

THE POINT-LINE COLLINEARITY GRAPH OF THE F_{24}' MAXIMAL 2-LOCAL GEOMETRY – THE FIRST THREE DISCS

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Abstract

The disc structure of the point-line collinearity graph for the maximal 2-local geometry associated with the largest simple Fischer group is investigated. For an arbitrary vertex of this graph the first three discs are determined. Additionally a fragment of the fourth disc is uncovered.

2020 Mathematics Subject Classification: 05E18, 20D08.

Keywords and phrases: point-line collinearity graph, maximal 2-local geometry, Fischer group, disc structure.

Submitted by Jinshan Zhang.

Received February 13, 2021

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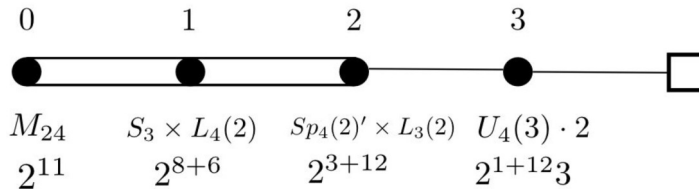
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1. Introduction and Main Results

The investigations of Fischer [5] into groups generated by 3-transpositions not only had an influence upon certain later work related to the classification of the finite simple groups but also unearthed three previously unknown sporadic groups, Fi_{22} , Fi_{23} , and Fi_{24} . The first two of these are simple while Fi_{24} , though not simple, has a simple subgroup Fi'_{24} of index 2. For more on these groups and 3-transposition groups in general, see the book by Aschbacher [1].

Along with many of the other sporadic simple groups, Fi_{22} , Fi_{23} , and Fi'_{24} possess minimal parabolic geometries and maximal 2-local geometries (see [9] and [10]). In the present paper we study the point-line collinearity graph \mathcal{G} of Γ , the maximal 2-local geometry for Fi'_{24} . This geometry has rank 4 and its associated diagram is



Many properties of Γ are itemized in Section 2. We recall that the vertices of \mathcal{G} are Γ_0 , the points of Γ and two points are adjacent in \mathcal{G} if they are incident with a common line. In [11, 12, 13] and [14] complete and detailed descriptions of the corresponding point-line collinearity graphs for Fi_{22} and Fi_{23} are presented.

For $x \in \Gamma_0$ and $i \in \mathbb{N}$, $\Delta_i(x)$ denotes the set of points of Γ_0 distance i from x . Let $G = Fi'_{24}$. Now G acts flag transitively on Γ and so, in studying \mathcal{G} , there is no loss in choosing and fixing a point a of Γ . Here we shall obtain properties of the first three discs of \mathcal{G} around a (that is, of $\Delta_1(a)$, $\Delta_2(a)$ and $\Delta_3(a)$) as well as describing a certain fragment of

$\Delta_4(a)$. In a subsequent paper [16], a complete description of \mathcal{G} is obtained - however the work in [16] is exclusively computer based, whereas this paper does not rely on any machine calculations. It is worth remarking that the notation and conventions used here and in [16] are compatible so as to allow a smooth transition between the two viewpoints. Earlier in [17], the second author obtained results on the structure of the first three discs of \mathcal{G} . The arguments given here will differ to some extent from those in [17] as we may now call upon results in [12, 13] and [14]. Further we are able to give more detail on adjacency within $\Delta_3(a)$.

We now present our main results - for notation we refer the reader to Section 2.

Theorem 1. (i) $\Delta_1(a)$ is a G_a -orbit of size 1518;

(ii) $\Delta_2(a)$ is the union of three G_a -orbits $\Delta_2^i(a)$ ($i = 1, 2, 3$) and $|\Delta_2(a)| = 1, 560, 504$;

(iii) $\Delta_3(a)$ is the union of ten G_a -orbits $\Delta_3^i(a)$ ($i = 1, \dots, 10$) and $|\Delta_3(a)| = 1, 400, 874, 432$; and

(iv) $\Delta_4(a) \cap \{x \in \Gamma_0 \mid \Omega_x \cap \Omega_a \neq \emptyset\}$ is the union of six G_a -orbits $\Delta_4^i(a)$ ($i = 1, \dots, 6$) and consists of 3, 992, 911, 872 points.

Tables 1 and 2 list the sizes of the above mentioned G_α -orbits $\Delta_j^i(a)$.

Table 1

$\Delta_j^i(a)$	Size of $\Delta_j^i(a)$
$\Delta_1(a)$	$2.3.11.23 = 1518$
$\Delta_2^1(a)$	$2^5.3.7.11.23 = 170, 016$
$\Delta_2^2(a)$	$2^8.3.7.11.23 = 1, 360, 128$
$\Delta_2^3(a)$	$2^3.3.5.11.23 = 30, 360$
$\Delta_3^1(a)$	$2^{12}.11.23 = 1, 036, 288$
$\Delta_3^2(a)$	$2^{10}.3^2.5.11.23 = 11, 658, 240$
$\Delta_3^3(a)$	$2^{12}.3.7.11.23 = 21, 762, 048$
$\Delta_3^4(a)$	$2^{12}.3.23 = 282, 624$
$\Delta_3^5(a)$	$2^{15}.3^2.7.11.23 = 522, 289, 152$
$\Delta_3^6(a)$	$2^{12}.3.5.7.11.23 = 108, 810, 240$
$\Delta_3^7(a)$	$2^9.3^2.5.7.11.23 = 40, 803, 840$
$\Delta_3^8(a)$	$2^6.5.7.11.23 = 566, 720$
$\Delta_3^9(a)$	$2^{13}.3^2.5.7.11.23 = 652, 861, 440$
$\Delta_3^{10}(a)$	$2^9.3^2.5.7.11.23 = 40, 803, 840$

Table 2

$\Delta_j^i(a)$	Size of $\Delta_j^i(a)$
$\Delta_4^1(a)$	$2^{16} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 = 746, 127, 360$
$\Delta_4^2(a)$	$2^{15} \cdot 3 \cdot 11 \cdot 23 = 24, 870, 912$
$\Delta_4^3(a)$	$2^{15} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 870, 481, 920$
$\Delta_4^4(a)$	$2^{19} \cdot 3^2 \cdot 7 \cdot 23 = 759, 693, 312$
$\Delta_4^5(a)$	$2^{18} \cdot 3 \cdot 11 \cdot 23 = 198, 967, 296$
$\Delta_4^6(a)$	$2^{18} \cdot 3 \cdot 7 \cdot 11 \cdot 23 = 1, 392, 771, 072$

Theorem 2. *Let $x \in \Delta_1(a)$. Then $G_{ax} \sim 2^{10} \cdot 2^4 : A_8$ (with $G_{ax}^{**} = (G_{ax}^{*x})_{x+a} \sim 2^4 : A_8$, an octad stabilizer) has 4 orbits on $\Gamma_1(x)$ with point distribution as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\{x + a\}$	1	$\{a\}2\Delta_1$
$\alpha_0(x, x + a)$	30	$\Delta_1 2\Delta_2^3$
$\alpha_2(x, x + a)$	448	$\Delta_1 2\Delta_2^2$
$\alpha_4(x, x + a)$	280	$\Delta_1 2\Delta_2^1$

Theorem 3. *Let $x \in \Delta_2^1(a)$. Then $G_{ax} \sim 2^7 \cdot 2^6 : (3 \times S_5)$ (with $G_{ax}^{**} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^6 : (3 \times S_5)$, where $\Lambda_1 = \Omega_a \cap \Omega_x$ is a tetrad and Λ_2 is the unique sextet of Ω_x containing Λ_1). Also $G_{ax} \leq G_{aX}$, where X is the unique hyperplane incident with both a and x . Further, G_{ax} has 6 orbits on $\Gamma_1(x)$ with point distribution as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{4,4^2}(x, \Lambda_1, \Lambda_2)$	5	$\Delta_1 2\Delta_2^1$
$\alpha_{0,4^2}(x, \Lambda_1, \Lambda_2)$	10	$\Delta_2^1 2\Delta_3^8$
$\alpha_{1,31^5}(x, \Lambda_1, \Lambda_2)$	320	$\Delta_2^1 2\Delta_3^6$
$\alpha_{2,2^4}(x, \Lambda_1, \Lambda_2)$	240	$\Delta_2^1 2\Delta_3^2$
$\alpha_{0,2^4}(x, \Lambda_1, \Lambda_2)$	120	$\Delta_2^1 2\Delta_3^7$
$\alpha_{3,31^5}(x, \Lambda_1, \Lambda_2)$	64	$\Delta_2^1 2\Delta_3^1$

Theorem 4. *Let $x \in \Delta_2^2(a)$. Then $G_{ax} \sim 2^5.2^4 : S_6$, $|\{a, x\}^\perp| = 1$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^4 : S_6$, where Λ_1 is the octad of Ω_x corresponding to $x + b$ (where $b = \{a, x\}^\perp$) and $\Lambda_2 = \Omega_2 \cap \Omega_b \cap \Omega_x$, a duad contained in Λ_1 . The number of G_{ax} -orbits on $\Gamma_1(x)$ is 8 with point distribution as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{8,2}(x, \Lambda_1, \Lambda_2) = \{x + b\}$	1	$\Delta_1 2\Delta_2^2$
$\alpha_{2,2}(x, \Lambda_1, \Lambda_2)$	16	$\Delta_2^2 \Delta_3^3 \Delta_3^4$
$\alpha_{4,2}(x, \Lambda_1, \Lambda_2)$	60	$\Delta_2^2 2\Delta_3^2$
$\alpha_{4,1}(x, \Lambda_1, \Lambda_2)$	160	$\Delta_2^2 2\Delta_3^6$
$\alpha_{2,1}(x, \Lambda_1, \Lambda_2)$	192	$\Delta_2^2 2\Delta_3^5$
$\alpha_{4,0}(x, \Lambda_1, \Lambda_2)$	60	$\Delta_2^2 2\Delta_3^{10}$
$\alpha_{2,0}(x, \Lambda_1, \Lambda_2)$	240	$\Delta_2^2 2\Delta_3^9$
$\alpha_{0,0}(x, \Lambda_1, \Lambda_2)$	30	$\Delta_2^2 2\Delta_3^7$

Theorem 5. Let $x \in \Delta_2^3(a)$. Then $G_{ax} \sim 2^9.2^6 : (L_3(2) \times 3)$ and $G_{ax}^{*x} \sim 2^6 : (L_3(2) \times 3)$, the derived subgroup of $\text{Stab}_{G_x^{*x}}\{\Lambda_1\}$, where Λ_1 is a trio of Ω_x . Also $G_{ax} \leq G_{a\pi}$, where π is the unique plane incident with both a and x . The number of G_{ax} -orbits on $\Gamma_1(x)$ is 3 with point distribution as follows:

Orbit	Size	Point distribution
$\alpha_{80^2}(x, \Lambda_1)$	3	$\Delta_1 2\Delta_2^3$
$\alpha_{4^2}(x, \Lambda_1)$	84	$\Delta_2^3 2\Delta_3^8$
$\alpha_{42^2}(x, \Lambda_1)$	672	$\Delta_2^3 2\Delta_3^{10}$

Now we move onto $\Delta_3(a)$ the third disc of a ; we caution that in the following results the point distribution is incomplete.

Theorem 6. Let $x \in \Delta_3^1(a)$. Then $G_{ax} \sim 2^2.L_3(4) : S_3$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1\} \sim L_3(4) : S_3$, where Λ_1 is a triad of Ω_x . The number of G_{ax} -orbits on $\Gamma_1(x)$ is 4, the point distribution of 3 of them are as follows:

Orbit	Size	Point distribution
$\alpha_3(x, \Lambda_1)$	21	$\Delta_2^1 2\Delta_3^1$
$\alpha_2(x, \Lambda_1)$	168	$\Delta_3^1 2\Delta_3^3$
$\alpha_1(x, \Lambda_1)$	360	$\Delta_3^1 2\Delta_4^1$

Theorem 7. Let $x \in \Delta_3^2(a)$. Then $G_{ax} \sim 2^4.2^3 : (L_3(2) \times 2)$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^3 : (L_3(2) \times 2)$, where Λ_1 is an octad and Λ_2 is a duad of Ω_x and $\Lambda_1 \cap \Lambda_2 = \emptyset$. The number of G_{ax} -orbits on $\Gamma_1(x)$ is 11, the point distribution of 6 of them are as follows:

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{0,2}(x, \Lambda_1, \Lambda_2)$	7	$\Delta_2^1 2\Delta_3^2$
$\alpha_{0,1}(x, \Lambda_1, \Lambda_2)$	16	$\Delta_3^2 2\Delta_4^2$
$\alpha_{4,2}(x, \Lambda_1, \Lambda_2)$	14	$\Delta_2^2 2\Delta_3^2$
$\alpha_{2,2}(x, \Lambda_1, \Lambda_2)$	56	$\Delta_3^2 2\Delta_3^3$
$\alpha_{4,1}(x, \Lambda_1, \Lambda_2)$	112	$\Delta_3^2 2\Delta_4^3$
$\alpha_{2,1}(x, \Lambda_1, \Lambda_2)$	224	$\Delta_3^2 2\Delta_4^1$

Theorem 8. *Let $x \in \Delta_3^3(a)$. Then $G_{ax} \sim 2.2^4 : S_6$ and $G_{ax}^{**} = \text{Stab}_{G_x^{**}}\{\Lambda_1, \Lambda_2\} \sim 2^4 : S_6$, where Λ_1 is an octad and Λ_2 is a duad of Ω_x and $\Lambda_2 \subseteq \Lambda_1$. The number of G_{ax} -orbits on $\Gamma_1(x)$ is 8, the point distribution of 5 of them are as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{8,2}(x, \Lambda_1, \Lambda_2) = \{\Lambda_1\}$	1	$\Delta_2^2 \Delta_3^3 \Delta_3^4$
$\alpha_{2,2}(x, \Lambda_1, \Lambda_2)$	16	$\Delta_3^1 2\Delta_3^3$
$\alpha_{4,2}(x, \Lambda_1, \Lambda_2)$	60	$\Delta_3^2 2\Delta_3^3$
$\alpha_{4,1}(x, \Lambda_1, \Lambda_2)$	160	$\Delta_3^3 2\Delta_4^6$
$\alpha_{2,1}(x, \Lambda_1, \Lambda_2)$	192	$\Delta_3^3 2\Delta_4^4$

