lambda_A(0) = 0$;

(5) $\lambda_A(a) = 0$ *iff* $a^n = 0$, *for some integer* $n \geq 1$;

(6) $\lambda_A(a^n) = \lambda_A(a)$, *for any integer* $n \geq 1$;

(7) $\rho(a) = \rho(b)$ *iff* $\lambda_A(a) = \lambda_A(b)$;

(8) $\lambda_A(a) = 0$ *iff* $a \leq \rho(0)$;

(9) *If* A *is semiprime, then* $\lambda_A(a) = 0$ *implies* $a = 0$.

For any $a \in A$ and $I \in Id(L(A))$, let us denote $a^* = \{\lambda_A(c) | c \in K(A), c \leq a\}$ and $I_* = \bigvee \{c \in K(A) | \lambda_A(c) \in I\}$.

Lemma 3.3 ([8]). *The following assertions hold*:

(1) *If* $a \in A$, *then* a^* *is an ideal of* $L(A)$ *and* $a \leq (a^*)_*$;

(2) *If* $I \in Id(L(A))$, *then* $(I_*)^* = I$;

(3) *If* $p \in Spec(A)$, *then* $(p^*)_* = p$ *and* $p^* \in Spec_{Id}(L(A))$;

(4) *If* $P \in Spec_{Id}(L(A))$, *then* $P_* \in Spec(A)$;

(5) *If* $c \in K(A)$, *then* $c^* = (\lambda_A(c)]$;

(6) *If* $c \in K(A)$ *and* $I \in Id(L(A))$, *then* $c \leq I_*$ *iff* $\lambda_A(c) \in I$;

(7) *If* $a \in A$ *and* $I \in Id(L(A))$, *then* $\rho(a) = (a^*)_*$, $a^* = (\rho(a))^*$ *and* $\rho(I_*) = I_*$;

(8) *If* $c \in K(A)$ *and* $p \in Spec(A)$, *then* $c \leq p$ *iff* $\lambda_A(c) \in p^*$.

Often the two previous lemmas shall be used in the proofs without mention.

According to Lemma 3.3, one can consider the following order-preserving functions: $\delta_A : Spec(A) \to Spec_{Id}(L(A))$ and $\epsilon_A : Spec_{Id}(L(A)) \to Spec(A)$, defined by $\delta_A(p) = p^*$ and $\epsilon_A(P) = P_*$, for all $p \in Spec(A)$ and $P \in Spec_{Id}(L(A))$.

Lemma 3.4 ([8, 14]). *The functions* δ_A *and* ϵ_A *are homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another.*

We also observe that δ_A and ϵ_A are also order-isomorphisms.

Lemma 3.5. *Let* A *be a semiprime coherent quantale and* $a, b \in A$. *Then* $a^{\perp} \leq b^{\perp}$, *if and only if* $(ab)^{\perp} = b^{\perp}$.

Proof. Since $(ab)^{\perp} \geq b^{\perp}$ it suffices to prove that $a^{\perp} \leq b^{\perp}$, if and only if $(ab)^{\perp} \leq b^{\perp}$. If $(ab)^{\perp} \leq b^{\perp}$, then $a^{\perp} \leq (ab)^{\perp} \leq b^{\perp}$. Conversely, assume that $a^{\perp} \leq b^{\perp}$, therefore $abc = 0$. Thus $cb \leq a^{\perp} \leq b^{\perp}$, so $cbb = 0$, hence we get $\lambda_A(bc) = \lambda_A(b) \wedge \lambda_A(c) = \lambda_A(cbb) = 0$. Since A is semiprime it follows that $bc = 0$. Thus $b(ab)^{\perp} = 0$, so one gets $(ab)^{\perp} \leq b^{\perp}$.

□

For a bounded distributive lattice L we shall denote by $B(L)$ the Boolean algebra of the complemented elements of L. It is well-known that $B(L)$ is isomorphic to the Boolean center $B(Id(L))$ of the frame $Id(L)$ (see [7, 17]).

Let us fix a coherent quantale A.

Proposition 3.6 ([8]). *The function* $\lambda_A|_{B(A)} : B(A) \to B(L(A))$ *is a Boolean isomorphism.*

If L is bounded distributive lattice and $I \in Id(L)$, then the annihilator of I is the ideal $Ann_L(I) = Ann(I) = \{x \in I \mid x \wedge y = 0, \text{ for all } y \in L\}$.

Lemma 3.7 ([14]). *If* $c \in K(A)$ *and* $p \in Spec(A)$, *then* $Ann(\lambda_A(c)) \subseteq p^*$, *if and only if* $c \to \rho(0) \le p$.

The next two propositions concern the behaviour of reticulation w.r.t. the annihilators.

Proposition 3.8 ([14]). *If a is an element of a coherent quantale, then* $Ann(a^*) = (a \to \rho(0))^*$; *if A is semiprime, then* $Ann(a^*) = (a^{\perp})^*$.

Proposition 3.9 ([14]). *Assume that A is a coherent quantale. If I is an ideal of* $L(A)$, *then* $(Ann(I))_* = I_* \to \rho(0)$; *if A is semiprime, then* $(Ann(I))_* = (I_*)^{\perp}$.

If A is a quantale then we denote by $Min(A)$ the set of *minimal m-prime* elements of A; $Min(A)$ is called the minimal prime spectrum of A. If $1 \in K(A)$ then for any $p \in Spec(A)$ there exists $q \in Min(A)$ such that $q \le p$.

Proposition 3.10 ([18]). *If A is semiprime coherent quantale and* $p \in Spec(A)$, *then* $p \in Min(A)$, *if and only if for all* $c \in K(A)$, $c \le p$ *implies* $c^{\perp} \not\le p$.

We denote by $Min_Z(A)$ (resp., $Min_F(A)$) the topological space obtained by restricting the topology of $Spec_Z(A)$ (resp., $Spec_F(A)$) to $Min(A)$. Then $Min_Z(A)$ is homeomorphic to the space $Min_{Id\,Z}(L(A))$ of minimal prime ideals in $L(A)$ with the Stone topology and $Min_F(A)$ is homeomorphic to the space $Min_{Id, F}(L(A))$ of minimal prime ideals in $L(A)$ with the flat topology (cf. Lemma 3.4). By [14], $Min_Z(A)$ is a zero-dimensional Hausdorff space and $Min_F(A)$ is a compact T_1 space.

4. Quantale Morphisms

Let A, B be two quantales and $u : A \to B$ be a function that preserves the arbitrary joins (in this case we have $u(0) = 0$). Let us consider the function $\tilde{u} : B \to A$ defined by $\tilde{u}(b) = \bigvee \{a \in A \,|\, u(a) \leq b\}$, for any $b \in B$. It is well-known that $\tilde{u}$ is the right adjoint of u: $u(a) \leq b$ iff $a \leq \tilde{u}(b)$, for all $a \in A$ and $b \in B$ (see, e.g., Lemma 3.1 of [25]).

Let A, B be two quantales. A function $f : A \to B$ is a morphism of quantales if it preserves the arbitrary joins and the multiplication (in this case we have $u(0) = 0$); f is an integral morphism if $f(1) = 1$. If $u(K(A)) \subseteq K(B)$, then we say that u preserves the compacts. If u is an integral quantale morphism that preserves the compacts then it is called a coherent quantale morphism. In a similar manner one defines the frame morphisms, integral frame morphisms, coherent frame morphism, etc. (cf. [3, 4]).

Let $f : R_1 \to R_2$ be a morphism of (unital) commutative rings. If I is an ideal of R_1, then I^e will denote the extension of I to R_2, i.e., the ideal $R_2 f(I)$ generated by $f(I)$ in R_2 (cf. [1], p.9). Then the function $f^\bullet : Id(R_1) \to Id(R_2)$, defined by $f^\bullet(I) = I^e$, for any $I \in Id(R_1)$, is a coherent quantale morphism.

Let $f : L_1 \to L_2$ be a morphism of bounded distributive lattices. If I is an ideal of L_1, then $f^{\bullet}(I)$ is the lattice ideal $(f(I)]$ generated by $f(I)$ in L_2. Then the function $f^{\bullet} : Id(L_1) \to Id(L_2)$, defined by $I \mapsto f^{\bullet}(I)$, for any $I \in Id(L_1)$, is a coherent frame morphism.

Lemma 4.1. *Let $u : A \to B$ be a quantale morphism. If $q \in Spec(B)$, then $\tilde{u}(q) \in Spec(A)$. The function $\tilde{u}|_{Spec(B)} : Spec(B) \to Spec(A)$ is continuous w.r.t. the Zariski and the flat topology.*

Proof. Let a, b be two elements of A such that $ab \le \tilde{u}(q)$. Thus $u(a)u(b) = u(ab) \le q$ (because $\tilde{u}$ is the right adjoint of u), hence $u(a) \le q$ or $u(b) \le q$. By applying again the adjointness property we get $a \le \tilde{u}(q)$ or $b \le \tilde{u}(q)$, so $\tilde{u}(q) \in Spec(A)$.