Theorem 9. *Let $x \in \Delta_3^4(a)$. Then $G_{ax} \sim 2.M_{22} : 2$ and $G_{ax}^{**} = \text{Stab}_{G_x^{**}}\{\Lambda_1\} \sim M_{22} : 2$, where Λ_1 is a duad of Ω_x . The number of G_{ax} -orbits on $\Gamma_1(x)$ is 3, the point distribution of 2 of them are as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_2(x, \Lambda_1)$	77	$\Delta_2^2 \Delta_3^3 \Delta_3^4$
$\alpha_1(x, \Lambda_1)$	352	$\Delta_3^4 2\Delta_4^5$

Theorem 10. *Let $x \in \Delta_3^5(a)$. Then $G_{ax} \cong G_{ax}^{*x} = \text{Stab}_{G_x^{*x}} \{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4 : A_5$, where $\Lambda_1 = O_1$, $\Lambda_2 = \{\infty\}$, and $\Lambda_3 = \{14\}$. The number of G_{ax} -orbits on $\Gamma_1(x)$ is 13, the point distribution of 9 of them are as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{8,1,1}(x, \Lambda_1, \Lambda_2, \Lambda_3) = \{O_1\}$	1	$\Delta_2^2 2\Delta_3^5$
$\alpha_{2,1,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	16	$\Delta_3^5 \Delta_4^4 \Delta_4^5$
$\alpha_{4,1,0}^{(1)}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	40	$\Delta_3^5 2\Delta_4^3$
$\alpha_{4,1,0}^{(2)}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	40	$\Delta_3^5 \Delta_4^5 \Delta_4^6$
$\alpha_{4,0,1}^{(1)}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	40	$\Delta_3^5 2\Delta_4^3$
$\alpha_{4,0,1}^{(2)}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	40	$\Delta_3^5 \Delta_4^5 \Delta_4^6$
$\alpha_{4,1,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	60	$\Delta_3^5 \Delta_4^1 \Delta_4^3$
$\alpha_{2,1,0}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	96	$2\Delta_3^5 \Delta_4^6$
$\alpha_{2,0,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	96	$2\Delta_3^5 \Delta_4^6$

Theorem 11. *Let $x \in \Delta_3^6(a)$. Then $G_{ax} \sim [2^6] : 3.S_4$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}} \{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4 : 3.S_4$, where Λ_1 is an octad of Ω_x , Λ_2 a tetrad contained in Λ_1 and Λ_3 a 1-element subset of Λ_2 . The number of G_{ax} -orbits on $\Gamma_1(x)$ is 16, the point distribution of 7 of them are as follows:*

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\{\Lambda_1\}$	1	$\Delta_2^1 2\Delta_3^6$
$\alpha_{4,4,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	4	$\Delta_2^2 2\Delta_3^6$
$\alpha_{4,1,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	16	$\Delta_3^6 \Delta_4^2 \Delta_4^3$
$\alpha_{2,2,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	48	$\Delta_3^6 2\Delta_4^3$
$\alpha_{4,3,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	48	$\Delta_3^6 2\Delta_4^1$
$\alpha_{2,1,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	64	$\Delta_3^6 \Delta_4^5 \Delta_4^6$
$\alpha_{4,2,1}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	72	$\Delta_3^6 2\Delta_4^1$

Theorem 12. (i) Let $x \in \Delta_3^7(a)$. Then $G_{ax} \sim [2^9].S_4$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim [2^6].S_4$, where $\Lambda_1 = O_1$, $\Lambda_2 = O_2$, and Λ_3 is the partition of O_1 given by $\{\infty, 14\}$, $\{0, 8\}$, $\{3, 20\}$, $\{15, 18\}$. The point distribution of 3 of the G_{ax} -orbits on $\Gamma_1(x)$ are as follows:

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{8,0,2^4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	1	$\Delta_2^1 2\Delta_3^7$
$\alpha_{0,8,0^4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	1	$\Delta_2^2 2\Delta_3^7$
$\alpha_{0,0,0^4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	1	$\Delta_2^2 2\Delta_3^7$

(ii) For $x \in \Delta_3^8(a)$, $G_{ax} \sim [2^{13}].3.3^2 : 4$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^6.3.3^2 : 4$, where $\Lambda_1 = S_0$, $\Lambda_2 = \{\infty, 14, 0, 8, 3, 20, 15, 18, 17, 4, 16, 10\}$, and $\Lambda_3 = \Omega_x \setminus \Lambda_2$. The point distribution of 3 of the G_{ax} -orbits on $\Gamma_1(x)$ are as follows:

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{4^2,8,0}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	3	$\Delta_2^3 2\Delta_3^8$
$\alpha_{4^2,0,8}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	3	$\Delta_2^3 2\Delta_3^8$
$\alpha_{4^2,4,4}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	9	$\Delta_2^1 2\Delta_3^8$

(iii) For $x \in \Delta_3^9(a)$, $G_{ax} \cong G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\} \sim 2.2^4 : S_4$, where $\Lambda_1 = O_1$, $\Lambda_2 = O_2$, $\Lambda_3 = O_3$, and $\Lambda_4 = \{\infty, 14\}$. The point distribution of the G_{ax} -orbit $\alpha_{8,0,0,2}(x, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ is $\Delta_2^2 2\Delta_3^9$.

(iv) For $x \in \Delta_3^{10}(a)$, $G_{ax} \sim [2^9].S_4$ and $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim [2^5].S_4$, where Λ_1 is the tetrad $\{\infty, 0, 3, 15\}$, Λ_2 is the duad $\{14, 8\}$, and Λ_3 is the duad $\{20, 18\}$. The point distributions of 2 of the G_{ax} -orbits on $\Gamma_1(x)$ are as follows:

<i>Orbit</i>	<i>Size</i>	<i>Point distribution</i>
$\alpha_{4,2,2}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	1	$\Delta_2^3 2\Delta_3^{10}$
$\alpha_{4,0,0}(x, \Lambda_1, \Lambda_2, \Lambda_3)$	4	$\Delta_2^2 2\Delta_3^{10}$

Theorem 13. (i) For $x \in \Delta_4^1(a)$, $G_{ax} \sim 2.L_3(2).2$, and $G_{ax}^{*x} \sim L_3(2).2$.

(ii) For $x \in \Delta_4^2(a)$, $G_{ax} \cong G_{ax}^{*x} \cong A_8$.

(iii) For $x \in \Delta_4^3(a)$, $G_{ax} \cong G_{ax}^{*x} \sim 2^6.3^2$.

(iv) For $x \in \Delta_4^4(a)$, $G_{ax} \cong G_{ax}^{*x} \cong L_2(11)$.

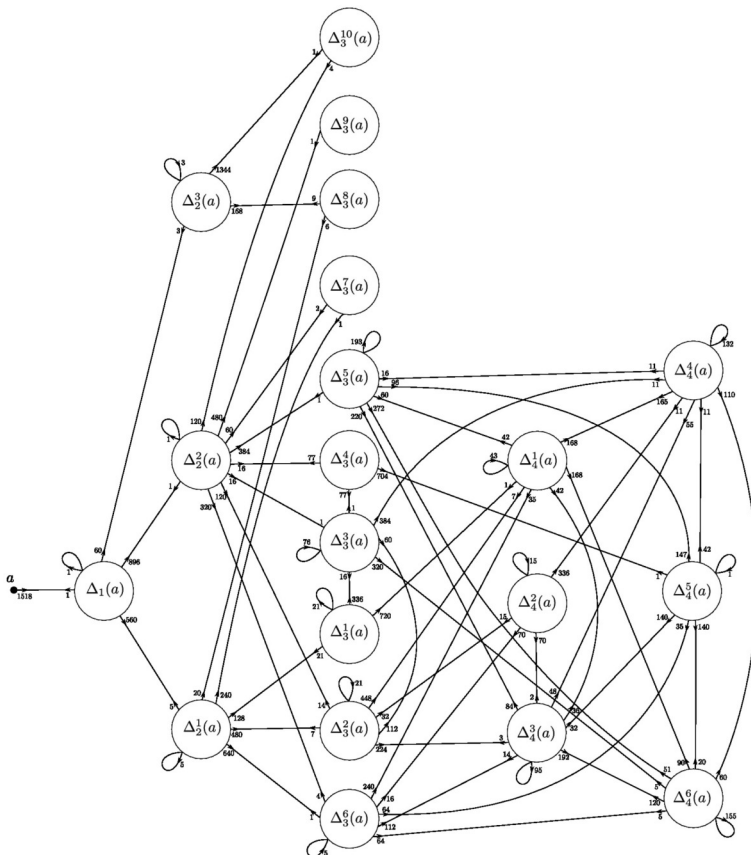
(v) For $x \in \Delta_4^5(a)$, $G_{ax} \cong G_{ax}^{*x} \cong A_7$.

(vi) For $x \in \Delta_4^6(a)$, $G_{ax} \cong G_{ax}^{*x} \sim (3 \times A_5).2$.

Since, for $t \in \Omega_a$, $G_{ax} = G_{ax}^t$ for all $x \in \Delta_4^i(a)$ ($i = 1, \dots, 6$), the point distributions given in Theorems 11-16 of [12] may be directly translated to give the point distributions for G_{ax} -orbits on $\Gamma_1(x)$ of those lines within Γ_0^t .

As mentioned earlier, in Section 2 we explain the notation we shall be using. Additionally, in (2.1), we summarize the properties we shall assume about Γ . In (2.2) we give the definition of the G_a -orbits $\Delta_i^j(x)$ (for $x \in \Gamma_0$). While Section 3 studies various properties of the point and hyperplane residues that we shall need to call upon later. Then Section 4 introduces, for X a hyperplane, the involutions $\tau(X)$ in G . These play an important part in many of our arguments, primarily because of Lemma 4.4. In Lemma 4.7 we quickly describe the first disc $\Delta_1(a)$. The second disc $\Delta_2(a)$ is examined in Section 5 with $\Delta_3(a)$ and part of $\Delta_4(a)$ being investigated in Sections 6 and 7.

We close this section by summarizing the collapsed adjacencies established in Theorems 1-13.



2. Notation and Γ

The maximal 2-local geometry Γ for $G = Fi'_{24}$ has rank 4 and we use Γ_i ($i = 0, 1, 2, 3$) to denote the objects of type i in Γ ; objects of type 0 (respectively, 1, 2, 3) will be referred to as points (respectively, lines, planes, hyperplanes). For $x \in \Gamma$, the residue of x , Γ_x , is defined to be $\{y \in \Gamma \mid x * y\}$, where $*$ is the symmetric incidence relation of Γ . Also, for $x \in \Gamma$, we set

$$Q(x) = \{g \in G_x \mid g \text{ fixes all objects in } \Gamma_x\},$$

and for $H \leq G_x$ we write H^{*x} for $HQ(x)/Q(x)$. If $\Sigma \subseteq \Gamma$ and $i \in \{0, 1, 2, 3\}$, then we set $\Gamma_i(\Sigma) = \{x \in \Gamma_i \mid x * y \text{ for all } y \in \Sigma\}$. The point-line collinearity graph \mathcal{G} of Γ has Γ_0 as its vertex set and for $x, y \in \Gamma_0$, x and y are adjacent in \mathcal{G} if they are collinear, that is if $\Gamma_1(x, y) \neq \emptyset$. For $x, y \in \Gamma_0$, put $\{x, y\}^\perp = \Delta_1(x) \cap \Delta_1(y)$. Also for $x \in \Gamma_0$, we define $Z_1(x) = \{g \in G \mid g \text{ fixes } \{x\} \cup \Delta_1(x) \text{ pointwise}\}$ - note that $Z_1(x) \trianglelefteq G_x$.

We take as our starting point the following properties of Γ .

(2.1) (i) G acts flag transitively on Γ .

(ii) Γ is a string geometry.

(iii) For $\ell \in \Gamma_1$, $|\Gamma_0(\ell)| = 3$ and if $x, y \in \Gamma_0(\ell)$ with $x \neq y$, then $\Gamma_1(x, y) = \{\ell\}$.

(iv) For $x \in \Gamma_0$, $G_x \sim 2^{11} \cdot M_{24}$ with $Q(x) \cong 2^{11}$, the dual of the Golay code module and $G_x^{*x} \cong M_{24}$. Moreover, Γ_x is isomorphic to the M_{24} maximal 2-local geometry.

(v) For $X \in \Gamma_3$, $G_X \sim 2_+^{1+12} \cdot 3.U_4(3).2$ with $Q(X) \sim 2_+^{1+12} \cdot 3$, $Z(G_X) = Z(O_2(Q(X))) \cong 2$ and $G_X^{*X} \sim U_4(3).2$. Also, Γ_X is isomorphic to a geometry for $U_4(3).2$ which is a subgeometry of the unitary geometry for $U_6(2)$.

In (2.1) and elsewhere we follow the ubiquitous ATLAS [2] in describing group structures - it is also a convenient source for information about Fi'_{24} and subgroups of M_{24} and $U_4(3).2$. In the situation of (2.1) we shall frequently denote ℓ by $x + y$ (to indicate we are viewing ℓ in Γ_x) or $y + x$ (to indicate we are viewing ℓ in Γ_y). See Section 3 for further details on the residue geometry in (2.1) (v).

Let $x \in \Gamma_0$ and let ℓ, π, X be, respectively, a line, plane and hyperplane in Γ_x . We remark that ℓ corresponds to an octad, π to a trio and X to a sextet (see [9] and [4]). For a further discussion of Γ_x and Γ_X , see Section 3. Other details of these geometries may be found in [6] and [17].