Let us assume that $c \in K(A)$. For any $p \in Spec(A)$ we have $p \in (\tilde{u})^{-1}(V_A(c))$ iff $c \le \tilde{u}(p)$ iff $p \in V_B(u(c))$, hence $(\tilde{u})^{-1}(V_A(c)) = V_B(u(c))$ and $(\tilde{u})^{-1}(D_A(c)) = D_B(u(c))$. Then the two functions $\tilde{u}|_{Spec(B)} : Spec_Z(B) \to Spec_Z(A)$ and $\tilde{u}|_{Spec(B)} : Spec_F(B) \to Spec_F(A)$ are continuous. □

In the rest of the section we shall assume that A, B are two coherent quantales.

Proposition 4.2 ([8])**.** *Let $u : A \to B$ be a coherent quantale morphism. Then there exists a morphism of bounded distributive lattices $L(u) : L(A) \to L(B)$ such that the following diagram is commutative*:

$$\begin{array}{ccc} K(A) & \xrightarrow{u|_{K(A)}} & K(B) \\ \lambda_A \downarrow & & \downarrow \lambda_B \\ L(A) & \xrightarrow{L(u)} & L(B) \end{array}$$

The following lemma is a well-known result in lattice theory.

Lemma 4.3. *Let* $f : L \to M$ *be a morphism of bounded distributive lattices. For each* $P \in Spec_{Id}(L)$, *there exists* $Q \in Min(M)$ *such that* $f^{-1}(Q) \subseteq P$.

Proof. For sake of completeness we shall present a short proof of this lemma. If $P \in Spec_{Id}(L)$, then $1 \in f(L - P)$ and $f(L - P)$ is closed under meet. Then there exists $Q' \in Spec(M)$ such that $Q' \bigcap f(L - P) = \emptyset$. Let Q be a minimal prime ideal of M such that $Q \subseteq Q'$, so it is easy to see that $f^{-1}(Q) \subseteq P$. □

Lemma 4.4. *Let* $u : A \to B$ *be a coherent quantale morphism. If* $p \in Spec(A)$, *then there exists* $q \in Min(B)$ *such that* $\tilde{u}(q) \leq p$.

Proof. Assume that $p \in Spec(A)$, hence $p^* \in Spec_{Id}(L(A))$. By Lemma 4.3, there exists $Q \in Min_{Id}(L(B))$ such that $(L(u))^{-1}(Q) \subseteq p^*$. Thus $Q = q^*$ for some $q \in Min(A)$, hence $(L(u))^{-1}(q^*) \subseteq p^*$.

We shall prove that $\tilde{u}(q) \leq p$. Let c be a compact element of A such that $c \leq \tilde{u}(q)$, so $u(c) \leq q$ (by the adjointness property). Therefore by using Proposition 4.2 we get $L(u)(\lambda_A(c)) = \lambda_B(u(c)) \in q^*$, hence $\lambda_A(c) \in (L(u))^{-1}(q^*) \subseteq p^*$. By using Lemma 3.3(8) one obtains $c \leq p$. We conclude that $\tilde{u}(q) \leq p$. □

Corollary 4.5. *Let* $u : A \to B$ *be a quantale morphism. If* $p \in Min(A)$, *then there exists* $q \in Min(B)$ *such that* $\tilde{u}(q) = p$.

Proposition 4.6. *Let* $u : A \to B$ *be a coherent quantale morphism and assume that the quantales* A, B *are semiprime.*

(1) *If* $a \in A$, *then* $(u(a))^{\perp_B} = ((L(u)(a^*))_*)^{\perp_B}$;

(2) *If* $I \in Id(L(A))$, *then* $Ann_{L(B)}(L(u)(I)) = Ann_{L(B)}((u(I_*))^*)$.

Proof. (1) For all $a \in A$ and $d \in K(B)$ we shall prove the following equivalence:

$$du(a) = 0 \text{ if and only if } d((L(u)(a^*))_*) = 0. \tag{4.1}$$

Assume that $du(a) = 0$. Let e be a compact element of B such that $\lambda_B(e) \in L(a)(a^*)$, so there exists $c \in K(A)$ such that $c \leq a$ and $\lambda_B(e) = L(u)(\lambda_A(c)) = \lambda_B(u(c))$ (the last equality is due by Proposition 4.2). Thus $du(c) \leq du(a) = 0$, so $\lambda_B(de) = \lambda_B(d) \wedge \lambda_B(e) = \lambda_B(d) \wedge \lambda_B(u(c)) = \lambda_B(du(c)) = 0$. Since B is semiprime, by Lemma 3.3(9) we have $de = 0$, so the equalities $d((L(u)(a^*))_*) = d(\bigvee\{e \in K(B) | \lambda_B(e) \in L(u)(a^*)\}) = \bigvee\{de | e \in K(B), \lambda_B(e) \in L(u)(a^*)\} = 0$ hold.

Conversely, let us suppose that $d((L(u)(a^*))_*) = 0$. Let c be a compact element of A such that $c \leq a$, so $u(c) \in K(B)$ and $\lambda_A(c) \in a^*$, therefore we get $\lambda_B(u(c)) = L(u)(\lambda_A(c)) \in L(u)(a^*)$. It follows that $u(c) \leq (L(u)\ (a^*))_*$, hence $du(c) = 0$. Recall that the map u-preserves the arbitrary joins. Then the following equalities hold:

$$du(a) = du(\bigvee\{c \in K(A) | c \leq a\}) = \bigvee\{du(c) | c \in K(A),\ c \leq a\} = 0.$$

We have proven the equivalence (4.1), therefore the equality $(u(a))^{\perp_B} = ((L(u)(a^*))_*)^{\perp_B}$ holds.

(2) We shall prove that for any $d \in K(B)$ the following equivalence holds:

$$\lambda_B(d) \in Ann_{L(B)}(L(u)(I)) \text{ if and only if } \lambda_B(d) \in Ann_{L(B)}((u(I_*))^*). \tag{4.2}$$

Assume that $\lambda_B(d) \in Ann_{L(B)}(L(u)(I))$ and $y \in (u(I_*))^*$, so there exists $e \in K(B)$ such that $e \leq u(I_*)$ and $y = \lambda_B(e)$. We remark that

$$u(I_*) = u(\bigvee\{c \in K(A) | \lambda_A(c) \in I\}) = \bigvee\{u(c) | c \in K(A), \lambda_A(c) \in I\}, \tag{4.3}$$

therefore from $e \leq u(I_*)$ and $e \in K(B)$ we get $e \leq u(c)$, for some $c \in K(A)$ such that $\lambda_A(c) \in I$. Then $L(u)(\lambda_A(c)) \in L(u)(I)$, hence one gets $\lambda_B(du(c)) = \lambda_B(d) \wedge \lambda_B(u(c)) = \lambda_B(d) \wedge L(u)(\lambda_A(c)) = 0$. Since the quantale B is semiprime we have $du(c) = 0$, so $de = 0$. Thus $\lambda_B(d) \wedge y = \lambda_B(d) \wedge \lambda_B(e) = \lambda_B(de) = 0$, so $\lambda_B(d) \in Ann_{L(B)}((u(I_*))^*)$.

Conversely, assume that $\lambda_B(d) \in Ann_{L(B)}((u(I_*))^*)$. For any $z \in L(u)(I)$, there exists $c \in K(A)$ such that $\lambda_A(c) \in K(A)$ and $z = L(u)(\lambda_A(c)) = \lambda_B(u(c))$. Thus $\lambda_B(u(c)) \in L(u)(I)$, hence by using (4.3) we get $u(c) \leq u(I_*)$, therefore $z = \lambda_B(u(c)) \in (u(I_*))^*$. By applying the hypothesis, we get $\lambda_B(d) \wedge z = 0$, hence $\lambda_B(d) \in Ann_{L(B)}(L(u)(I))$.

□

5. Baer Quantale Morphisms

Let $f : R \to Q$ be a morphism of commutative rings. Following [22], we say that f is a Baer ring morphism if for all $I, j \in Id(R)$, $Ann_R(I) = Ann_R(J)$ implies $Ann_Q(I^e) = Ann_Q(J^e)$; f is said to be a w-Baer ring morphism if for all $x, y \in R$, $Ann_R(x) = Ann_R(y)$ implies $Ann_Q(f(x)) = Ann_Q(f(y))$.

Let $f : L \to M$ be a morphism of bounded distributive lattices. We say that f is a Stone lattice morphism if for all $I, J \in Id(L)$, $Ann_L(I) = Ann_L(J)$ implies $Ann_M(I^e) = Ann_M(J^e)$; f is said to be a w-Stone lattice morphism if for all $x, y \in M$, $Ann_L(x) = Ann_L(y)$ implies $Ann_M(f(x)) = Ann_M(f(y))$.

The previous definitions will be extended from ring and lattice morphisms to quantale morphisms. Let $u : A \to B$ be a quantale morphism. Then u is said to be a Baer quantale morphism if for all $a, b \in A$, $a^{\perp_A} = b^{\perp_A}$ implies $(u(a))^{\perp_B} = (u(b))^{\perp_B}$; u is said to be a w-Baer quantale morphism if for all $a, b \in K(A)$, $a^{\perp_A} = b^{\perp_A}$ implies $(u(a))^{\perp_B} = (u(b))^{\perp_B}$. The notions of Baer and w-Baer frame morphisms are defined in a similar manner.