Before introducing an alternative way of viewing Γ we note, in passing, that $|\Gamma_0| = 2, 503, 413, 946, 215$ and that the permutation rank of G on Γ_0 is 120 [7].

Let \mathcal{T} denote the set of transpositions in Fi_{24} . It is a fact that a maximal set B of pairwise commuting transpositions has $|B| = 24$ and $Stab_G(B) \sim 2^{11} \cdot M_{24}$. Such a set is called a base in [2] and G is transitive on the set of bases. Since Fi'_{24} has only one conjugacy class of subgroups isomorphic to $2^{11} \cdot M_{24}$ we may identify Γ_0 with the set of bases in a way which is compatible with the G -action. For $x \in \Gamma_0$ we use Ω_x to denote the base identified with x . Now Ω_x carries a copy of the Steiner system $S(24, 8, 5)$ preserved by $Stab_G(\Omega_x)$. Indeed an octad of Ω_x corresponds to a line in Γ_x (such an octad is contained in precisely three bases and incidence between points and lines corresponds to containment of bases and octads). Therefore $x, y \in \Gamma_0$ are adjacent in \mathcal{G} if and only if $\Omega_x \cap \Omega_y$ is an octad of both Ω_x and Ω_y .

For $t \in \mathcal{T}$ put $\Gamma_0^t = \{x \in \Gamma_0 \mid t \in \Omega_x\}$. So the points in Γ_0^t correspond to all the bases which contain the fixed transposition t . Also put $G^t = C_G(t)$. Then $G^t \cong Fi_{23}$ and Γ_0^t is the set of points of the Fi_{23} geometry scrutinized in [12], [13], [14] (see especially Section 1). Further, if \mathcal{G}^t denotes the point-line collinearity graph of this Fi_{23} geometry, then we see that for $x, y \in \Gamma_0^t$, x and y are adjacent in \mathcal{G}^t if and only if x and y

are adjacent in \mathcal{G} . This observation gives us access to a rich vein of geometric information from [12, 13, 14]. So, in studying \mathcal{G} , we may view Γ geometrically working within residues or regard Γ_0 as living in the world of transpositions. In our arguments we adopt whichever viewpoint is the most efficacious. We shall also frequently call upon data given in [15] and accordingly will denote result (i.j) in [15] by $\mathbf{O}(i.j)$. We carry along the notational conventions of [4]. So \mathcal{S}_0 and \mathcal{T}_0 denote the standard sextet and standard trio and O_1, O_2, O_3 are the heavy blocks of the MOG. Additionally we adapt the notation in [15] in the following manner. Let $x \in \Gamma_0$. In Γ_x the lines correspond to the octads of the M_{24} maximal 2-local geometry so to indicate we are working in Γ_x we write $\alpha_i(x, \Lambda_1)$ instead of just α_i (see $\mathbf{O}(2.1)$), with a similar convention for the other orbits itemized in [15].

(2.2) Let x be a point in Γ .

(i) $\Delta_2^1(x) = \{y \in \Gamma_0 \mid \text{there exists } b \in \{x, y\}^\perp \text{ such that } b + y \in \alpha_4(b, b + x)\}$.

(ii) $\Delta_2^2(x) = \{y \in \Gamma_0 \mid \text{there exists } b \in \{x, y\}^\perp \text{ such that } b + y \in \alpha_2(b, b + x)\}$.

(iii) $\Delta_2^3(x) = \{y \in \Gamma_0 \mid \text{there exists } b \in \{x, y\}^\perp \text{ such that } b + y \in \alpha_0(b, b + x)\}$.

(iv) $\Delta_3^1(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^1(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{3,31^5}(c, \Omega_x \cap \Omega_c, \mathcal{S}_{cx})\}$.

(v) $\Delta_3^2(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{4,2}(c, c + b, \mathcal{D}_{cx}), \text{ where } \{b\} = \{x, c\}^\perp\}$.

(vi) $\Delta_3^3(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{2,2}(c, c + b, \mathcal{D}_{cx}), \text{ where } \{b\} = \{x, c\}^\perp \text{ and for } t \in \mathcal{D}_{cx}, c \text{ is the unique point in } \Gamma_0^t \cap \Delta_2^2(x) \cap \Delta_1(y)\}$.

(vii) $\Delta_3^4(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{2,2}(c, c + b, \mathcal{D}_{cx}), \text{ where } \{b\} = \{x, c\}^\perp \text{ and for } t \in \mathcal{D}_{cx}, \text{ there are 77 points in } \Gamma_0^t \cap \Delta_2^2(x) \cap \Delta_1(y)\}$.

(viii) $\Delta_3^5(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{2,1}(c, c + b, \mathcal{D}_{cx}), \text{ where } \{b\} = \{x, c\}^\perp\}$.

(ix) $\Delta_3^6(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{4,1}(c, c + b, \mathcal{D}_{cx}), \text{ where } \{b\} = \{x, c\}^\perp\}$.

(x) $\Delta_3^7(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^1(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{0,2^4}(c, \Omega_x \cap \Omega_c, \mathcal{S}_{cx})\}$.

(xi) $\Delta_3^8(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^3(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_4(c, \mathcal{T}_{cx})\}$.

(xii) $\Delta_3^9(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{2,0}(c, c + b, \mathcal{D}_{cx}), \text{ where } \{b\} = \{x, c\}^\perp\}$.

(xiii) $\Delta_3^{10}(x) = \{y \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^3(x) \cap \Delta_1(y) \text{ such that } c + y \in \alpha_{42^2}(c, \mathcal{T}_{cx})\}$.

(xiv) $\Delta_4^1(x) = \{y \in \Gamma_0 \mid \text{there exists } d \in \Delta_3^1(x) \cap \Delta_1(y) \text{ such that } d + y \in \alpha_1(d, \mathcal{T}_{dx})\}$.

(xv) $\Delta_4^2(x) = \{y \in \Gamma_0 \mid \text{there exists } d \in \Delta_3^2(x) \cap \Delta_1(y) \text{ such that } d + y \in \alpha_{0,1}(d, \mathcal{O}_{dx}, \mathcal{D}_{dx})\}$.

(xvi) $\Delta_4^3(x) = \{y \in \Gamma_0 \mid \text{there exists } d \in \Delta_3^2(x) \cap \Delta_1(y) \text{ such that } d + y \in \alpha_{4,1}(d, \mathcal{O}_{dx}, \mathcal{D}_{dx})\}$.

(xvii) $\Delta_4^4(x) = \{y \in \Gamma_0 \mid \text{there exists } d \in \Delta_3^3(x) \cap \Delta_1(y) \text{ such that } d + y \in \alpha_{2,1}(d, d + b, \mathcal{D}_{dx}), \text{ where } \{b\} = \Delta_1(d) \cap \Delta_2^2(x)\}$.

(xviii) $\Delta_4^5(x) = \{y \in \Gamma_0 \mid \text{there exists } d \in \Delta_3^4(x) \cap \Delta_1(y) \text{ such that } d + y \in \alpha_1(d, \mathcal{D}_{dx})\}$.

(xix) $\Delta_4^6(x) = \{y \in \Gamma_0 \mid \text{there exists } d \in \Delta_3^3(x) \cap \Delta_1(y) \text{ such that } d + y \in \alpha_{4,1}(d, d + b, \mathcal{D}_{dx}), \text{ where } \{b\} = \Delta_1(d) \cap \Delta_2^2(x)\}$.

In (2.2) the letters \mathcal{O} , \mathcal{D} , \mathcal{S} , \mathcal{T} (with appropriate subscripts) stand for, respectively, particular octads, duads, sextets and trios of certain bases. Their exact description will emerge later, and will tie in with the data given in [15].

Remark. In fact

$$\Delta_3^8(x) = \bigcup_{X \in \Gamma_3(x)} \Gamma_0(X) \cap \Delta_3(x).$$

See [17] for further details.

Let $x \in \Gamma_0$ and $t \in \Omega_x$. Set $\Delta_1(x)^t = \Delta_1(x) \cap \Gamma_0^t$ and for $i = 1, 2$, let $\Delta_2^i(x)^t = \Delta_2^i(x) \cap \Gamma_0^t$. For $i = 1, \dots, 6$, we set

$$\Delta_3^i(x)^t = \Delta_3^i(x) \cap \Gamma_0^t,$$

and

$$\Delta_4^i(x)^t = \Delta_4^i(x) \cap \Gamma_0^t.$$

Further we put $Q(x)^t = Q(x) \cap G^t$. The above notation is set up so as $\Delta_j^i(x)^t$ corresponds to the $\Delta_j^i(x)$ as given in [12; (2.15)] for the point-line collinearity graph \mathcal{G}^t .

(2.3) Let $x \in \Gamma_0$.

(i) $\Delta_1(x) = \bigcup_{t \in \Omega_x} \Delta_1(x)^t$, $\Delta_2^i(x) = \bigcup_{t \in \Omega_x} \Delta_2^i(x)^t$ ($i = 1, 2$) and $\Delta_j^i(x) = \bigcup_{t \in \Omega_x} \Delta_j^i(x)^t$ ($i = 1, \dots, 6, j = 3, 4$).

(ii) For each $t \in \Omega_x$, $Q(x) = Q(x)^t$.

(iii) $\Delta_1(x)$, $\Delta_2^1(x)$, $\Delta_2^2(x)$, $\Delta_3^i(x)$ and $\Delta_4^i(x)$ ($i = 1, \dots, 6$) are all distinct G_x -orbits.

(iv) If $t \in \Omega_x$ and $y \in \Gamma_0^t$, then $[G_{xy} : G_{xy}^t] \leq 24$.

Proof. Part (i) follows from (2.2) and (ii) holds because $Q(x)$ centralizes all transpositions t in Ω_x . Since G_x acts transitively on the 24 transpositions in Ω_x and, by [12], $\Delta_1(x)^t$, $\Delta_2^1(x)^t$, $\Delta_2^2(x)^t$, $\Delta_3^i(x)^t$, $\Delta_4^i(x)^t$ are all G_x^t -orbits (of different sizes) we infer that (iii) holds. Because $|\Omega_x| = 24$ the G_{xy} -orbit of t can have size at most 24, whence we have (iv). \square

3. The Point and Hyperplane Residues

Recall that we shall employ the same notational conventions as in [15] for the subscripts of α . Suppose that $x \in \Gamma_0$, $\ell \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. By (2.1) we may identify ℓ with an octad of Ω_x and X with a sextet of Ω_x . So, for example, $\alpha_{4^2}(x, X)$ denotes the set of octads (lines) which cut the sextet X in 4^2 , and $\alpha_2(x, \ell)$ is the set of octads

(lines) which intersect the octad ℓ in two elements. Also we define $\beta_0(x, X)$, $\beta_1(x, X)$, $\beta_3(x, X)$ to be the set of sextets of Ω_x (not equal to X) which have, respectively, exactly 0, 1 and 3 octads which are also incident with X . Additionally, we define the following subsets of $\Gamma_3(x)$:

$$\delta_1(x, \ell) = \{Y \in \Gamma_3(x) \mid \ell \in \alpha_{4^2}(x, Y)\},$$

$$\delta_2(x, \ell) = \{Y \in \Gamma_3(x) \mid \ell \in \alpha_{2^4}(x, Y)\},$$

$$\delta_3(x, \ell) = \{Y \in \Gamma_3(x) \mid \ell \in \alpha_{1^5_3}(x, Y)\}.$$

Lemma 3.1. *Let $x \in \Gamma_0$, $\ell \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$.*

(i) *The $G_{x\ell}$ -orbits on $\Gamma_1(x)$ are $\{\ell\}$, $\alpha_0(x, \ell)$, $\alpha_2(x, \ell)$ and $\alpha_4(x, \ell)$, where $|\alpha_0(x, \ell)| = 30$, $|\alpha_2(x, \ell)| = 448$, and $|\alpha_4(x, \ell)| = 280$.*

(ii) *The $G_{x\ell}$ -orbits on $\Gamma_3(x)$ are $\delta_1(x, \ell)$, $\delta_2(x, \ell)$ and $\delta_3(x, \ell)$, where $|\delta_1(x, \ell)| = 35$, $|\delta_2(x, \ell)| = 840$, and $|\delta_3(x, \ell)| = 896$.*

Proof. See [3] or [4].