A morphism $f : R \to Q$ of commutative rings is a Baer (resp., w-Baer) ring morphism if and only if $f^{\bullet} : Id(R) \to Id(Q)$ is a Baer (resp., w-Baer) quantale morphism. A morphism $f : L \to M$ of bounded distributive lattices is a Stone (resp., w-Stone) lattice morphism if and only if $f^{\bullet} : Id(L) \to Id(M)$ is a Baer (resp., w-Baer) frame morphism.

The class of Baer (resp., w-Baer) quantale morphisms is closed under the composition. The following result generalizes the main part of Proposition 8 of [22].

Proposition 5.1. *Let us consider two semiprime coherent quantales* A, B *and a coherent quantale morphism* $u : A \to B$. *Then the following assertions are equivalent*:

(1) *u is a Baer quantale morphism*;

(2) *For any* $a \in A$, $a^{\perp_A} = 0$ *implies* $(u(a))^{\perp_B} = 0$;

(3) *For all* $a, b \in A$, $a^{\perp_A} \leq b^{\perp_A}$ *implies* $(u(a))^{\perp_B} \leq (u(b))^{\perp_B}$;

(4) *For any* $a \in A$, $(u(a^{\perp_A}))^{\perp_B \perp_B} = (u(a))^{\perp_B}$;

(5) *For any* $b \in B$, $(b^{\perp_B}|_A)^{\perp_A} = 0$ *implies* $b = 0$, *whenever* $b^{\perp_B}|_A$ *is the element of A defined by* $b^{\perp_B}|_A = \bigvee\{c \in A \,|\, u(c)b = 0\}$.

Proof. We shall assume that a and b are arbitrary elements of the quantale A.

(1) $\Rightarrow$ (2) Assume that u is a Baer quantale morphism. If $a^{\perp_A} = 0$, then $a^{\perp_A} = 1^{\perp}$, hence $(u(a))^{\perp_B} = (u(1))^{\perp_B} = 1^{\perp_B} = 0$.

(2) $\Rightarrow$ (4) Since $(a \vee a^{\perp_A})^{\perp_A} = a^{\perp_A} \wedge a^{\perp_A \perp_A} = 0$, by applying the hypothesis (2) we get $(u(a \vee a^{\perp_A}))^{\perp_B} = 0$, hence $(u(a))^{\perp_B} \wedge (u(a^{\perp_A}))^{\perp_B} =$

$(u(a) \vee u(a^{\perp_A}))^{\perp_B} = (u(a \vee a^{\perp_A}))^{\perp_B} = 0$. Thus $(u(a))^{\perp_B}(u(a^{\perp_A}))^{\perp_B} = 0$, so we get the inequality $(u(a^{\perp_A}))^{\perp_B \perp_B} \geq (u(a))^{\perp_B}$.

On the other hand, $u(a)u(a^{\perp_A}) = u(aa^{\perp_A}) = u(0) = 0$ implies $u(a) \leq (u(a^{\perp_A}))^{\perp_B}$, therefore the converse inequality $(u(a^{\perp_A}))^{\perp_B \perp_B} \leq (u(a))^{\perp_B}$ holds. Thus u is a Baer quantale morphism.

(4) $\Rightarrow$ (1) If $a^{\perp_A} = b^{\perp_A}$, then

$$(u(a))^{\perp_B} = (u(a^{\perp_A}))^{\perp_B \perp_B} = (u(b^{\perp_A}))^{\perp_B \perp_B} = (u(b))^{\perp_B}.$$

(3) $\Rightarrow$ (1) Obviously.

(1) $\Rightarrow$ (3) By using Lemma 3.5, the following implications hold:

$$a^{\perp_A} \leq b^{\perp_A} \Rightarrow (ab)^{\perp_A} = b^{\perp_A} (u(ab))^{\perp_B} = (u(b))^{\perp_B}.$$

Observing that $u(ab) = u(a)u(b) \leq u(a)$ implies $(u(a))^{\perp_B} \leq (u(ab))^{\perp_B}$ it follows that $a^{\perp_A} \leq b^{\perp_A}$ implies $(u(a))^{\perp_B} \leq (u(b))^{\perp_B}$.

(2) $\Rightarrow$ (5) Assume that $(b^{\perp_B}|_A)^{\perp_A} = 0$ so $(u(b^{\perp_B}|_A))^{\perp_B} = 0$ (by applying the hypothesis (2)). We remind that u preserves arbitrary joins, hence the following equalities hold: $bu(b^{\perp_B}|_A) = bu(\bigvee\{c \in A | u(c)b = 0\} = b\bigvee\{u(c) | u(c)b = 0\} = \bigvee\{bu(c) | c \in A, u(c)b = 0\} = 0$. Therefore $b \leq (u(b^{\perp_B}|_A))^{\perp_B} = 0$, so $b = 0$.

(5) $\Rightarrow$ (2) Assume that $a \in A$ and $a^{\perp_A} = 0$. For any $c \in B$, the following implications hold:

$$c \leq (u(a))^{\perp_B} \Rightarrow cu(a) = 0 \Rightarrow a \leq c^{\perp_B}|_A \Rightarrow (c^{\perp_B}|_A)^{\perp_A}$$

$$\leq a^{\perp_A} = 0 \Rightarrow c = 0.$$

In particular, we obtain $(u(a))^{\perp_B} = 0$.

□

For the rest of section, let us fix two semiprime coherent quantales A, B and a coherent quantale morphism $u : A \to B$. We shall describe the way in which the reticulation functor $L(\cdot)$ transforms the Baer (resp., w-Baer) quantale morphisms into the Stone (resp., w-Stone) lattice morphisms.

Theorem 5.2. *The following assertions are equivalent*:

(1) u *is a Baer quantale morphism*;

(2) $L(u)$ *is a Stone lattice morphism*.

Proof. (1) $\Rightarrow$ (2) Let I, J be two ideals of $L(A)$ such that $Ann_{L(A)}(I) = Ann_{L(A)}(J)$. Recall that the quantales A and B are semiprime. According to Proposition 3.9, $(I_*)^{\perp_A} = (Ann_{L(A)}(I))_* = (Ann_{L(A)}(J))_* = (J_*)^{\perp_A}$, therefore, by applying the hypothesis that u is a Baer quantale morphism, it follows that $(u(I_*))^{\perp_B} = (u(J_*))^{\perp_B}$. Then by using Proposition 3.8, we get

$$Ann_{L(B)}((u(I_*))^*) = ((u(I_*))^{\perp_B})^* = ((u(J_*))^{\perp_B})^* = Ann_{L(B)}((u(J_*))^*).$$

By Proposition 4.6(2) we get $Ann_{L(B)}(L(u)(I)) = Ann_{L(B)}(L(u)(J))$, so $L(u)$ is a Stone lattice morphism.

(2) $\Rightarrow$ (1) Let a, b be two elements of A such that $a^{\perp_A} = b^{\perp_A}$. By Proposition 3.8, we have $Ann_{L(A)}(a^*) = (a^{\perp_A})^* = (b^{\perp_A})^* = Ann_{L(A)}(b^*)$. Since $L(u)$ is a Stone lattice morphism we get $Ann_{L(B)}(L(u)(a^*)) = Ann_{L(B)}(L(u)(b^*))$, therefore $(Ann_{L(B)}(L(u)(a^*)))_* = (Ann_{L(B)}(L(u)(b^*)))_*$. In accordance with Proposition 3.9 we get $((L(u)(a^*))_*)^{\perp_B} = ((L(u)(b^*))_*)^{\perp_B}$, hence $(u(a))^{\perp_B} = (u(b))^{\perp_B}$ (by using Proposition 4.6(1)).

□

Lemma 5.3. *For any* $c \in K(A)$, $c^{\perp_A} = (\rho(c))^{\perp_A}$.

Proof. Firstly we observe that $c \leq \rho(c)$ implies $c^{\perp_A} \geq (\rho(c))^{\perp_A}$. In order to show that $c^{\perp_A} \leq (\rho(c))^{\perp_A}$, let us consider a compact element c of A such that $d \leq c^{\perp_A}$, hence $dc = 0$. Let c be a compact element of A such that $e^n \leq c$ for some integer $n \geq 1$, therefore $de^n \leq dc = 0$. It follows that the following equalities hold: $d\rho(c) = d(\bigvee\{e \in K(A) | e^n \leq c,$ for some integer $n \geq 1\}) = \bigvee\{de \in K(A) | e^n \leq c,$ for some integer $n \geq 1\} = 0$. Thus $d \leq (\rho(c))^{\perp_A}$, so we conclude that $c^{\perp_A} \leq (\rho(c))^{\perp_A}$.

Theorem 5.4. *The following assertions are equivalent*:

(1) *u is a w-Baer quantale morphism*;

(2) *$L(u)$ is a w-Stone lattice morphism*.