Lemma 3.2. *Let $x \in \Gamma_0$ and $X \in \Gamma_3(x)$ (so in Γ_x , X may be identified with a sextet in Ω_x). Then the orbits of G_{xX} on $\Gamma_1(x)$ (the octads of Ω_x) are $\alpha_{4^2}(x, X)$, $\alpha_{1^5_3}(x, X)$, and $\alpha_{2^4}(x, X)$. Moreover $|\alpha_{4^2}(x, X)| = 15$, $|\alpha_{1^5_3}(x, X)| = 384$, and $|\alpha_{2^4}(x, X)| = 360$.*

Proof. Since $G_{xX}^* \sim 2^6 : 3.S_6$, the stabilizer of the sextet X , this follows from [3]. □

Lemma 3.3. *For $x \in \Gamma_0$ and $X \in \Gamma_3(x)$, the G_{xX} -orbits on $\Gamma_3(x)$ are $\{X\}$, $\beta_0(x, X)$, $\beta_1(x, X)$, and $\beta_3(x, X)$. Further $|\beta_0(x, X)| = 1440$, $|\beta_1(x, X)| = 240$, and $|\beta_3(x, X)| = 90$.*

Proof. See [3]. □

Lemma 3.4. *Let $x \in \Gamma_0$, and $X, Y \in \Gamma_3(x)$.*

(i) *Suppose $Y \in \beta_3(x, X)$. Of the fifteen octads in Ω_x incident with X , three are in $\alpha_{4^2}(x, Y)$ and twelve are in $\alpha_{2^4}(x, Y)$.*

(ii) *Suppose $Y \in \beta_1(x, X)$. Of the fifteen octads in Ω_x incident with X , one is in $\alpha_{4^2}(x, Y)$, six are in $\alpha_{2^4}(x, Y)$ and eight are in $\alpha_{1^5_3}(x, Y)$.*

(iii) *Suppose $Y \in \beta_0(x, X)$. Of the fifteen octads in Ω_x incident with X , seven are in $\alpha_{2^4}(x, Y)$ and eight are in $\alpha_{1^5_3}(x, Y)$.*

Proof. Since G_x is transitive on $\Gamma_3(x)$ we may suppose X is the standard sextet. Then, in view of Lemma 3.2, for parts (i) (ii) and (iii), respectively we may take

$$Y = \begin{array}{|c|c|c|c|} \hline \times & \times & - & - & * & * \\ \hline \times & \times & - & - & * & * \\ \hline \circ & \circ & + & + & \square & \square \\ \hline \circ & \circ & + & + & \square & \square \\ \hline \end{array}, \quad Y = \begin{array}{|c|c|c|c|} \hline \times & \circ & - & - & - & - \\ \hline \circ & \times & + & + & + & + \\ \hline \circ & \times & * & * & * & * \\ \hline \circ & \times & \square & \square & \square & \square \\ \hline \end{array},$$

and

$$Y = \begin{array}{|c|c|c|c|} \hline \times & \times & \times & - & * & + \\ \hline \times & - & - & - & \circ & \square \\ \hline \circ & + & * & \square & * & \circ \\ \hline \square & * & + & \circ & \square & + \\ \hline \end{array}.$$

It is now straightforward to check the result. □

Lemma 3.5. *Let $x \in \Gamma_0$, $m \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. If $m \notin \Gamma_1(X)$, then there exists $Y \in \beta_3(x, X) \cup \{X\}$ such that $m \in \alpha_{15_3}(x, Y)$.*

Proof. Since $m \notin \Gamma_1(X)$, $m \notin \alpha_{4_2}(x, X)$. Hence, by Lemma 3.2, $m \in \alpha_{15_3}(x, X) \cup \alpha_{2_4}(x, X)$. If $m \in \alpha_{15_3}(x, X)$, then we let $Y = X$. So now we assume that $m \in \alpha_{2_4}(x, X)$. Let t_1 and t_2 be tetrads of X such that $|t_1 \cap m| = 2$ and $|t_2 \cap m| = 0$. Now choose a tetrad t_3 such that $|t_3 \cap t_1| = |t_3 \cap t_2| = 2$ and $|t_3 \cap m| = 1$. Letting Y be the unique sextet containing t_3 , we have $Y \in \beta_3(x, X)$ and $m \in \alpha_{15_3}(x, Y)$, so proving the lemma. \square

The balance of this section considers the hyperplane residue of Γ . Set $H = U_4(3).2 \cong G_X^{*X}$ where $X \in \Gamma_3$. We consider H as a subgroup of $U_6(2)$, and let V denote the 6-dimensional $GF(4)$ unitary module. Now there are 693 isotropic 1-subspaces of V (see [2]) and H has two orbits on these 1-spaces, say \mathcal{P} and \mathcal{Q} with $|\mathcal{P}| = 567$ and $|\mathcal{Q}| = 126$. Of the 6237 isotropic 2-subspaces of V , 2835 of them have three 1-subspaces in \mathcal{P} and two 1-subspaces in \mathcal{Q} -denote this set by \mathcal{L} . Among the 891 isotropic 3-subspaces, 567 contain exactly one 1-subspace in \mathcal{Q} ; call this set \mathcal{R} . We define a geometry $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ where $\Lambda_0 = \mathcal{P}$, $\Lambda_1 = \mathcal{L}$ and $\Lambda_2 = \mathcal{R}$, where incidence is symmetrized inclusion. This geometry is an example of a GAB (see [6]) and we have

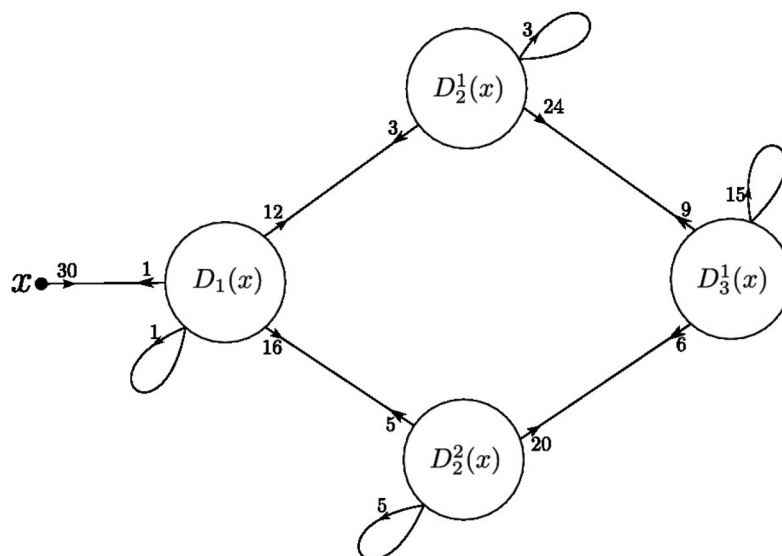
Lemma 3.6. *For $X \in \Gamma_3$, Γ_X is isomorphic to Λ .*

Our next result lists some properties of Λ we shall require later on.

Lemma 3.7. *Let $x \in \Lambda_0$.*

(i) *The H_x -orbits on Λ_0 are $D_1(x)$, $D_2^1(x)$, $D_2^2(x)$ and $D_3^1(x)$, where $|D_1(x)| = 30$, $|D_2^1(x)| = 120$, $|D_2^2(x)| = 96$, and $|D_3^1(x)| = 320$.*

(ii) *The point-line collinearity graph of Λ is as follows*



(iii) *We have $H_x \sim 2^4.S_6$ with $O_2(H) \cong 2^4$.*

(iv) *If $g \in O_2(H_x)$, $g \neq 1$, then g interchanges $\Lambda_0(\ell) / \{x\}$ for 8 lines ℓ incident with x and fixes $\Lambda_0(\ell)$ for the other 7 lines incident with x .*

Proof. See either [6], [8] or Section 3 of [17].

4. Involutions

In this section, we explore the combinatorial relationship between G and the residue geometries as it relates to the action of G on Γ .

Lemma 4.1. *Let $x \in \Gamma_0$ and $X \in \Gamma_3(x)$. Then*

(i) *$Q(x) \cap Q(X) \cong 2^7$ and $Q(x)^{*X} (\cong 2^4) \leq G_{xX}^{*X} \sim 2^4.S_6$; and*

(ii) *$Z_1(x) = 1$.*

Proof. First we note that $Q(x) \not\leq Q(X)$. For $Q(x) \leq Q(X)$ gives, by (2.1)(v), $Q(x) \leq O_2(Q(X)) \cong 2_+^{1+12}$. Since $Q(x)$ is elementary abelian of order 2^{11} , this is impossible. So $1 \neq Q(x)^{*X} \trianglelefteq G_{xX}^{*X} \cong 2^4 S_6$, using Lemma 3.7(iii). Since the 2^4 is an irreducible S_6 -module we must have $Q(x)^{*X} \cong 2^4$. Hence $Q(x) \cap Q(X) \cong 2^7$ and part (i) holds.

Since $Q(x)$ is an irreducible G_x -module and $Z_1(x) \trianglelefteq G_x$, either $Z_1(x) = 1$ or $Z_1(x) = Q(x)$. If $Z_1(x) = Q(x)$, then $Z_1(x)^{*X} = O_2(G_{xX}^{*X})$ by part (i). However, from Lemma 3.7(iv), every non-trivial element of $O_2(G_{xX}^{*X})$ moves some point in $\Gamma_X \cap \Delta_1(x)$ whereas $Z_1(x)$ fixes all points in $\Delta_1(x)$ by definition, a contradiction. Thus $Z_1(x) = 1$. \square

For $X \in \Gamma_3$, we use $\tau(X)$ to denote the involution in $Z(G_X)$; recall that $|Z(G_X)| = 2$ by (2.1)(v). Now let $x \in \Gamma_0(X)$. In Γ_x we may identify X with a sextet (of Ω_x) whose tetrads are T_1, \dots, T_6 , and we have, for each $i \in \{1, \dots, 6\}$,

$$\tau(X) = \prod_{t \in T_i} t.$$

(We note that $\tau(X)$ is a tetra-transposition in the language of [2; p207].) Also observe, as $C_G(\tau(X)) = G_X$, for $X, Y \in \Gamma_3$, $\tau(X) = \tau(Y)$ if and only if $X = Y$.

Let $x \in \Gamma_0$. In Ω_x consider a duad (that is, a 2-element subset), say $D = \{t_1, t_2\}$. Then $\delta(D) = t_1 t_2$ is referred to as a bi-transposition in [2]. Every involution in G is conjugate in G to either $\tau(X)$ or $\delta(D)$.

Lemma 4.2. *Let $x \in \Gamma_0$, $X \in \Gamma_3(x)$ and D be a duad of Ω_x . Then*

- (i) $\tau(X), \delta(D) \in Q(x)$;
- (ii) $C_G(\tau(X)) \sim 2_+^{1+12}.3 \cdot U_4(3).2$, $C_G(\delta(D)) \sim 2 \cdot F_{22} : 2$; and
- (iii) $Q(x) \setminus \{1\} = \tau(X)^{G_x} \cup \delta(D)^{G_x}$ with $|\tau(X)^{G_x}| = 1771$ and $|\delta(D)^{G_x}| = 276$.

Proof. The definitions of $\tau(X)$, $\delta(D)$ and (2.1)(iv), (v) give part (i). For part (ii), see [2]. Part (iii) follows from the definition of $\tau(X)$, $\delta(D)$ and properties of the Golay co-code. \square

Our next lemma concerns sextet lines whose definition we recall. For $x \in \Gamma_0$, let $X_1, X_2, X_3 \in \Gamma_3(x)$, if for all $i, j, 1 \leq i < j \leq 3$, we have $X_i \in \beta_3(x, X_j)$, then $\{X_1, X_2, X_3\}$ is called a sextet line of Ω_x .

Lemma 4.3. *Suppose that $x \in \Gamma_0$ and $\{X_1, X_2, X_3\}$ is a sextet line of Ω_x . Then $\tau(X_1)\tau(X_2) = \tau(X_3)$.*

Proof. Since, for $X \in \Gamma_3$,

$$\tau(X) = \prod_{t \in T} t,$$

for any tetrad T of X , the lemma follows immediately. \square

Lemma 4.4. *Let $x \in \Gamma_0$, $\ell \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. Then $\tau(X)$ interchanges the points in $\Gamma_0(\ell) \setminus \{x\}$ if and only if $\ell \in \alpha_{15_3}(x, X)$.*

Proof. Since G_x is transitive on $\Gamma_3(x)$ we may in Γ_x , without loss of generality, suppose X is the standard sextet. Now let Y be the sextet

×	○	-	-	-	-
○	×	+	+	+	+
○	×	*	*	*	*
○	×	□	□	□	□

By Lemma 3.4(ii), of the 15 octads incident with Y , one is in $\alpha_{4^2}(x, X)$, eight are in $\alpha_{1^5_3}(x, X)$ and six are in $\alpha_{2^4}(x, X)$. Since $\tau(X) \in Z(G_{xX})$, if $\tau(X)$ fixes $\Gamma_0(\ell)$ (point-wise) for some $\ell \in \alpha_{4^2}(x, X)$ (respectively, $\alpha_{1^5_3}(x, X)$, $\alpha_{2^4}(x, X)$), then, by Lemma 3.2 $\tau(X)$ fixes $\Gamma_0(\ell)$ (point-wise) for all $\ell \in \alpha_{4^2}(x, X)$ (respectively, $\alpha_{1^5_3}(x, X)$, $\alpha_{2^4}(x, X)$). Because G_x is transitive on $\Gamma_3(x)$ and, by Lemma 4.1(ii), $Z_1(x) = 1$, $\tau(X)^{*Y} \neq 1$. So, by Lemmas 4.1(i) and 4.2(i), $1 \neq \tau(X)^{*Y} \in Q(x)^{*Y} = O_2(G_{xY}^*)$. Then $\tau(X)^{*Y}$ (and $\tau(X)$) fixes $\Gamma_0(\ell)$ (point-wise) for exactly 7 of the lines $\ell \in \Gamma_1(x, Y)$ by Lemma 3.7(iv). Therefore $\tau(X)$ interchanges the points in $\Gamma_0(\ell) \setminus \{x\}$ only when $\ell \in \alpha_{1^5_3}(x, X)$. \square

Lemma 4.5. *Let $x \in \Gamma_0$ and $X, Y \in \Gamma_3(x)$ with $X \neq Y$. Then $Y \in \beta_3(x, X)$ if and only if $\tau(Y) \in Q(X)$.*

Proof. If $Y \in \beta_0(x, X) \cup \beta_1(x, X)$, then there exists $\ell \in \alpha_{1^5_3}(x, Y)$ by consulting the MOG in [4], and so, by Lemma 4.4, $\tau(Y)$ does not fix $\Gamma_0(\ell)$ point-wise. Therefore $\tau(Y) \notin Q(X)$. While if $Y \in \beta_3(x, X)$, then $\Gamma_1(x, X) \subseteq \alpha_{4^2}(x, Y) \cup \alpha_{2^4}(x, Y)$ and hence $\tau(Y)$ fixes $\Gamma_0(\ell)$ point-wise for all $\ell \in \Gamma_1(x, X)$ by Lemma 4.4. Since, by Lemmas 4.1(i) and 4.2(i), $\tau(Y)^{*X} \in Q(x)^{*X} = O_2(G_{xX}^*)$, Lemma 3.7(iv) implies $\tau(Y)^{*X} = 1$. So $\tau(Y) \in Q(X)$ as desired. \square

Lemma 4.6. *Let x, y, z be distinct points of Γ_0 such that $\{x, y, z\}$ is a triangle in \mathcal{G} . Then $z \in \Gamma_0(x + y)$ (or, in other words, $\{x, y, z\} = \Gamma_0(\ell)$ for some $\ell \in \Gamma_1$).*

Proof. We have that $\Omega_x \cap \Omega_y$ and $\Omega_z \cap \Omega_y$ are octads in Ω_y . Let $t \in \Omega_x \cap \Omega_z$. Then t centralizes the transpositions in $\Omega_x \cap \Omega_y$ and $\Omega_z \cap \Omega_y$ and so either $\Omega_x \cap \Omega_y = \Omega_z \cap \Omega_y$ or $t \in \Omega_y$. In either case we get $\Omega_x \cap \Omega_y = \Omega_y \cap \Omega_z = \Omega_x \cap \Omega_z$. \square

Lemma 4.7. (i) $|\Delta_1(a)| = 1518 = 2 \cdot 3 \cdot 11 \cdot 23$;

(ii) $\Delta_1(a)$ is a G_a -orbit; and

(iii) if $x \in \Delta_1(a)$, then $G_{ax} \sim 2^{10} \cdot 2^4 : A_8$ (with $G_{ax}^{*x} = G_{xx+a}^{*x}$, an octad stabilizer).