Proof. (1) $\Rightarrow$ (2) Assume that u is a w-Baer quantale morphism and c, d are two compact elements of A such that $Ann_{L(A)}(\lambda_A(c)) = Ann_{L(A)}(\lambda_A(d))$, so, by applying Proposition 3.9, the following equalities hold:

$$((\lambda_A(c)])^{\perp_A} = (Ann_{L(A)}((\lambda_A(c)]))_* = (Ann_{L(A)}((\lambda_A(d)]))_* = ((\lambda_A(d)])^{\perp_A}.$$

We remark that $(\lambda_A(c)]_* = (c^*)_* = \rho(c)$ and $(\lambda_A(d)]_* = (d^*)_* = \rho(d)$ (cf. Lemma 3.3(5) and (7)), hence, by using Lemma 5.4, we get $c^{\perp_A} = (\rho(c))^{\perp_A} = (\rho(d))^{\perp_A} = c^{\perp_A}$. Then $(u(c))^{\perp_B} = (u(d))^{\perp_B}$ (by the hypothesis (1)), hence, by using Proposition 3.8, the following hold:

$$Ann_{L(B)}((u(c))^*) = ((u(c))^{\perp_B})^* = ((u(d))^{\perp_B})^* = Ann_{L(B)}((u(d))^*).$$

By Proposition 4.2 and Lemma 3.3(5), we obtain the following equalities hold: $Ann_{L(B)}(L(u)(\lambda_A(c))) = Ann_{L(B)}(\lambda_B(u(c))) = Ann_{L(B)}((u(c))^*) = Ann_{L(B)}((u(d))^*) = \cdots = Ann_{L(B)}(L(u)(\lambda_A(d)))$. We conclude that $L(u)$ is a w-Stone lattice morphism.

(2) $\Rightarrow$ (1) Assume that $L(u)$ is a w-Stone lattice morphism. Let c, d be two compacts of A such that $c^{\perp_A} = d^{\perp_A}$. By using Lemma 3.3(5), the following equalities hold: $Ann_{L(A)}(\lambda_A(c)) = Ann_{L(A)}(c^*) = (c^{\perp_A})^* = (d^{\perp_A})^* = Ann_{L(A)}(d^*) = Ann_{L(A)}(\lambda_A(d))$. Applying the hypothesis and Proposition 4.2, we get $Ann_{L(B)}(\lambda_B(u(c))) = Ann_{L(B)}(L(u)(\lambda_B(c))) = Ann_{L(B)}(L(u)(\lambda_B(d))) = Ann_{L(B)}(\lambda_B(u(d)))$.

In virtue of Proposition 3.8, we get $((u(c))^{\perp_B})^* = Ann_{L(B)}(\lambda_B(u(c))) = Ann_{L(B)}(\lambda_B(u(d))) = ((u(d))^{\perp_B})^*$, therefore $\rho((u(c))^{\perp_B}) = (((u(c))^{\perp_B})^*)_* = (((u(d))^{\perp_B})^*)_* = \rho((u(d))^{\perp_B})$. By using Lemma 5.3 we obtain $(u(c))^{\perp_B} = (u(c))^{\perp_B}$, so u is a w-Baer quantale morphism.

□

6. Minimalisant Quantale Morphisms

Let $f : R \to S$ be a morphism of commutative rings. By [22], f is said to be a minimalisant ring morphism (= m-ring morphism) if for each $Q \in Min(S)$, $f^{-1}(Q) \in Min(R)$. This notion can be generalized to the quantale framework: a quantale morphism $u : A \to B$ is said to be a minimalisant quantale morphism (= m-quantale morphism) if for each $q \in Min(B)$, $\tilde{u}(q) \in Min(A)$. It is clear that a ring morphism $f : R \to S$ is an m-ring morphism if and only if $f^{\bullet} : Id(R) \to Id(S)$ is an m-quantale morphism.

Let $f : L \to M$ be a morphism of bounded distributive lattices. Then f is said to be a minimalisant lattice morphism (= m-lattice morphism) if for each $Q \in Min(M)$, $f^{-1}(Q) \in Min(L)$. We remark that $f : L \to M$ is an m-lattice morphism if and only if $f^{\bullet} : Id(L) \to Id(M)$ is an m-frame morphism.

If $u : A \to B$ is an m-quantale morphism then we consider the function $\Gamma : Min(B) \to Min(A)$ defined by $\Gamma(q) = \tilde{u}(q)$, for any $q \in Min(B)$.

Lemma 6.1. *Assume that u is an m-quantale morphism.*

(1) *If* $c \in K(A)$, *then* $\Gamma^{-1}(V_A(c) \bigcap Min(A)) = V_B(u(c)) \bigcap Min(B)$;

(2) Γ *is a continuous map w.r.t. the Zariski and the flat topologies.*

Let us fix two semiprime coherent quantales A, B and a coherent quantale morphism $f : R \to S$ According to Proposition 4.2, one can consider the lattice morphism $L(u) : L(A) \to L(B)$.

Lemma 6.2. *If* $q \in Spec(B)$, *then* $(L(u))^{-1}(q^*) = (\tilde{u}(q))^*$.

Proof. Assume that $x \in (L(u))^{-1}(q^*)$ so there exists $c \in K(A)$ such that $x = \lambda_A(c)$ and $L(u)(\lambda_A(c)) \in q^*$. According to Proposition 4.2, we get $u(\lambda_A(c)) \in q^*$, hence $u(c) \leq q$ (by Lemma 3.3(8)). By using the adjointness property we get $c \leq \tilde{u}(q)$, hence $x = \lambda_A(c) \in (\tilde{u}(q))^*$. Thus we obtain the inclusion $(L(u))^{-1}(q^*) \subseteq (\tilde{u}(q))^*$.

In order to prove the converse inclusion $(\tilde{u}(q))^* \subseteq (L(u))^{-1}(q^*)$, assume that d is a compact element of A such that $d \leq \tilde{u}(q)$. Then we have $u(d) \leq q$, hence $L(u)(\lambda_A(d)) = (\lambda_B(u(d)) \in q^*$, i.e., $\lambda_A(d)) \in (L(u))^{-1}(q^*)$. Thus the inclusion $(\tilde{u}(q))^* \subseteq (L(u))^{-1}(q^*)$ follows.

□

Proposition 6.3. *The following assertions are equivalent*:

(1) *u is an m-quantale morphism*;

(2) *$L(u)$ is an m-lattice morphism.*

Proof. (1) $\Rightarrow$ (2) Let Q be a minimal prime ideal of the lattice $L(B)$, so $Q = q^*$ for some $q \in Min(B)$. By the hypothesis that u is an m-quantale morphism we have $\tilde{u}(q) \in Min(A)$, so $(\tilde{u}(q))^* \in Min_{Id}(L(A))$. In accordance with Lemma 6.2, we have $(L(u))^{-1}(Q) = (L(u))^{-1}(q^*) = (\tilde{u}(q))^*$, so $(L(u))^{-1}(Q)$ is a minimal prime ideal of the lattice $L(A)$. Then $L(u)$ is an m-lattice morphism.

(2) $\Rightarrow$ (1) Assume that $L(u)$ is an m-lattice morphism. Let q be a minimal prime element of the quantale B, hence $q^* \in Min_{Id}(L(B))$. By taking into account the hypothesis, it follows that $(L(u))^{-1}(q^*) \in Min_{Id}(L(A))$. By using Lemma 6.2, we get $(\tilde{u}(q))^* \in Min_{Id}(L(A))$, hence $\tilde{u}(q) \in Min(A)$. We conclude that u is an m-quantale morphism.

□

Lemma 6.4. *The following assertions are equivalent*:

(1) *u is an m-quantale morphism*;

(2) *For all* $q \in Min(A)$ *and* $c \in K(A)$, $c \leq q$ *implies* $c^{\perp_A} \not\leq \tilde{u}(q)$;

(3) *For all* $q \in Min(A)$ *and* $c \in K(A)$, $c \leq q$ *if and only if* $c^{\perp_A} \not\leq \tilde{u}(q)$.

Proof. By Proposition 3.10.

□

Proposition 6.5. *If* $u : A \to B$ *is an m-quantale morphism, then it is a w-Baer quantale morphism.*

Proof. Let c, d be two compact elements of A such that $c^{\perp_A} = d^{\perp_A}$. Assume by absurdum that $(u(c))^{\perp_B} \neq (u(d))^{\perp_B}$, so $(u(c))^{\perp_B} \not\leq (u(d))^{\perp_B}$ or $(u(d))^{\perp_B} \not\leq (u(c))^{\perp_B}$. For example, suppose that $(u(c))^{\perp_B} \not\leq (u(d))^{\perp_B}$, so there exists $e \in K(B)$ such that $e \leq (u(c))^{\perp_B}$ and $e \not\leq (u(d))^{\perp_B}$. Then $eu(d) \neq 0$ so there exists $q \in Min(B)$ such that $eu(d) \leq q$ (because $\bigwedge Min(B) = 0$), hence $e \not\leq q$ and $u(d) \not\leq q$. Since u is an m-quantale morphism we have $\tilde{u}(q) \in Min(A)$.

From $e \not\leq q$ and $e \leq (u(c))^{\perp_B}$ we get $u(c) \leq e^{\perp_B} \leq q$, hence $c \leq \tilde{u}(q)$ (because $\tilde{u}$ is the right adjoint of u). Since $\tilde{u}(q) \in Min(A)$, by applying Proposition 3.10 we get $c^{\perp_A} \not\leq \tilde{u}(q)$.

From $u(d) \not\leq q$ we get $d \not\leq \tilde{u}(q)$ (by the adjointness property), hence $c^{\perp_A} = d^{\perp_A} \leq \tilde{u}(q)$. We obtained a contradiction, hence $(u(c))^{\perp_B} = (u(d))^{\perp_B}$, so u is a w-Baer quantale morphism.

□

Remark 6.6. According to Proposition 2.4 of [22], if $f : R \to S$ is a w-Baer ring morphism and $Min_Z(R)$ is compact, then f is a minimalisant ring morphism. An open question is if this assertion can be generalised to quantale morphisms: if $u : A \to B$ is a w-Baer quantale morphism and $Min_Z(A)$ is compact, then is u a minimalisant quantale morphism?.

7. Quasi *r*-Quantale Morphisms

Let $f : Q \to S$ be a morphism of commutative rings. We say that f is an r-ring morphism (resp., a quasi r-ring morphism) if for each $Q \in Min(S)$ and for each finitely generated ideal J of S such that $J \nsubseteq Q$ there exists an element $a \in R$ (resp., a finitely generated ideal I of R) such that $f(a) \notin Q$ and $Ann_S(s) \subseteq Ann_S(I^e)$ (resp., $I^e \nsubseteq Q$ and $Ann_S(J) \subseteq Ann_S(I^e)$).

The notion of quasi r-ring morphism can be extended to quantale theory: a quantale morphism $u : A \to B$ is said to be a quasi r-quantale morphism if for all $q \in Min(B)$ and $d \in K(A)$ such that $d \nleq q$ there exists $c \in K(A)$ such that $u(c) \nleq q$ and $d^{\perp_B} \leq (u(c))^{\perp_B}$. In a similar way, we define the notion of quasi r-frame morphism.

Let us fix two semiprime coherent quantales A, B and a coherent quantale morphism $u : A \to B$.

Proposition 7.1. *Let $u : A \to B$ be a quasi r-quantale morphism. Then u is an m-quantale morphism and the function $\Gamma : Min(B) \to Min(A)$ is bijective.*

Proof. Assume by absurdum that $u : A \to B$ is not an m-quantale morphism, so there exists $q \in Min(B)$ such that $\tilde{u}(q) \notin Min(A)$. Then there exists $p \in Min(A)$ such that $p < \tilde{u}(q)$. According to Corollary 4.5, there exists $r \in Min(B)$ such that $p = \tilde{u}(r)$, so the minimal prime elements q, r of the quantale B are distinct (assuming that $r = q$ we get $p = \tilde{u}(r) = \tilde{u}(q)$, contradicting $p < \tilde{u}(q)$). Thus there exists $d \in K(B)$ such that $d \leq r$ and $d \nleq q$, hence there exists $c \in K(A)$ such that $u(c) \nleq q$ and $d^{\perp_B} \leq (u(c))^{\perp_B}$ (because u is a quasi r-quantale morphism).

By Proposition 3.10, $d \leq r$ and $r \in Min(A)$ imply $d^{\perp_B} \nleq r$, hence $(u(c))^{\perp_B} \nleq r$, therefore $u(c) \leq r$ (because r is an m-prime element). By using the adjointness property we have $c \leq \tilde{u}(r) = p < \tilde{u}(q)$, so $u(c) < q$. We obtained a contradiction, so u is an m-quantale morphism.

According to Corollary 4.5, for any $p \in Min(A)$ there exists $q \in Min(B)$ such that $\tilde{u}(q) = p$, so Γ is surjective.

Assume that $q_1, q_2 \in Min(B)$ and $q_1 \neq q_2$, so there exists $d \in K(B)$ such that $d \leq q_1$ and $d \nleq q_2$. Since u is a quasi r-quantale morphism there exists $c \in K(A)$ such that $u(c) \nleq q_2$ and $d^{\perp_B} \leq (u(c))^{\perp_B}$.

In accordance with Corollary 4.5, the following implications hold:

$$d \leq q_1 \Rightarrow d^{\perp_B} \nleq q_1 \Rightarrow (u(c))^{\perp_B} \nleq q_1 \Rightarrow u(c) \leq q_1 \Rightarrow c \leq \tilde{u}(q_1).$$

Since $u(c) \nleq q_2$ implies $c \nleq \tilde{u}(q_2)$ it follows that $\tilde{u}(q_1) \neq \tilde{u}(q_2)$, so Γ is injective.

$\square$

Theorem 7.2. *The following assertions are equivalent*:

(1) *u is a quasi r-quantale morphism*;

(2) *The function $\Gamma : Min_Z(B) \to Min_Z(A)$ is a homeomorphism.*

Proof. (1) $\Rightarrow$ (2) According to Proposition 7.1, we know that Γ is a continuous bijective map. A basic open subset of $Min_Z(B)$ has the form $D_B(d)$, where d is a compact element of B. We shall prove that $\Gamma(D_B(d)) \bigcap Min(B))$ is an open subset of $Min_Z(A)$.

We observe that $\Gamma(D_B(d) \bigcap Min(B)) = \{\tilde{u}(q) \mid q \in Min(B), d \not\leq q\}$. Let $\tilde{u}(q)$ be a point of $\Gamma(D_B(d) \bigcap Min(B))$, i.e., $q \in Min(B)$ and $d \not\leq q$. Since u is a quasi r-quantale morphism, there exists $c \in K(A)$ such that $u(c) \not\leq q$ and $d^{\perp_B} \leq ((u(c))^{\perp_A}$.

We shall show that $\tilde{u}(q) \in D_A(c) \bigcap Min(A) \subseteq \Gamma(D_B(d) \bigcap Min(B))$. From $u(c) \not\leq q$ one obtains $c \leq \tilde{u}(q)$, so $\tilde{u}(q) \in D_A(c) \bigcap Min(A)$. Now let us consider that $p \in D_A(c) \bigcap Min(A)$, so $p \in Min(A)$ and $c \not\leq p$. By taking into account Corollary 4.5, from $p \in Min(A)$ it follows that $p = \tilde{u}(r)$, for some $r \in Min(A)$.

The following implications hold:

$$c \not\leq p = \tilde{u}(r) \Rightarrow u(c) \not\leq r \Rightarrow (u(c))^{\perp_B} \leq r \Rightarrow d^{\perp_B} \leq r \Rightarrow d \not\leq r \Rightarrow r \in D_B(d).$$

Since $p = \tilde{u}(r) = \Gamma(r)$, we obtain the following inclusion:

$$D_A(c) \bigcap Min(A) \subseteq \Gamma(D_B(d) \bigcap Min(B)).$$

(2) $\Rightarrow$ (1) Assume that $\Gamma : Min_Z(B) \to Min_Z(A)$ is a homeomorphism and consider the elements $q \in Min(B), d \in K(B)$ such that $d \not\leq q$. Thus $q \in D_B(d) \bigcap Min(B)$, so $\Gamma(q)$ is an element of the open subset $\Gamma(D_B(d) \bigcap Min(B))$ of $Min_Z(A)$. Therefore there exists a compact element c of A such that $\Gamma(q) \in D_A(c) \bigcap Min(A) \subseteq \Gamma(D_B(d) \bigcap Min(B))$, hence $c \not\leq \Gamma(q) = \tilde{u}(q)$. By the adjointness property we have $u(c) \not\leq q$.

In order to prove that $d^{\perp_B} \leq (u(c))^{\perp_B}$, consider a compact element e of A such that $e \leq d^{\perp_B}$, so $ed = 0$. For any $r \in Min(B)$, we have two possibilities:

• If $u(c) \leq r$, then $eu(c) \leq r$;

• If $u(c) \not\leq r$, then $c \not\leq \tilde{u}(r)$, so $\tilde{u}(r) \in D_A(c) \bigcap Min(A)$. It follows that $\Gamma(r) = \tilde{u}(r) \in \Gamma(D_B(d) \bigcap Min(B))$, hence $r \in D_B(d)$ (because Γ is bijective), i.e., $d \not\leq r$. Thus $e \leq d^{\perp_B} \leq r$, so $eu(c) \leq e \leq r$.

Then $eu(c) \leq \bigwedge Min(B) = 0$, so $eu(c) = 0$, i.e., $e \leq (u(c))^{\perp_B}$. We conclude that $d^{\perp_B} \leq (u(c))^{\perp_B}$, hence u is a quasi r-quantale morphism.