Proof. (i) Since $|\Gamma_0(\ell) \setminus \{a\}| = 2$ for any $\ell \in \Gamma_1(x)$, $|\Delta_1(a)| = 2|\Gamma_1(a)| = 1518$.

(ii) For $\ell \in \Gamma_1(a)$ we can find $X \in \Gamma_3(a)$ such that $\ell \in \alpha_{1^5_3}(a, X)$. Hence by Lemma 4.4, $Q(a)$ is transitive on $\Gamma_0(\ell) \setminus \{a\}$. Since G_a is transitive on $\Gamma_1(a)$, (ii) holds.

(iii) We have $G_{ax} \leq G_{xx+a}$ because $x + a$ is the unique line in $\Gamma_1(a, x)$ and $[G_{xx+a} : G_{ax}] \leq 2$ as $|\Gamma_0(x + a) \setminus \{x\}| = 2$. Hence as $Q(a)$ is transitive on $|\Gamma_0(x + a) \setminus \{x\}|$ we obtain (iii). \square

Combining Lemma 4.7 and **O(2.1)** with the definitions of $\Delta_2^1(a)$, $\Delta_2^2(a)$, and $\Delta_2^3(a)$ given in (2.2) we obtain Theorem 2.

Lemma 4.8. *Let $y \in \Delta_1(x)$ where $x \in \Gamma_0$. Then*

(i) $|Q(x) \cap Q(y)| = 2^6$; and

(ii) for $X \in \Gamma_3(x)$, $\tau(X) \in Q(y)$, if and only if $X \in \Gamma_3(y)$.

Proof. Since $O_2(G_{xy}^{*y})$ is an irreducible 4-dimensional A_8 -module over $GF(2)$, $Q(x)_y^{*y} = 1$ or $O_2(G_{xy}^{*y})$. Suppose $Q(x)_y^{*y} = 1$ and so $Q(x)_y = Q(x) \cap Q(y)$. Let $X \in \Gamma_3(x)$ with $x + y \in \alpha_{2^4}(x, X)$. Then $\tau(X) \in Q(x)_y \subseteq Q(y)$. Therefore,

$$|Q(y)| \geq 1771 + 840 = 2611$$

by Lemma 3.1(ii). This contradicts $|Q(y)| = 2^{11}$ from (2.1)(iv). So $|Q(x)_y^{*y}| = 2^4$ and then part (i) follows from Lemma 4.7(iii). For part (ii), if $X \in \Gamma_3(y)$ then $\tau(X) \in Q(y)$ by Lemma 4.2(i). Suppose that $X \notin \Gamma_3(y)$ and $\tau(X) \in Q(y)$. Since $x + y \notin \Gamma_1(X)$, we then have $x + y \in \alpha_{2^4}(x, X) \cup \alpha_{1^5_3}(x, X)$. Suppose that $x + y \in \alpha_{2^4}(x, X)$. Since G_{xy}^{*x} is transitive on the set of hyperplanes $\delta_2(x, x + y) = \{Y \in \Gamma_3(x) \mid x + y \in \alpha_{2^4}(x, Y)\}$ by Lemma 3.1(ii) and $\tau(X) \in Q(y)$ we have $\tau(Y) \in Q(y)$ for all $Y \in \delta_2(x, x + y)$. Then

$$|Q(x) \cap Q(y)| \geq 35 + 840 = 875.$$

This contradicts part (i). By a similar argument, if $x + y \in \alpha_{1^5_3}(x, X)$ we get

$$|Q(x) \cap Q(y)| \geq 35 + 896 = 933,$$

again giving a contradiction. This proves part (ii). \square

5. The Second Disc, $\Delta_2(a)$

We begin by defining certain subsets of $\Delta_2(a)$ as follows.

$$\tilde{\Delta}_2^1(a) = \{x \in \Delta_2(a) \mid \Gamma_3(a, x) \neq \emptyset = \Gamma_2(a, x)\},$$

$$\tilde{\Delta}_2^2(a) = \{x \in \Delta_2(a) \mid \Gamma_3(a, x) = \emptyset\},$$

$$\tilde{\Delta}_2^3(a) = \{x \in \Delta_2(a) \mid \Gamma_3(a, x) \neq \emptyset \neq \Gamma_2(a, x)\}.$$

An immediate consequence of these definitions is

Lemma 5.1. *For $1 \leq j < k \leq 3$, $\tilde{\Delta}_2^j(a) \cap \tilde{\Delta}_2^k(a) = \emptyset$ and $\bigcup_{i=1}^3 \tilde{\Delta}_2^i(a) = \Delta_2(a)$.*

Lemma 5.2. *Suppose $x \in \Delta_2(a)$ with $X \in \Gamma_3(a, x)$. Then $\{a, x\}^\perp \subseteq \Gamma_0(X)$.*

Proof. Let $b \in (a, x)^\perp$ and assume that $b \notin \Gamma_0(X)$. Then $a + b \notin \Gamma_1(X)$ as Γ is a string geometry. Using Lemma 3.5, we can find $Y \in \beta_3(a, X) \cup \{X\}$ for which $a + b \in \alpha_{1,5_3}(a, Y)$. By Lemma 4.5, $\tau(Y) \in Q(x)$ which implies that $\tau(Y) \in Q(a)_x$. Since $\tau(Y)$ does not fix b by Lemma 4.4 we get a triangle $\{x, b, b^{\tau(Y)}\}$ which then forces $a = x$ by Lemma 4.6. From this contradiction we infer that $b \in \Gamma_0(X)$, so proving the lemma. \square

Lemma 5.3. *For $i = 1, 2, 3$, $\tilde{\Delta}_2^i(a) = \Delta_2^i(a)$.*

Proof. Let $b \in \{a, x\}^\perp$. Using MOG information in Ω_b , Lemma 5.2 implies that $\tilde{\Delta}_2^i(a) = \Delta_2^i(a)$ for $i = 1, 2, 3$. \square

Lemma 5.4. *Let $x \in \Delta_2^1(a)$. Then there is a unique hyperplane in $\Gamma_3(a, x)$.*

Proof. Let $X, Y \in \Gamma_3(a, x)$ and $b \in \{a, x\}^\perp$. Then $b \in \Gamma_0(X) \cap \Gamma_0(Y)$ by Lemma 5.2. If $X \neq Y$, then $b + x, b + a \in \Gamma_0(X) \cap \Gamma_0(Y)$ and $\Gamma_2(b + x, b + a) \neq \emptyset$ by considering MOG information in Γ_b . Hence $x \notin \tilde{\Delta}_2^1(x)$, whereas $\tilde{\Delta}_2^1(x) = \Delta_2^1(x)$ by Lemma 5.3. Thus we conclude $X = Y$ and the lemma is proved. \square

Let the unique hyperplane in Lemma 5.4 be denoted by $X(a, x)$ (respectively, $X(x, a)$) if we regard $X(a, x) \in \Gamma_3(a)$, (respectively $X(x, a) \in \Gamma_3(x)$). Of course $X(a, x) = X(x, a)$.

Lemma 5.5. *Let $x \in \Delta_2^1(a)$. Then $|\{a, x\}^\perp| = 5$ and, for each $b \in \{a, x\}^\perp$, the octad $a + b$ in Ω_a contains a fixed tetrad of the sextet $X(a, x)$.*

Proof. By Lemma 5.2, for every $b \in \{a, x\}^\perp$, $b \in \Gamma_0(X(a, x))$ and so $a + b \in \Gamma_1(X(a, x))$. Working in the residue geometry of $X(a, x)$ and using Lemma 3.7(ii) we get $|\{a, x\}^\perp| = 5$. Since $\Gamma_2(a, x) = \emptyset$ by Lemma 5.3, in Ω_a , the five octads $\{a + b \mid b \in \{a, x\}^\perp\}$ must intersect in the same tetrad of the sextet $X(a, x)$. \square

Note that $x \in \Delta_2^1(a)$ implies $a \in \Delta_2^1(x)$. We denote the fixed tetrad in Ω_a (respectively, Ω_x) described in Lemma 5.5 by $t(a, x)$ (respectively, $t(x, a)$).

Lemma 5.6. (i) $|\Delta_2^1(a)| = 2^5 \cdot 3 \cdot 7 \cdot 11 \cdot 23$.

(ii) $\Delta_2^1(a)$ is a G_a -orbit.

(iii) For $x \in \Delta_2^1(a)$ and $G_{ax}^{*x} \sim 2^6 : (3 \times S_5)$ is the stabilizer in G_x^{*x} of $X(x, a)$ and $t(x, a)$ and $|Q(x)_a| = 2^7$.

Proof. By Lemma 3.7(i), for any $X \in \Gamma_3(a)$, $|\Gamma_0(X) \cap \Delta_2^1(a)| = 96$ and so by Lemma 5.4 we get $|\Delta_2^1(a)| = 96 \cdot |\Gamma_3(a)| = 2^5 \cdot 3 \cdot 7 \cdot 11 \cdot 23$, proving part (i).

For part (ii), let $b \in \Delta_1(a)$ and $x \in \Delta_2^1(a) \cap \Delta_1(b)$. Then in Ω_b , $b + a \in \alpha_4(b, b + x)$. Since $\alpha_4(b, b + x)$ is a G_{ab}^{*b} -orbit it is enough to show that there exists $g \in G_{ab}$ with $x^g = x'$, where $\Gamma_0(b + x) = \{b, x, x'\}$. In Ω_b , we can choose a sextet Y incident with the octad $b + a$ such that $a + x \in \alpha_{153}(x, Y)$. Then by Lemma 4.4, $\tau(Y) \in (Q(a) \cap Q(b)) \setminus G_x$ and so $\tau(Y)$ is the required element of G_{ab} .

For $t \in \Omega_a \cap \Omega_b \cap \Omega_x$, a, x are vertices of \mathcal{G}^t with $x \in \Delta_2^1(a)^t$. Hence $Q(x)_a = Q(x)_a^t \cong 2^7$ by Theorem 3 of [12]. Since, by parts (i) and (ii), $|G_{ax}| = 2^{16} \cdot 3^2 \cdot 5$, Lemmas 5.4 and 5.5 yield part (iii). □

We now turn to $\Delta_2^2(a)$.

Lemma 5.7. Let $x \in \Delta_2^2(a)$ and $b \in \{a, x\}^\perp$. Then

(i) $|\Delta_1(b) \cap \Delta_2^2(a)| = 2^7 \cdot 7$ with G_{ab} transitive on $\Delta_1(b) \cap \Delta_2^2(a)$; and

(ii) $|\{a, x\}^\perp| = 1$ or 2 .

Proof. Since $\Gamma_3(a, x) = \emptyset$ by Lemma 5.3, we have $b + a \in \alpha_2(b, b + x)$. So by Lemma 3.1(i), $|\Delta_1(b) \cap \Delta_2^2(a)| = 2 \times 448 = 2^7 \cdot 7$. Let $x' \in \Gamma_0(b + x) \setminus \{b, x\}$. We can choose $Y \in \Gamma_3(b + a)$ with $b + x \in \alpha_{15_3}(b, Y)$. By Lemma 4.4, $x^{\tau(Y)} = x'$ and so G_{ab} is transitive on $\Delta_1(b) \cap \Delta_2^2(a)$ because $\alpha_2(b, b + x)$ is a G_{ab} -orbit by Lemma 3.1(i).

Using (i), [2] and the fact that $G_{ab} \sim [2^{14}].A_8$ by Lemma 4.7(iii) we must have $G_{abx} \sim [2^9].S_6$ or $[2^{10}].A_6$. In either case G_{abx}^{*a} is contained in the stabilizer in Ω_a of a duad δ contained in the octad $a + b$. We now show that for every $c \in \{a, x\}^\perp$, the octad $a + c$ in Ω_a contains δ . Assume, for a contradiction that for some $c \in \{a, x\}^\perp$, $a + c$ does not contain δ . Since $\Gamma_3(a, x) = \emptyset$, we must have $a + c \in \alpha_2(a, a + b)$. Using MOG information, there are exactly 15 sextets in $\Gamma_3(a, b)$ that each have a tetrad containing δ . Let T denote this set of 15 sextets.