□

Let $f : L \to M$ be a morphism of bounded distributive lattices. Then f is called an r-lattice morphism if for all $Q \in Min_{Id}(M)$ and $y \in M$ such that $y \notin Q$ there exists $x \in L$ such that $f(x) \notin Q$ and $Ann_M(y) \subseteq Ann_M(f(x))$. We remark that f is an r-lattice morphism if and only if $f^{\bullet} : Id(L) \to Id(M)$ is a quasi r-frame morphism.

Theorem 7.3. *The following assertions are equivalent*:

(1) $u : A \to B$ *is a quasi r-quantale morphism*;

(2) $L(u) : L(A) \to L(B)$ *is an r-lattice morphism*.

Proof. (1) $\Rightarrow$ (2) Let us consider an element $y \in L(B)$ and a minimal prime ideal Q of $L(B)$ such that $y \notin Q$. We have to show that there exists $x \in L(A)$ such that $L(u)(x) \notin Q$ and $Ann_{L(B)}(y) \subseteq Ann_{L(B)}(L(u)(x))$.

Let us take a compact element d of B and a minimal m-prime element q of B such that $y = \lambda_A(d)$ and $Q = q*$, hence $\lambda_A(d) \notin q*$. By Lemma 3.3(8) we have $d \not\leq q$, hence, by applying the hypothesis that u is a quasi r-quantale morphism it follows that there exists $c \in K(A)$ such that $u(c) \not\leq q$ and $d^{\perp_B} \leq (u(c))^{\perp_B}$.

Denote $x = \lambda_A(c)$ and assume by absurdum that $L(u)(x) \in Q$, hence by Proposition 4.2 we have $\lambda_B(u(c)) = L(u)(u(c)) = L(u)(x) \in q*$. By Lemma 3.3(8) we get $u(c) \leq q$, contradicting $u(c) \not\leq q$. Then one obtains $L(u)(x) \notin Q$. By applying Proposition 3.8, it results that $Ann_{L(B)}(y) = Ann_{L(B)}(\lambda_B(d)) = Ann_{L(B)}(d^*) = (d^{\perp_B})^*$. Similarly, $Ann_{L(B)}(L(u)(x)) = Ann_{L(B)}(L(u)(\lambda_A(c))) = Ann_{L(B)}(\lambda_B(u(c))) = Ann_{L(B)}$ $((u(c))^*) = ((u(c))^{\perp_B})^*$.

Since $(\cdot)^*$ is an order-preserving map, from the inequality $d^{\perp_B} \leq$ $(u(c))^{\perp_B}$ we get $(d^{\perp_B})^* \subseteq ((u(c))^{\perp_B})^*$, so $Ann_{L(B)}(y) \subseteq Ann_{L(B)}(L(u)(x))$. We conclude that $L(u)$ is an r-lattice morphism.

(2) $\Rightarrow$ (1) Assume that $q \in Min(B)$ and d is a compact element of B such that $d \not\leq q$. We have to prove that there exists $c \in K(A)$ such that $u(c) \not\leq q$ and $d^{\perp_B} \leq (u(c))^{\perp_B}$. We observe that $q^* \in Min_{Id}(L(B))$, so $d \not\leq q$ implies $\lambda_B(d) \notin q^*$ (cf. Lemma 3.3(8)). By applying the hypothesis that $L(u) : L(A) \to L(B)$ is an r-lattice morphism, there exists $c \in K(A)$ such that $L(u)(\lambda_A(c)) \notin q^*$ and $Ann_{L(B)}(\lambda_B(d)) \subseteq Ann_{L(B)}(L(u)(\lambda_A(c)))$, hence $\lambda_B(u(c)) = L(u)(\lambda_A(c)) \notin q^*$, hence $(d^{\perp_B})^* = Ann_{L(B)}(d^*) =$ $Ann_{L(B)}(\lambda_B(d)) \subseteq Ann_{L(B)}(L(u)(\lambda_A(c))) = Ann_{L(B)}(\lambda_A(u(c))) = Ann_{L(B)}$ $((u(c))^*) = ((u(c))^{\perp_B})^*$.

From $\lambda_B(u(c)) \notin q*$ we get $u(c) \not\leq q$. By using Lemmas 5.3 and 3.3(7), from the inclusion $(d^{\perp_B})^* \subseteq ((u(c))^{\perp_B})^*$ one obtains

$$d^{\perp_B} = \rho(d^{\perp_B}) = ((d^{\perp_B})^*)_* \leq (((u(c))^{\perp_B})^*)_* = \rho((u(c))^{\perp_B}) = (u(c))^{\perp_B}.$$

It follows that u is a quasi r-quantale morphism.

□

8. Quasi r^*-Quantale Morphisms

Let $f : Q \to S$ be a morphism of commutative rings. We say that f is an r^*-ring morphism (resp., a quasi r^*-ring morphism) if for each $Q \in Min(S)$ and for each finitely generated ideal J of S such that $J \subseteq Q$ there exists an element $a \in R$ (resp., a finitely generated ideal I of R) such that $f(a) \in Q$ and $Ann_S(s) \subseteq Ann_S(I^e)$ (resp., $I^e \subseteq Q$ and $Ann_S(J) \subseteq Ann_S(I^e)$).

The notion of quasi r-ring morphism can be extended to quantale theory: a quantale morphism $u : A \to B$ is said to be a quasi r^*-quantale morphism if for all $q \in Min(B)$ and $d \in K(A)$ such that $d \leq q$ there exists $c \in K(A)$ such that $u(c) \leq q$ and $(u(c))^{\perp_B} \leq d^{\perp_B}$.

Let us fix two semiprime coherent quantales A, B and a coherent quantale morphism $u : A \to B$.

Proposition 8.1. *Let $u : A \to B$ be a quasi r^*-quantale morphism. Then u is an m-quantale morphism and the function $\Gamma : Min(B) \to Min(A)$ is bijective.*

Proof. In order to prove that u is an m-quantale morphism assume that $q \in Min(B)$, hence $\tilde{u}(q) \in Spec(A)$ (cf. Lemma 4.1). Then there exists $p \in Min(A)$ such that $p \leq \tilde{u}(q)$. According to Corollary 4.5, there exists $r \in Min(B)$ such that $p = \tilde{u}(r)$. Assume by absurdum that the minimal prime elements q, r of the quantale B are distinct, so there exists $d \in K(B)$ such that $d \leq r$ and $d \not\leq q$. Since u be a quasi r^*-quantale morphism, from $d \leq r$ it follows that there exists $c \in K(A)$ such that $u(c) \leq q$ and $(u(c))^{\perp_B} \leq d^{\perp_B}$. Thus $c \leq \tilde{u}(r) = p \leq \tilde{u}(q)$, so $u(c) \leq q$, therefore $(u(c))^{\perp_B} \not\leq q$ (cf. Proposition 3.10). On the other hand, $d \not\leq q$ implies $d^{\perp_B} \leq q$, so $(u(c))^{\perp_B} \leq d^{\perp_B} \leq q$. We obtained a contradiction, so $r = q$, therefore $\tilde{u}(q) = \tilde{u}(r) = p \in Min(A)$. It follows that u is an m-quantale morphism.

According to Corollary 4.5, for any $p \in Min(A)$ there exists $q \in Min(B)$ such that $\tilde{u}(q) = p$, so Γ is surjective. Assume that $q_1, q_2 \in Min(B)$ and $q_1 \neq q_2$, so there exists $d \in K(B)$ such that $d \leq q_1$ and $d \not\leq q_2$. Since u is a quasi r^*-quantale morphism there exists $c \in K(A)$ such that $u(c) \leq q_1$ and $(u(c))^{\perp_B} \leq d^{\perp_B}$. From $d \not\leq q_2$ we get $d^{\perp_B} \leq q_2$, hence $(u(c))^{\perp_B} \leq q_2$. By applying Proposition 3.10 we get $u(c) \not\leq q_2$, hence q_1 and q_2 are distinct. Then Γ is injective.

□

Theorem 8.2. *The following assertions are equivalent*:

(1) *u is a quasi r^*-quantale morphism*;

(2) *The function $\Gamma : Min_F(B) \to Min_F(A)$ is a homeomorphism.*

Proof. (1) $\Rightarrow$ (2) We know that $\Gamma : Min_F(B) \to Min_F(A)$ is a continuous map, and by Proposition 8.1, it is bijective. It remains to show that Γ is an open map. A basic open subset of $Min_F(B)$ has the form $V_B(d)$, where d is a compact element of B. We shall prove that $\Gamma(V_B(d) \bigcap Min(B))$ is an open subset of $Min_F(A)$.

Consider a point $\in \Gamma(V_B(d) \bigcap Min(B))$, so there exists $q \in V_B(d)$ such that $p = \tilde{u}(q)$. Since u is a quasi r-quantale morphism, from $d \le q$ it follows that there exists $c \in K(A)$ such that $u(c) \le q$ and $((u(c)))^{\perp_A} \le d^{\perp_B}$. We shall show that $p \in V_A(c) \bigcap Min(A) \subseteq \Gamma(V_B(d) \bigcap Min(B))$. From $c \le \tilde{u}(q) = p$ one obtains $p \in V_A(c) \bigcap Min(A)$.

Now let us consider a point $r \in V_A(c) \bigcap Min(A)$, so $r \in Min(A)$ and $c \le r$. By Corollary 4.5, there exists $s \in Min(B)$ such that $r = \tilde{u}(s)$, so $c \le r = \tilde{u}(s)$ implies $u(c) \le s$. In accordance with Proposition 3.10, we get $(u(c))^{\perp_B} \not\le s$, hence $d^{\perp_B} \not\le s$, therefore $d \le s$ (because s is m-prime). This implies $s \in V_B(d) \bigcap Min(B)$, hence $r = \Gamma(s) \in \Gamma(V_B(d) \bigcap Min(B))$. We conclude that $p \in V_A(c) \bigcap Min(A) \subseteq \Gamma(V_B(d) \bigcap Min(B))$, so $\Gamma(V_B(d) \bigcap Min(B))$ is an open subset of $Min_F(A)$.

(2) $\Rightarrow$ (1) Assume that $\Gamma : Min_F(B) \to Min_F(A)$ is a homeomorphism and consider the elements $q \in Min(B)$ and $d \in K(B)$ such that $d \le q$. Thus $q \in V_B(d) \bigcap Min(B)$, so $\tilde{u}(q) = \Gamma(q)$ is an element of the open subset $\Gamma(V_B(d) \bigcap Min(B))$ of $Min_F(A)$. Thus, there exists a compact element c of A such that $\tilde{u}(q) \in V_A(c) \bigcap Min(A) \subseteq \Gamma(V_B(d) \bigcap Min(B))$. From $\tilde{u}(q) \in V_A(c)$ we get $c \le \tilde{u}(q)$, hence $u(c) \le q$.

In order to prove that $(u(c))^{\perp_B} \le d^{\perp_B}$, consider a compact element y of A such that $y \le (u(c))^{\perp_B}$, so $yu(c) = 0$. For any $r \in Min(B)$, we have two possibilities:

• If $d \le r$, then $dy \le r$;

• If $d \not\le r$, then $\tilde{u}(r) \not\le \Gamma(V_B(d) \bigcap Min\,(B))$ (because Γ is a bijection), hence $\tilde{u}(r) \notin V_B(c) \bigcap Min\,(A)$, i.e., $c \not\le \tilde{u}(r)$. Thus $u(c) \not\le r$, so $(u(c))^{\perp_B} \le r$. It follows that $y \le r$, so $dy \le r$.

Then we obtain $dy \le \bigwedge Min(B) = 0$, so $dy = 0$, i.e., $y \le d^{\perp_B}$. We conclude that $(u(c))^{\perp_B} \le d^{\perp_B}$, hence u is a quasi $r*$-quantale morphism.

□

Let $f : L \to M$ be a morphism of bounded distributive lattices. Then f is called an $r*$-lattice morphism if for all $Q \in Min_{Id}(M)$ and $y \in Q$ there exists $x \in L$ such that $f(x) \in Q$ and $Ann_M(f(x)) \subseteq Ann_M(y)$. We remark that f is an $r*$-lattice morphism if and only if $f^{\bullet} : Id(L) \to Id(M)$ is a quasi $r*$-frame morphism.

Theorem 8.3. *The following assertions are equivalent*:

(1) $u : A \to B$ *is a quasi* $r*$*-quantale morphism*;

(2) $L(u) : L(A) \to L(B)$ *is an* $r*$*-lattice morphism.*

Proof. (1) $\Rightarrow$ (2) Let us consider an element $y \in L(B)$ and a minimal prime ideal Q of $L(B)$ such that $y \in Q$. We have to show that there exists $x \in L(A)$ such that $L(u)(x) \in Q$ and $Ann_{L(B)}(L(u)(x)) \subseteq Ann_{L(B)}(y)$.

Let us take a compact element d of B and a minimal m-prime element q of B such that $y = \lambda_B(d)$ and $Q = q^*$, hence $\lambda_B(d) \in q^*$. By Lemma 3.3(8) we have $d \leq q$, hence, by applying the hypothesis that u is a quasi $r*$-quantale morphism it follows that there exists $c \in K(A)$ such that $u(c) \leq q$ and $(u(c))^{\perp_B} \leq d^{\perp_B}$. From $u(c) \leq q$ we get $L(u)(\lambda_A(c)) = \lambda_B(u(c)) \in q^* = Q$. If we denote $x = \lambda_A(c)$ then it follows that $L(u)(x) \in Q$.

By using the proof of Theorem 7.3, we have $Ann_{L(B)}(y) = (d^{\perp_B})^*$ and $Ann_{L(B)}(L(u)(x)) = ((u(c))^{\perp_B})^*$. Since $(\cdot)^*$ is an order-preserving map, from the inequality $(u(c))^{\perp_B} \leq d^{\perp_B}$ we get the inclusion $((u(c))^{\perp_B})^* \subseteq (d^{\perp_B})^*$, so $Ann_{L(B)}(L(u)(x)) \subseteq Ann_{L(B)}(y)$. We conclude that $L(u)$ is an $r*$-lattice morphism.

(2) $\Rightarrow$ (1) Assume that $q \in Min(B)$ and d is a compact element of B such that $d \leq q$. We have to prove that there exists $c \in K(A)$ such that $u(c) \leq q$ and $(u(c))^{\perp_B} \leq d^{\perp_B}$. We observe that $q^* \in Min_{Id}(L(B))$, so $d \leq q$ implies $\lambda_B(d) \in q^*$ (cf. Lemma 3.3(8)). By applying the hypothesis that $L(u) : L(A) \to L(B)$ is an $r*$-lattice morphism, there exists $c \in K(A)$ such that $L(u)(\lambda_A(c)) \in q^*$ and $Ann_{L(B)}(L(u)(\lambda_A(c))) \subseteq Ann_{L(B)}(\lambda_B(d))$. By using Proposition 4.2 we get $\lambda_B(u(c)) = Ł(u)(\lambda_A(c)) \in q^*$, therefore $u(c) \leq q$ (by Lemma 3.3(8)).

On the other hand, we have

$$(d^{\perp_B})^* = Ann_{L(B)}(d^*) = Ann_{L(B)}(\lambda_B(d)) \supseteq Ann_{L(B)}(L(u)(\lambda_A(c)))$$

$$= Ann_{L(B)}(\lambda_A(u(c))) = Ann_{L(B)}((u(c))^*) = ((u(c))^{\perp_B})^*.$$

Recall that $(\cdot)_*$ is an order-preserving map. Then by using Lemmas 5.3 and 3.3(7), from the $(d^{\perp_B})^* \supseteq ((u(c))^{\perp_B})^*$ one obtains

$$d^{\perp_B} = \rho(d^{\perp_B}) = ((d^{\perp_B})^*)_* \geq (((u(c))^{\perp_B})^*)_* = \rho((u(c))^{\perp_B}) = (u(c))^{\perp_B}.$$

It follows that u is a quasi $r*$-quantale morphism.

□

9. Quasi Rigid Quantale Morphisms

Let $f : R \to S$ be a morphism of commutative rings. Then f is said to be a quasi rigid ring morphism if for each finitely generated ideal J of S there exists a finitely generated ideal I of R such that $Ann_S(f^\bullet(I)) = Ann_S(J)$.

Let A, B be two coherent frames. According to [4, 6], a frame morphism $u : A \to B$ is said to be a *rigid frame morphism* if for each $d \in K(B)$ there exists $c \in K(A)$ such that $(u(c))^{\perp_B} = d^{\perp_B}$.

Let A, B be two coherent quantales and $u : A \to B$ a quantale morphism. Then u is said to be a *quasi rigid quantale morphism* if for each $d \in K(B)$ there exists $c \in K(A)$ such that $(u(c))^{\perp_B} = d^{\perp_B}$.

If $f : R \to S$ is a morphism of commutative rings, then f is a quasi rigid ring morphism if and only if $f^\bullet : Id(R) \to Id(S)$ is a quasi rigid quantale morphism.

Let us fix two semiprime coherent quantales A, B and a coherent quantale morphism $u : A \to B$.

Proposition 9.1. *If $u : A \to B$ is a quasi rigid quantale morphism, then it is a quasi r-quantale morphism and a quasi $r*$-quantale morphism.*

Proof. Assume that $q \in Min(B)$ and $d \in K(B)$. By hypothesis, there exists $c \in K(A)$ such that $(u(c))^{\perp_B} = d^{\perp_B}$. According to Proposition 3.10, the following implications hold:

(a) $d \not\leq q \Rightarrow d^{\perp_B} \leq q \Rightarrow (u(c))^{\perp_B} \leq q \Rightarrow (u(c))^{\perp_B} \not\leq q$;

(b) $d \leq q \Rightarrow d^{\perp_B} \not\leq q \Rightarrow (u(c))^{\perp_B} \not\leq q \Rightarrow (u(c))^{\perp_B} \leq q$.

Thus u is a quasi r-quantale morphism and a quasi $r*$-quantale morphism.