We can take $Y_1, Y_2, Y_3 \in T$ forming a sextet line. Since $\tau(Y_1)\tau(Y_2) = \tau(Y_3)$ by Lemma 4.3 we must have $\tau(Y_i) \in G_x$ for each $i = 1, 2, 3$. Because G_{abx} is transitive on T it follows that $\tau(Y) \in G_x$ for each $Y \in T$. Since $a + c$ does not contain δ we must have $a + c \in \alpha_{15_3}(Y)$ for some $Y \in T$ and then $x^{\tau(Y)} \neq x$. Lemma 4.5 now implies that $a = x$, a contradiction. Part (ii) follows because we cannot find three octads in Ω_a , intersecting pairwise in exactly δ . \square

Lemma 5.8. *Let $x \in \Delta_2^2(a)$. Then*

- (i) $\Delta_2^2(a)$ is a G_a -orbit;
- (ii) $|\{a, x\}^\perp| = 1$, $|\Delta_2^2(a)| = 2^8.3.7.11.23$ and G_{ab} is transitive on $\Delta_1(b) \cap \Delta_2^2(a)$, where $\{a, x\}^\perp = \{b\}$; and
- (iii) $G_{ax}^{*x} \sim 2^4 : S_6$ is the stabilizer in Ω_x of the octad $x + b$ and the duad $\Omega_a \cap \Omega_b \cap \Omega_x$, where $\{a, x\}^\perp = \{b\}$.

Proof. Part (i) follows from Lemma 5.7(i) and the fact that $\Delta_1(a)$ is a G_a -orbit.

Suppose that $|\{a, x\}^\perp| \neq 1$. Then $\{a, x\}^\perp = \{b, c\}$ with $b \neq c$ by Lemma 5.7(ii). Lemma 4.5 rules out $d(b, c) = 1$. If $c \in \Delta_2^1(b) \cup \Delta_2^3(b)$ ($= \tilde{\Delta}_2^1(b) \cup \tilde{\Delta}_2^3(b)$), then $b, c \in \Gamma_0(X)$ for some $X \in \Gamma_3$ whence, by Lemma 5.2, $a, x \in \Gamma_0(X)$. However $\Gamma_3(a, x) = \emptyset$, and therefore $x \in \Delta_2^2(b)$. Hence $a + c \in \alpha_2(a, a + b)$. From Theorem 4 of [12] $Q(a)_x \cong 2^5$ and so, as $G_{abx} \sim [2^9].S_6$ or $[2^{10}].A_6$, $G_{abx}^{*a} \sim [2^4].S_6$ or $[2^5].A_6$. In particular $2^8 \mid |G_{abx}^{*a}|$. Clearly $G_{abx} = G_{abxc}$ and so $G_{abxc}^{*a} = G_{abx}^{*a}$. Since $a + c \in \alpha_2(a, a + b)$, G_{abxc}^{*a} leaves a dodecad of Ω_a invariant whence G_{abxc}^{*a} is isomorphic to a subgroup of M_{12} . But $2^8 \mid |G_{abxc}^{*a}|$ yields a contradiction. Thus, we conclude that $|\{a, x\}^\perp| = 1$, and consequently for $b \in \{a, x\}^\perp$

$$|\Delta_2^2(a)| = \frac{|\Delta_1(b) \cap \Delta_2^2(a)| |\Delta_1(a)|}{|\{a, x\}^\perp|} = 2^8.3.7.11.13.$$

Part (iii), using $Q(x)_a \cong 2^5$, follows readily. \square

Lemma 5.9. *Let $x \in \Delta_2^3(a)$. Then there is a unique element $\Lambda(a, x) \in \Gamma_2(a, x)$ and for every $b \in \{a, x\}^\perp$, $b \in \Gamma_0(\Lambda(a, x))$.*

Proof. By definition, $\Gamma_2(a, x) \neq \emptyset$. Let $b \in \{a, x\}^\perp$ with $b + a \in \alpha_0$ ($b, b + x$) and let $\Lambda(a, x)$ be the unique element of $\Gamma_2(b + a, b + x)$. Suppose $b' \in \{a, x\}^\perp$ with $b' \notin \Gamma_0(\Lambda(a, x))$. In Ω_b there are seven sextets $X_i (i = 1, \dots, 7)$ in $\Gamma_3(b + a, b + x)$ and by Lemma 5.2 $b' \in \Gamma_0(X_i)$ for each $i = 1, \dots, 7$. Therefore, in $\Omega_{b'}$ there exists a trio $\Lambda \in \Gamma_2(b' + a, b' + x, X_i)$ for each $i = 1, \dots, 7$. Considering the situation in Ω_a we must have $\Lambda = \Lambda(a, x)$ and the lemma is proved. \square

We follow our earlier notational convention and also denote the unique plane in Lemma 5.9 by $\Lambda(x, a)$ if we are viewing $\Lambda(x, a)$ as a trio in Γ_x .

Lemma 5.10. *Let $x \in \Delta_2^3(a)$. Then $|\Gamma_3(a, x)| = 7$ and $|\{a, x\}^\perp| = 3$.*

Proof. By Lemma 5.2, for $X \in \Gamma_3$, $X \in \Gamma_3(a, x)$, if and only if $X \in \Gamma_3(\Lambda(a, x))$. The result now follows from Lemma 5.9 because in Γ_X there are three points collinear with a and x and in Γ_a , $|\Gamma_3(\Lambda(a, x))| = 7$. \square

Lemma 5.11. *Let $x \in \Lambda_2^3(a)$. Then*

(i) $|\Delta_2^3(a)| = 2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 23$;

(ii) $\Delta_2^3(a)$ is a G_a -orbit; and

(iii) $G_{ax}^{*x} \sim 2^6 : (L_3(2) \times 3)$ is a subgroup of index 2 of the stabilizer in Ω_x of the trio $\Lambda(x, a)$ and $|Q(x)_a| = 2^9$.

Proof. Since $|\{a, x\}^\perp| = 3$ by Lemma 5.10, $|\alpha_0(b, b+a)| = 30$ ($b \in \{a, x\}^\perp$) and, by Lemma 4.7(i), $|\Delta_1(a)| = 2.3.11.23$, we calculate that $|\Delta_2^3(a)| = 2^3.3.5.11.23$.

For part (ii), let $b \in \Delta_1(a)$ with $\Lambda \in \Gamma_2(a, b)$ and $X \in \Gamma_3(\Lambda)$. Then $G_{aX\Lambda}^* \sim 2^4 \cdot (S_4 \times 2)$ and is transitive on the four points in $\Delta_2^3(a) \cap \Delta_1(b) \cap \Gamma_0(\Lambda)$. Then G_a is transitive on $\Delta_2^3(a)$ because $\Gamma_2(a)$ and $\Delta_1(a)$ are G_a -orbits.

By Lemma 5.10 $\{a, x\}^\perp = \{b_1, b_2, b_3\}$. Also, using Lemma 5.9, $G_{ax}^* \leq G_{ax\Lambda(a, x)}^* \sim 2^6 : (L_3(2) \times S_3)$. Let $1 \leq i < j \leq 3$. Then $a + b_i$ and $a + b_j$ are disjoint octads as they are both incident with the trio $\Lambda(a, x)$. Choose a tetrad δ of Ω_a which intersects $a + b_i$ in two elements and $a + b_j$ in one element, and let Y denote the sextet of Ω_a with δ a tetrad of Y . Then $a + b_i \in \alpha_{2^4}(a, Y)$ and $a + b_j \in \alpha_{1^5_3}(a, Y)$. Hence, by Lemma 4.4, $\tau(Y) \in Q(a)_{b_i} \setminus Q(a)_{b_j}$. Thus $Q(a)_{b_i} \neq Q(a)_{b_j}$ for $1 \leq i < j \leq 3$. Further $Q(a)_x \leq Q(a)_{b_i}$ ($1 \leq i \leq 3$), for $Q(a)_x \not\leq Q(a)_{b_i}$ yields that $|\{a, x\}^\perp \cap \Gamma_0(a + b_i)| = 2$ whereas no two points of $\{a, x\}^\perp$ are collinear. So, as $[Q(a) : Q(a)_{b_i}] = 2$ and $Q(a)_{b_i} \neq Q(a)_{b_j}$ for $i \neq j$, we have $[Q(a) : Q(a)_x] \geq 2^2$. Consequently using part (i) either $G_{ax}^* \sim 2^6 : (L_3(2) \times 3)$ with $|Q(a)_x| = 2^9$ or $G_{ax}^* \sim 2^6 : (L_3(2) \times S_3)$ with $|Q(a)_x| = 2^8$. Suppose the latter holds. Let ξ be the element of order 3 in the S_3 direct factor of G_{ax}^* . Then, as ξ permutes the three octads $\{a + b_i | i = 1, 2, 3\}$ and $Q(a)_{b_i} \neq Q(a)_{b_j}$ ($i \neq j$), ξ must act non-trivially on $Q(a) / Q(a)_x$. But then λ centralizes $Q(a) / Q(a)_x$, where λ is an element of G_{ax}^* of order 7, a contradiction as $|C_{Q(a)}(\lambda)| = 2^2$. Thus, as $a \in \Delta_2^3(x)$, we obtain $G_{ax}^* \sim 2^6 : (L_3(2) \times 3)$ and $|Q(x)_a| = 2^9$, so proving (iii). \square

Lemma 5.6 combined with (2.2) proves Theorem 3 except for the octad orbits $\alpha_{2,2^4}(x, \Lambda_1, \Lambda_2)$, $\alpha_{1,31^5}(x, \Lambda_1, \Lambda_2)$, and $\alpha_{0,4^2}(x, \Lambda_1, \Lambda_2)$. The first two will be settled by Theorems 7 and 11 and the data in **O**(2.2), while the last one follows from Theorem 12(ii). Theorem 4, apart from the octad orbits $\alpha_{0,0}(x, \Lambda_1, \Lambda_2)$ and $\alpha_{4,0}(x, \Lambda_1, \Lambda_2)$, follows from Lemma 5.8 and (2.2). The remaining two orbits are dealt with by Theorem 12(i), (iv) and **O**(2.3). Finally Lemma 5.11 and (2.2) deliver Theorem 5.

6. Theorems 6-11 and 13

Lemma 6.1. *Suppose that $x \in \Gamma_0$ and that $\Omega_a \cap \Omega_x \neq \emptyset$. Let $t \in \Omega_a$ and let Δ denote the G_a -orbit of x . Set $k = |\{s \in \Omega_a \mid x \in \Gamma_0^s\}|$. Then*

$$k|\Delta| = 24|\Delta \cap \Gamma_0^t|.$$

Proof. Since Δ is a G_a -orbit and G_a acts transitively on Ω_a , $|\Delta \cap \Gamma_0^s|$ is the same for all $s \in \Omega_a$. Furthermore, we also have that $|\{s \in \Omega_a \mid y \in \Gamma_0^s\}|$ is the same for all $y \in \Delta$. Because $\Omega_a \cap \Omega_x \neq \emptyset$ we note that $k \neq 0$. Now counting in two ways the number of elements in

$$|\{(s, y) \in \Omega_a \times \Delta \mid y \in \Gamma_0^s\}|$$

yields, as $|\Omega_a| = 24$, the lemma. \square

For $x \in \Gamma_0$ and $s \in \Omega_x$, G_x^s denotes the stabilizer of x in $G^s \cong Fi_{23}$. So $G_x^s \sim 2^{11}M_{23}$. Also recall that $Q(x)^s$ denotes the normal elementary abelian subgroup of G_x^s of order 2^{11} .

Lemma 6.2. *For $x \in \Gamma_0$ and $s \in \Omega_x$, $Q(x)^s = Q(x)$.*

Proof. Since $2^{11}M_{23} \sim G_x^s \leq G_x \sim 2^{11}M_{24}$, the subgroup structure of M_{24} forces $Q(x)^s = Q(x)$. \square

Lemma 6.3. (i) If $x \in \Delta_2^1(a)$, then $|\{s \in \Omega_a | x \in \Gamma_0^s\}| = 4$.

(ii) If $x \in \Delta_2^2(a)$, then $|\{s \in \Omega_a | x \in \Gamma_0^s\}| = 2$.

Proof. Let $x \in \Delta_2^1(a)$ and set $k = |\{s \in \Omega_a | x \in \Gamma_0^s\}|$. Observe that, for $t \in \Omega_a$, $\Delta_2^1(a) \cap \Gamma_0^t = \Delta_2^1(a)^t$. Since G_a is transitive on $\Delta_2^1(a)$, Lemmas 5.6(ii) and 6.1 imply that

$$k|\Delta_2^1(a)| = 24|\Delta_2^1(a)^t|,$$

where t is some fixed transposition in Ω_a . From Lemma 5.6(i) and Table 1 of [12], $|\Delta_2^1(a)| = 2^5 \cdot 3 \cdot 7 \cdot 11 \cdot 23$ and $|\Delta_2^1(a)^t| = 2^4 \cdot 7 \cdot 11 \cdot 23$, and therefore $k = 4$.

A similar argument, using Lemma 5.8 instead of Lemma 5.6, establishes part (ii). \square

Lemma 6.4. For $i = 1, \dots, 6$, $\Delta_3^i(a)$ is a G_a -orbit and, for $t \in \Omega_a$, $\Delta_3^i(a) \cap \Gamma_0^t = \Delta_3^i(a)^t$.

Proof. Let $x \in \Delta_3^1(a)$ and $t \in \{s \in \Omega_a | x \in \Gamma_0^s\} = \Omega_a \cap \Omega_x$. From Lemma 5.6 and Theorem 3 of [12], $|G_{ax}| = 2^{16} \cdot 3 \cdot 5$ and $|G_{ax}^t| = 2^{14} \cdot 3 \cdot 5$. So $[G_{ax} : G_{ax}^t] = 4$ and hence, by Lemma 6.3(i), G_{ax} is transitive on $\{s \in \Omega_a | x \in \Gamma_0^s\}$. Because G_{ax}^t is transitive on $\Delta_3^1(a)^t = \Delta_3^1(a) \cap \Gamma_0^t$, we conclude that G_a is transitive on $\Delta_3^1(a)$.

The remaining sets $\Delta_3^i(a)$ ($i = 2, \dots, 6$) are defined from $\Delta_2^2(a)$. Now similar arguments may be employed for these sets as $[G_{ax} : G_{ax}^t] = 2$ for $x \in \Delta_2^2(a)$ (where $t \in \{s \in \Omega_a | x \in \Gamma_0^s\}$) and, by Lemma 6.3(ii) $|\{s \in \Omega_a | x \in \Gamma_0^s\}| = 2$. \square

Theorem 6.5. *Let $x \in \Delta_3^i(a)$.*

(i) *If $i = 1$, then $G_{ax} \sim 2^2.L_3(4) : S_3$, $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1\} \sim L_3(4) : S_3$,*

where Λ_1 is a triad of Ω_x and $|\Delta_3^1(a)| = 2^{12}.11.23$.