□

Corollary 9.2. *Assume that* $u : A \to B$ *is a quasi rigid quantale morphism. Then* $\Gamma : Min_Z(B) \to Min_Z(A)$ *and* $\Gamma : Min_F(B) \to Min_F(A)$ *are homeomorphisms.*

Proof. We apply Theorems 7.2 and 8.2.

□

Theorem 9.3. *If* $u : A \to B$ *is a quasi* $r*$*-quantale morphism, then the following assertions are equivalent*:

(1) $u : A \to B$ *is a quasi rigid quantale morphism*;

(2) Γ *maps basic open sets of* $Min_F(B)$ *to basic open sets of* $Min_F(A)$.

Proof. (1) $\Rightarrow$ (2) A basic open set of $Min_F(B)$ has the form $V_B(d) \bigcap Min(B)$, where $d \in K(B)$. By the hypothesis that u is a quasi rigid quantale morphism we have $(u(c))^{\perp_B} = d^{\perp_B}$, for some $c \in K(A)$. In order to show that Γ maps basic open sets of $Min_F(B)$ to basic open sets of $Min_F(A)$ it suffices to prove that the equality $\Gamma(V_B(d) \bigcap Min(B)) = V_A(c) \bigcap Min(A)$ holds.

Assume that $p \in \Gamma(V_B(d) \bigcap Min(B))$, hence there exists $q \in Min(B)$ such that $d \leq q$ and $p = \tilde{u}(q)$. According to Proposition 3.10, the following implications hold: $d \leq q \Rightarrow d^{\perp_B} \not\leq q \Rightarrow (u(c))^{\perp_B} \not\leq q \Rightarrow u(c) \leq q \Rightarrow c \leq \tilde{u}(q) \Rightarrow \tilde{u}(q) \in V_A(c) \Rightarrow p = \tilde{u}(q) \in V_A(c) \bigcap Min(A)$. It follows that $\Gamma(V_B(d) \bigcap Min(B)) \subseteq V_A(c) \bigcap Min(A)$.

In order to prove that $V_A(c) \bigcap Min(A) \subseteq \Gamma(V_B(d) \bigcap Min(B))$ we assume that $p \in V_A(c) \bigcap Min(A)$. By using Corollary 4.5, there exists $q \in Min(B)$ such that $p = \tilde{u}(q) = \Gamma(q)$. By using Proposition 3.10, the following implications hold: $p \in V_A(c) \Rightarrow c \leq p = \tilde{u}(q) \Rightarrow u(c) \leq q \Rightarrow (u(c))^{\perp_B} \not\leq q \Rightarrow d^{\perp_B} \not\leq q \Rightarrow d \leq q \Rightarrow q \in V_B(d) \bigcap Min(B)$. It follows that $p \in \Gamma(V_B(d) \bigcap Min(B))$, so we get the desired inclusion.

(2) $\Rightarrow$ (1) Assume that $d \in K(B)$. By using the hypothesis that Γ maps basic open sets of $Min_F(B)$ to basic open sets of $Min_F(A)$, there exists $c \in K(A)$ such that $\Gamma(V_B(d) \bigcap Min(B)) = V_A(c) \bigcap Min(A)$. By Lemma 6.1(1), we have $\Gamma^{-1}(V_A(c) \bigcap Min(A)) = V_B(u(c)) \bigcap Min(B)$.

According to Proposition 8.1, from the hypothesis that u is a quasi $r*$-quantale morphism it follows that Γ is bijective, therefore $V_B(d) \bigcap Min(B) = \Gamma^{-1}(V_A(c) \bigcap Min(A)) = V_B(u(c)) \bigcap Min(B)$.

Thus for any $q \in Min(B)$. the following equivalences hold: $d^{\perp_B} \leq q$ iff $d \not\leq q$ iff $u(c) \not\leq q$ iff $(u(c))^{\perp_B} \leq q$ (we used Proposition 3.10). It follows that $V_B(d^{\perp_B}) \bigcap Min(B) = V_B((u(c))^{\perp_B}) \bigcap Min(B)$. By applying Proposition 2.5, we get

$$d^{\perp_B} = \bigwedge(V_B(d^{\perp_B}) \bigcap Min(B)) = \bigwedge(V_B((u(c))^{\perp_B}) \bigcap Min(B)) = (u(c))^{\perp_B},$$

therefore u is a quasi rigid quantale morphism.

□

Let $f : L \to M$ be a morphism of bounded distributive lattices. Then f is said to be a *rigid lattice morphism* if for any $y \in M$ there exists $x \in R$ such that $Ann_M(f(x)) = Ann_M(y)$. It is clear that f is a rigid lattice morphism if and only if $f^\bullet : Id(L) \to Id(M)$ is a rigid frame morphism.

Theorem 9.4. *The following assertions are equivalent*:

(1) $u : A \to B$ *is a quasi rigid quantale morphism*;

(2) $L(u) : L(A) \to L(B)$ *is a rigid lattice morphism*.

Proof. (1) $\Rightarrow$ (2) Assume that $y \in L(B)$ so $y = \lambda_B(d)$ for some $d \in K(B)$. According to the hypothesis that u is a quasi rigid quantale morphism there exists $c \in K(A)$ such that $(u(c))^{\perp_B} = d^{\perp_B}$. By applying Propositions 4.2 and 3.8, the following equalities hold:

$$Ann_{L(B)}(L(u)(x)) = Ann_{L(B)}(L(u)(\lambda_A(c))) = Ann_{L(B)}(\lambda_B(u(c)))$$

$$= Ann_{L(B)}(((u(c))^*) = ((u(c))^{\perp_B})^* = (d^{\perp_B})^*$$

$$= Ann_{L(B)}(d^*) = Ann_{L(B)}(\lambda_B(d)) = Ann_{L(B)}(y).$$

Thus $L(u)$ is a rigid lattice morphism.

(2) $\Rightarrow$ (1) Let d be a compact element of B, so there exists $c \in K(A)$ such that $Ann_{L(B)}(L(u)(\lambda_A(c)) = Ann_{L(B)}(\lambda_B(d))$ (because $L(u)$ is a rigid lattice morphism). By using Proposition 3.8 and Lemma 3.3(5), we get $((u(c))^{\perp_B})^* = Ann_{L(B)}(\lambda_B(u(c))) = Ann_{L(B)}(\lambda_B(d)) = (d^{\perp_B})^*$.

According to Lemmas 5.3 and 3.3(7), the following equalities hold:

$$(u(c))^{\perp_B} = \rho((u(c))^{\perp_B})) = (((u(c))^{\perp_B})^{*})_{*} = ((d^{\perp_B})^{*})_{*} = \rho((d)^{\perp_B})) = d^{\perp_B}.$$

We conclude that u is a quasi rigid quantale morphism.

□

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■

10. Appendix

Let $R \subseteq S$ be an extension of commutative rings. In ([5], p. 1805) were defined the following types of ring extensions:

- $R \subseteq S$ is a rigid (resp., a quasi rigid) extension if for any $y \in S$ there exists an element $x \in R$ (resp., a finitely generated ideal I of R) such $Ann_S(y) = Ann_S(x)$ (resp., $Ann_S(y) = Ann_S(I^e)$);

- $R \subseteq S$ is an r-extension (resp., a quasi r-extension) if for all $Q \in Min(S)$ and $y \in S - Q$ there exists an element $x \in R$ (resp., a finitely generated ideal I of R) such that $x \notin Q$ and $Ann_S(y) \subseteq Ann_S(x)$ (resp., $I^e \nsubseteq Q$ and $Ann_S(y) \subseteq Ann_S(I^e)$);

- $R \subseteq S$ is an $r*$-extension (resp., a quasi $r*$-extension) if for all $Q \in Min(S)$ and $y \in S - Q$ there exists an element $x \in R$ (resp., a finitely generated ideal I of R) such that $x \in Q$ and $Ann_S(y) \supseteq Ann_S(x)$ (resp., $I^e \subseteq Q$ and $Ann_S(y) \supseteq Ann_S(I^e)$).

Remark 10.1. For any ring extension $R \subseteq S$, the following equivalences hold:

- $R \subseteq S$ is quasi rigid iff for any finitely generated ideal J of S there exists a finitely generated ideal I of R such that $Ann_S(J) = Ann_S(I^e)$;

- $R \subseteq S$ is a quasi r-extension iff for each $Q \in Min(S)$ and for each finitely generated ideal J of S such that $J \nsubseteq Q$ there exists a finitely generated ideal I of R such that $I^e \nsubseteq Q$ and $Ann_S(J) \subseteq Ann_S(I^e)$;

- $R \subseteq S$ is a quasi $r*$-ring extension iff for each $Q \in Min(S)$ and for each finitely generated ideal J of S such that $J \subseteq Q$ there exists a finitely generated ideal I of R such that $I^e \subseteq Q$ and $Ann_S(J) \supseteq Ann_S(I^e)$.

The previous remark allows us to define the corresponding types of ring morphisms, then to formulate the definition of quasi rigid, quasi r- and quasi $r*$-quantale morphisms (see Sections 7-9). We observe that the notions of rigid, r- and $r*$-quantale morphisms cannot be defined in the framework of quantales.