(ii) *If $i = 2$, then $G_{ax} \sim 2^4.2^3 : (L_3(2) \times 2)$, $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^3 : (L_3(2) \times 2)$, where Λ_1 is an octad, Λ_2 is a duad of Ω_x with $\Lambda_1 \cap \Lambda_2 = \emptyset$, and $|\Delta_3^2(a)| = 2^{10}.3^2.5.11.23$.*

(iii) *If $i = 3$, then $G_{ax} \sim 2.2^4 : S_6$, $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2\} \sim 2^4 : S_6$, where Λ_1 is an octad, Λ_2 is a duad of Ω_x with $\Lambda_2 \subseteq \Lambda_1$, and $|\Delta_3^3(a)| = 2^{12}.3.7.11.23$.*

(iv) *If $i = 4$, then $G_{ax} \sim 2.M_{22} : 2$, $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1\} \cong M_{22} : 2$, where Λ_1 is a duad of Ω_x and $|\Delta_3^4(a)| = 2^{12}.3.23$.*

(v) *If $i = 5$, then $2^4 : A_5 \sim G_{ax} \cong G_{ax}^{*x} \leq \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\}$, where Λ_1 is an octad of Ω_x , $|\Lambda_2| = |\Lambda_3| = 1$ with $\Lambda_2 \cup \Lambda_3 \subseteq \Lambda_1$, and $|\Delta_3^5(a)| = 2^{15}.3^2.7.11.23$.*

(vi) *If $i = 6$, then $G_{ax} \sim [2^6].3.S_4$, $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1, \Lambda_2, \Lambda_3\} \sim 2^4.3.S_4$, where Λ_1 is an octad of Ω_x , $|\Lambda_2| = 4$, $|\Lambda_3| = 1$, $\Lambda_3 \subseteq \Lambda_2 \subseteq \Lambda_1$, and $|\Delta_3^6(a)| = 2^{12}.3.5.7.11.23$.*

Proof. (i) Let $t \in \Omega_a$. From Lemma 6.4, $\Delta_3^1(a)$ is a G_a -orbit and $\Delta_3^1(a) \cap \Gamma_0^t = \Delta_3^1(a)^t$. For $x \in \Delta_3^1(a)$, let $k = |\{s \in \Omega_a \mid x \in \Gamma_0^s\}|$. Using Lemma 6.1, we obtain

$$k|\Delta_3^1(a)| = 24|\Delta_3^1(a)^t|.$$

By the definition of $\Delta_3^1(a)$, there exists $y \in \Delta_2^1(a)$ such that $y + x \in \alpha_{3,315}(x, \Lambda_1, \Lambda_2)$. Now consulting Theorem 3, we see that $\Lambda_1 = \Omega_a \cap \Omega_y$, and hence $|\Omega_a \cap \Omega_x| \geq 3$. So $k \geq 3$. Therefore, as $|\Delta_3^1(a)^t| = 2^9 \cdot 11 \cdot 23$ by Table 1 of [12],

$$\begin{aligned} |\Delta_3^1(a)| &= \frac{24|\Delta_3^1(a)^t|}{k} = \frac{24 \cdot 2^9 \cdot 11 \cdot 23}{k} \\ &\leq \frac{24 \cdot 2^9 \cdot 11 \cdot 23}{3} = 2^{12} \cdot 11 \cdot 2. \end{aligned}$$

Supposing that $x \in \Gamma_0^t$. Then $G_{ax} \sim 2^2.L_3(4) : 2$ by Theorem 5 of [12]. Since $\Delta_3^1(a)$ is a G_a -orbit, $|\Delta_3^1(a)|$ must divide $[G_a : G_{ax}^t] = 2^{12} \cdot 3 \cdot 11 \cdot 23$. Bearing in mind the possible overgroups of $L_3(4) : 2$ in $M_{24} \cong G_x^{*x}$ and Lemma 6.2, we get that $[G_{ax} : G_{ax}^t] = 3$. Thus $|\Delta_3^1(a)| = 2^{12} \cdot 11 \cdot 23$ with $k = 3$ and $G_{ax} \sim 2^2.L_3(4) : S_3$ with $G_{ax}^{*x} = \text{Stab}_{G_x^{*x}}\{\Lambda_1\}$, Λ_1 being the triad $\{t\} \cup D(x, a)$. (With $D(a, x)$ as in Theorem 5 of [12].) This establishes (i).

Parts (ii)-(vi) may be proved in a similar fashion. For these cases we may extract $k = |\{s \in \Omega_a \mid x \in \Gamma_0^s\}|$ (for $x \in \Delta_3^i(x)$, $i = 2, \dots, 6$) from [12]. Recall that in the Fi_{23} geometry, a hyperplane is just a transposition with points of this geometry being sets of 23 pairwise commuting transpositions. For $x \in \Delta_3^i(a)^t$, $t \in \Omega_a$, where $i \in \{2, 3, 4\}$, a and x are incident with a unique hyperplane of the Fi_{23} geometry (see Section 1 of [12]) - so for $i \in \{2, 3, 4\}$, $k = 2$. Whereas, for $x \in \Delta_3^i(a)^t$, $i \in \{5, 6\}$, a and x are not incident with a common hyperplane of the Fi_{23} geometry. Thus $k = 1$ for $i \in \{5, 6\}$. So knowing k we can make effective use of Lemma 6.1. We observe that for $x \in \Delta_3^i(a) \cap \Gamma_0^t$ ($t \in \Omega_a$) we have

$G_{ax} = G_{ax}^t$ for $i = 3, 5, 6$. While $[G_{ax} : G_{ax}^t] = 2$ for $i = 2, 3, 4$. In these latter cases, we must also call on the services of Lemma 6.2 in order to deduce that G_{ax} has shape, respectively, $2^4.2^3 : (L_3(2) \times 2)$, $2^2.2^4 : S_6$ and $2.M_{22} : 2$.

We are now in a position to verify Theorems 6-13. For Theorem 6, Theorem 6.5(i) gives G_{ax} and G_{ax}^{*x} for $x \in \Delta_3^1(a)$. We must discover the point distribution of the G_{ax} line orbits $\alpha_i(x, \Lambda_1)$ ($i = 1, 2, 3$), three of the G_{ax}^{*x} -orbits on lines, see [15]. Let $y \in \Delta_1(x)$ be such that $x + y \in \alpha_1(x, \Lambda_1)$. Now we may further assume y is chosen so as $x, y \in \Gamma_0^t$ for some $t \in \Omega_a$. Then, by Theorem 5 of [12], $x + y \in \alpha_0(x, D(x, a))$ (seen within Γ_0^t) with $x + y$ having point distribution $\Delta_3^{1t} 2\Delta_4^{1t}$. Since $\alpha_1(x, \Lambda_1)$ is a G_{ax} -orbit and $\Delta_4^1(a)^t \subseteq \Delta_4^1(a)$, we conclude that lines in $\alpha_1(x, \Lambda_1)$ have point distribution $\Delta_3^1 2\Delta_4^1$. Similarly, we see that $\alpha_2(x, \Lambda_1)$ has point distribution $\Delta_3^1 2\Delta_3^3$ and $\alpha_3(x, \Lambda_1)$ has point distribution $\Delta_2^1 2\Delta_3^1$.

The same kind of arguments work for $\Delta_3^2(a)$, $\Delta_3^3(a)$, $\Delta_3^4(a)$, $\Delta_3^5(a)$ and $\Delta_3^6(a)$, so we omit the details. \square

The same strategy as employed in this section will reveal G_{ax} and orbit sizes for $x \in \Delta_4^i(a)$, $i = 1, \dots, 6$. Note that in all these cases $k = |\{s \in \Omega_a \mid x \in \Gamma_0^s\}| = 1$ as a and x cannot be incident with a common hyperplane in the Fi_{23} geometry, as the point-line collinearity graph of the Fi_{22} geometry has diameter 3 (see Appendix 1 of [11]).

7. Proof of Theorem 12

The orbits considered in Theorem 12 do not lie within a Fi_{23} residue and so we cannot apply the same reasoning as in Section 6. Recall that for any $X \in \Gamma_3$, Γ_X is isomorphic to the geometry for $U_4(3).2$ described in [6].

We define

$$\widetilde{\Delta}_3^8(a) = \{x \in \Gamma_0 \mid \Gamma_3(a, x) \neq \emptyset \text{ and } d(a, x) = 3\}.$$

Lemma 7.1. $\Delta_3^8(a) = \widetilde{\Delta}_3^8(a)$.

Proof. If $x \in \widetilde{\Delta}_3^8(a)$ and $X \in \Gamma(a, x)$, using information about the geometry Γ_X given in Lemma 3.7(ii), there exists $c \in \Delta_2(a) \cap \Delta_1(a)$ with $c + x \in \alpha_{42}(c, T)$, where $T \in \Gamma_2(a, c)$. By (2.2) $c \in \Delta_2^3(a)$ and $x \in \Delta_3^8(a)$. Conversely if $x \in \Delta_3^8(a)$ we must have $\Gamma_3(a, x) \neq \emptyset$ by **O**(2.4) and $d(a, x) = 3$ by Lemma 5.2. So $x \in \widetilde{\Delta}_3^8(a)$ as required. \square

Lemma 7.2. If $x \in \Delta_3^8(a)$, then $|\Gamma_3(a, x)| = 1$.

Proof. Let $x \in \Delta_3^8(a)$ and assume that $X, Y \in \Gamma_3(a, x)$ with $X \neq Y$. Using information about the $U_4(3).2$ geometry described in Lemma 3.7(ii), for every $\ell \in \Gamma_1(a, X)$, there exists $b \in \Gamma_0(\ell)$ with $b \in \Delta_2(x) \cap \Delta_1(a)$. If $Y \notin \beta_3(a, X)$, then there is some $b \in \Delta_2(x) \cap \Delta_1(a)$ with $a + b \in \alpha_{15_3}(a, Y)$ by Lemma 3.4. Therefore Lemma 4.4 implies that $\tau(Y)$ does not fix b . Since $\tau(Y) \in Q(a)_x$, $b^{\tau(Y)} \in \Gamma_0(a + b) \cap \Delta_2(x)$. However, as $\Gamma_0(a + b) \subseteq \Gamma_0(X)$, Lemma 3.7(ii) implies that $a \in \Delta_1(x)$, a contradiction. Hence $Y \in \beta_3(a, X)$. In Γ_a , there are three octads ℓ incident with X and Y and for one of these, we can find $y \in \Gamma_0(\ell) \cap \Delta_2^1(a)$. Since $X, Y \in \Gamma_3(a, y)$ we now have a contradiction to Lemma 5.4, and so $X = Y$ as asserted. \square

Lemma 7.3. *Let $c_1 \in \Delta_2^2(a)$ and $c_2 \in \Delta_2(a) \cap \Delta_1(c_1)$. Then*

(i) $c_2 \in \Delta_2^2(a)$; and

(ii) if $y \in \Gamma_0(c_1 + c_2) \setminus \{c_1, c_2\}$, then $y \in \Delta_1(a)$.

Proof. (i) Suppose that $c_2 \in \Delta_2^1(a) \cup \Delta_2^3(a)$, and argue for a contradiction. Then, by definition of $\Delta_2^1(a)$ and $\Delta_2^3(a)$, there exists $X \in \Gamma_3(a, c_2)$. Since $c_1 \in \Delta_2^2(a)$, $|\{a, c_1\}^\perp| = 1$. Let $\{a, c_1\}^\perp = \{b\}$. If $b \in \Gamma_0(X)$, then, using Lemma 5.2, $c_1 \in \{b, c_2\}^\perp \subseteq \Gamma_0(X)$ and so $X \in \Gamma_3(a, c_2)$, whereas $\Gamma_3(a, c_2) = \emptyset$. Thus $b \notin \Gamma_0(X)$ and as a consequence $a + b \notin \Gamma_1(X)$. Hence $a + b \in \alpha_{2^4}(a, X) \cup \alpha_{1^5_3}(a, X)$. Assume that $a + b \in \alpha_{2^4}(a, X)$. Then $\tau(X) \in Q(a)_b$ by Lemma 4.4. Since $X \notin \Gamma_3(b)$, $\tau(X) \notin Q(b)$ by Lemma 4.8(ii). So $\tau(X) \in Q(a)_b \setminus Q(b)$ and hence $1 \neq \tau(X)^{*b} \in Q(a)^{*b} = O_2(G_{ba}^{*b})$. Since $b + c_1 \in \alpha_2(b, b + a)$ we then infer that $\tau(X)^{*b}$ does not leave the octad $b + c_1$ invariant. Hence $\tau(X) \notin G_{c_1}$. However $\tau(X) \in Q(c_2)$ and so we obtain a triangle $\{b, c_1, c_1^{\tau(X)}\}$ with $c_1^{\tau(X)} \in \Gamma_0(c_1 + c_2)$. Lemma 4.6 forces $b = c_2$, a contradiction. Thus we have shown that $a + b \notin \alpha_{2^4}(a, X)$ and so $a + b \in \alpha_{1^5_3}(a, X)$. By Lemma 4.4, $b^{\tau(X)} \neq b$. If $c_1^{\tau(X)} = c_1$, then $\{b, b^{\tau(X)}, c_1\}$ is a triangle, whence $a = c_1$ by Lemma 4.6. Thus $c_1^{\tau(X)} \neq c_1$. Since $c_1^{\tau(X)} \in \Gamma_0(c_1 + c_2)$, this gives $\{b, c_1^{\tau(X)}\} \subseteq \{b^{\tau(X)}, c_1\}^\perp$ which, as $b^{\tau(X)} \in \Delta_2^2(c_1)$, contradicts Lemma 5.8(ii) (note that $b = c_1^{\tau(X)}$ would give $c_2 \in \Gamma_0(b + c_1)$ and then $c_2 \in \Delta_2^2(a)$). With this contradiction we have established part (i).

(ii) Let $\{a, c_i\}^\perp = \{b_i\}$ for $i = 1, 2$. Suppose (ii) is false and argue for a contradiction. We first claim that $d(b_1, c_2) = 2 = d(b_2, c_1)$. If, say, $d(b_1, c_2) = 1$, then $\{b_1, c_1, c_2\}$ is a triangle and so, as $c_1, c_2 \in \Delta_2(a)$, Lemma 4.6 yields that $y = b_1 \in \Delta_1(a)$. Thus $d(b_1, c_2) = 1$ and, similarly, $d(b_2, c_1) = 2$. In particular, this gives $b_1 \neq b_2$. Further, $d(b_1, b_2) = 2$. For $d(b_1, b_2) = 1$ implies $b_2 \in \Gamma_0(a + b_1)$ by Lemma 4.6 and then $\{b_1, c_2\} \subseteq \{b_2, c_1\}^\perp$. This contradicts Lemma 5.8(ii) as $b_2 \in \Delta_2^2(c_1)$.

If $b_1 \in \Delta_2^1(b_2) \cup \Delta_2^3(b_2)$, then by part (i) (with b_1 in place of a) $c_1 \notin \Delta_2^2(b_2)$. Therefore $c_1 \in \Delta_2^1(b_2) \cup \Delta_2^3(b_2)$. Consequently $a \in \Delta_2^2(c_1)$ and $b_2 \in \Delta_2^1(c_1) \cup \Delta_2^3(c_1)$ which is contrary to part (i) (with c_1 in place of a). Thus $b_1 \notin \Delta_2^1(b_2) \cup \Delta_2^3(b_2)$ and hence $b_1 \in \Delta_2^2(b_2)$. Similar arguments show that $c_1 \in \Delta_2^2(b_2)$ and $c_2 \in \Delta_2^2(b_1)$. By considering the elements of $\Gamma_3(b_1, c_2)$ as sextets in Ω_{b_1} and using Lemma 4.4 there exists $Y \in \Gamma_3(b_1, c_1)$ with $\tau(Y) \in G_a$. Suppose that $\tau(Y) \notin G_{c_2}$. Since $\tau(Y)$ fixes the line $c_1 + c_2$, Lemma 4.6 implies that $b_2^{\tau(Y)} \neq b_2$ and $\tau(Y) \notin Q(a)$. Therefore $1 \neq \tau(Y)^{*a} \in O_2(G_{ab_1}^{*a})$. This means that, in Ω_a , the octads $a + b_2^{\tau(Y)}$, $a + b_1$ and $a + b_2$ intersect pairwise in the same duad. However, we see from the MOG [4] that this is impossible. Thus, we have shown that $\tau(Y) \in G_{c_2}$. Since $b_1 \in \Delta_2^2(c_2)$, $Y \notin \Gamma_3(c_2)$ and so $\tau(Y) \notin Q(c_2)$ by Lemma 4.8(ii). Then $1 \neq \tau(Y)^{*c_2} \in O_2(G_{c_2c_1}^{*c_2})$. Since $c_2 + b_2 \in \alpha_2(c_2, c_2 + c_1)$, $\tau(Y)^{*c_2}$ does not fix $c_2 + b_2$. This contradicts Lemma 5.8(ii) and hence gives part (ii), completing the proof of Lemma 7.3.

□

Lemma 7.4. (i) $\Delta_3^8(a)$ is a G_a -orbit and $|\Delta_3^8(a)| = 2^6.5.7.11.23$.

(ii) For $x \in \Delta_3^8(a)$, $G_{ax} \sim [2^{13}].3.3^2 : 4$ and $G_{ax}^{*x} \sim 2^6 : 3.3^2 : 4$ is the stabilizer in G_x^{*x} of the sextet $X \in \Gamma_3(a, x)$ and the partition of Ω_x into $\Sigma = \{\infty, 14, 0, 8, 3, 20, 15, 18, 17, 4, 16, 10\}$ and its complement (where X is identified with the standard sextet in Ω_x).

(iii) $|\Delta_2^1(a) \cap \Delta_1(x)| = 6$ and $|\Delta_2^3(a) \cap \Delta_1(x)| = 9$.

(iv) Let $x \in \Delta_3^8(a)$ and $\{X\} = \Gamma_3(a, x)$. If $\{a, b, c, x\}$ is a path of length 3 in \mathcal{G} , then $b, c \in \Gamma_0(X)$. Moreover $\Delta_2^2(a) \cap \Delta_1(x) = \emptyset$.

Proof. Let $x \in \Delta_3^8(a)$. By Lemma 7.2, $\Gamma_3(a, x) = \{X\}$. Observe that $\Gamma_0(X) \cap \Delta_3^8(a) = D_3^1(a)$ by Lemmas 7.1, and 3.7(ii). Since G_a is transitive on $\Gamma_3(a)$ and, by Lemma 3.7(i), $D_3^1(a)$ is a G_{ax} -orbit, we see that $\Delta_3^8(a)$ is a G_a -orbit. Also, as $|D_3^1(a)| = 320$ by Lemma 3.7(i),

$$\begin{aligned} |\Delta_3^8(a)| &= |\Gamma_3(a)| |\Gamma_0(X) \cap \Delta_3^8(a)| \\ &= 7.11.23.320 = 2^6.5.7.11.23. \end{aligned}$$

So (i) holds.

Clearly we have $G_{ax} \leq G_{axX}$ and so $G_{ax}^{*x} \leq G_{axX}^{*x} \sim 2^6 3S_6$. Also, by part (i), $|G_{ax}| = 2^{15}.3^3$. We now look at $Q(a)_x$. Using Lemma 4.5, as $a, x \in \Gamma_0(X)$, gives $\langle \tau(Y) | Y \in \beta_3(a, X) \rangle \leq Q(a)_x$. Hence, by Lemma 3.3, $|Q(a)_x| \geq 2^7$. Now select $y \in \Delta_2^1(a) \cap \Gamma_0(X) (= D_2^2(a))$ with $y \in \Delta_1(x)$. Suppose $Q(a)_x \not\leq Q(a)_y$, and let $g \in Q(a)_x \setminus Q(a)_y$. Then $y^g \neq y$ and $y^g \in \Delta_1(x) \cap \Delta_2^1(a) \cap \Gamma_0(X)$. Let $b \in \{a, y\}^\perp$ (and note that $y \in \Gamma_0(X)$). Since $g \in Q(a)$, $b^g \in \Gamma_0(a + b)$. If $b \neq b^g$, then Lemma 3.7(ii) forces

$a \in \Delta_1(x)$ whereas $d(a, x) = 3$. Thus $b = b^g$ and consequently $\{a, y\}^\perp = \{a, y^g\}^\perp$. Looking in $\Gamma_0(X)$ we see this is impossible. Hence we infer that $Q(a)_x \leq Q(a)_y$. By Theorem 3, $|Q(a)_y| = 2^7$ and therefore $|Q(a)_x| = 2^7$. Since $a \in \Delta_3^8(x)$, we also get $|Q(x)_a| = 2^7$, and so $|G_{ax}^{*x}| = 2^8 \cdot 3^3$. Since G_{ax}^{*x} contains a Sylow 3-subgroup of G_{axX}^{*x} and the only subgroup of S_6 of order $3^2 2^\alpha$ are subgroups of $3^2 : 4$ we see that $G_{ax}^{*x} \sim 2^6 : 3 \cdot 3^2 : 4$, which completes the proof of (ii).

Consulting Lemma 3.7(ii) we see $|\Delta_2^1(a) \cap \Delta_1(x) \cap \Gamma_0(X)| = 6$ and $|\Delta_2^3(a) \cap \Delta_1(x) \cap \Gamma_0(X)| = 9$. If $|\Delta_2^3(a) \cap \Delta_1(x)| > 9$, then for $y \in \Delta_2^3(a)$ the lines in $\alpha_{4^2 2}(y, \mathcal{T}(a, y))$ must be incident with at least one point in $\Delta_3^8(a)$. Let $k = |\Delta_1(x) \cap \Delta_2^3(a)|$. Using part (i), Lemma 5.11 and **O**(2.4) we calculate that $k = 36 + 9$ or $72 + 9$. Now, by **O**(2.11), there are no line orbits (apart from $\alpha_{4^2, 8}(x, X)$ and $\alpha_{4^2, 4^2}(x, X)$) of size ≤ 72 . Thus we conclude that $|\Delta_2^3(a) \cap \Delta_1(x)| = 9$. A similar argument, using $|\Delta_3^8(a)|$, $|\Delta_2^1(a)|$ and **O**(2.11) shows that $|\Delta_2^1(a) \cap \Delta_1(x)| = 6$ - note that all the line orbits from $y \in \Delta_2^1(a)$ have already been accounted for except $\alpha_{0, 2^4}(y, \Lambda_1, \Lambda_2)$.

Suppose (iv) is false, and argue for a contradiction. Then, by Lemma 5.2, $b, c \notin \Gamma_0(X)$. By Lemma 3.5, there exists $Y \in \beta_3(a, X) \cup \{X\}$ with $a + b \in \alpha_{15_3}(a, Y)$. Set $\tau = \tau(Y)$. By Lemma 4.5 $\tau \in Q(X)$ and so $a^\tau = a$ and $x^\tau = x$. Also, from Lemma 4.4, $b \neq b^\tau \in \Gamma_0(a + b)$. Note that $b, b^\tau \in \Delta_2(x)$ and that b and b^τ are in the same G_x -orbit. Lemma 7.1 implies that $a \in \Delta_3^8(x)$. If $b \in \Delta_2^1(x) \cup \Delta_2^3(x)$, then part (iii) (with a and x

interchanged) yields that $b \in \Gamma_0(X)$. Thus $b, b^\tau \in \Delta_2^2(x)$. Using Lemma 7.3(ii) (with x in place of a) we infer that $a \in \Delta_1(x)$, a contradiction. That $\Delta_2^2(a) \cap \Delta_1(x) = \emptyset$ follows from Lemma 3.7(ii). \square

We now consider the set

$$\Delta_3^{10}(a) = \{x \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^3(a) \cap \Delta_1(x) \text{ such that } c+x \in \alpha_{422}(c, \mathcal{T}_{ca})\},$$

where \mathcal{T}_{ca} is the unique element of $\Gamma_2(a, c)$.

Lemma 7.5. $\Delta_3^{10}(a) \subseteq \Delta_3(a)$ and $\Delta_3^{10}(a) \cap \Delta_3^8(a) = \emptyset$ and so $\Gamma_3(a, x) = \emptyset$.

Proof. Let $x \in \Delta_3^{10}(a)$ and $c \in \Delta_2^3(a) \cap \Delta_1(x)$ such that $c+x \in \alpha_{422}(c, \mathcal{T}_{ca})$. If $x \in \Delta_3^8(a)$, then there exists $X \in \Gamma_3(a, x)$ and, by Lemma 7.4(iv), in Ω_c , the octad $c+x$ would intersect \mathcal{T}_{ca} in 4^2 , a contradiction. So $\Delta_3^{10}(a) \cap \Delta_3^8(a) = \emptyset$ and $\Gamma_3(a, x) = \emptyset$. If $x \in \Delta_1(a)$, then $x \in \{a, c\}^\perp$ and so $x \in \Gamma_0(X)$ for each $X \in \Gamma_3(a, c)$, a contradiction. Suppose that $x \in \Delta_2(a)$. Then Lemma 7.3 gives that $x \in \Delta_2^1(a) \cup \Delta_2^3(a)$. However this contradicts $\Gamma_3(a, x) = \emptyset$ again. Therefore $x \in \Delta_3(a)$ by definition. \square

We now turn to $\Delta_3^7(a)$. Recall from (2.2) that

$$\Delta_3^7(a) = \{x \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^1(a) \cap \Delta_1(x) \text{ such that}$$

$$c+y \in \alpha_{0,2^4}(c, \Omega_c \cap \Omega_a, \mathcal{S}_{ca})\},$$

where $\Omega_c \cap \Omega_a$ is the tetrad of Ω_a described in Lemma 5.5 and \mathcal{S}_{ca} is the sextet in Ω_a corresponding to the unique element of $\Gamma_3(a, c)$.

The next result shows the link between $\Delta_3^7(a)$ and $\Delta_3^{10}(a)$.

Lemma 7.6. For any $x \in \Gamma_0$, $x \in \Delta_3^{10}(a)$, if and only if $a \in \Delta_3^7(a)$.

Proof. Let $x \in \Delta_3^{10}(a)$ and let $c \in \Delta_2^3(a) \cap \Delta_1(x)$ with $c + x \in \alpha_{4,2}(c, \mathcal{T}_{ca})$. If $\{a, c\}^\perp = \{b_1, b_2, b_3\}$ we may suppose that $b_1 \in \Delta_2^1(x)$ and $b_2, b_3 \in \Delta_2^2(a)$. In Ω_{b_1} , the octad $b_1 + c$ is incident with the sextet $X(b_1, x)$, where $X(b_1, x)$ is the unique element of $\Gamma_3(b_1, x)$ (see Lemma 5.4). Also $(b_1 + a) \cap (b_1 + c) = \emptyset$ as octads in Ω_{b_1} because $c \in \Delta_2^3(a)$. Therefore $b_1 + a \in \alpha_{2,4}(b_1, X(b_1, x))$ and $|(b_1 + a) \cap t(b_1, x)| = 0$, where T is the tetrad contained in $b_1 + d$ for all $d \in \{b_1, x\}^\perp$. Therefore $a \in \Delta_3^7(x)$ by definition.

Conversely assume $a \in \Delta_3^7(x)$ and let $b \in \Delta_2^1(x) \cap \Delta_1(a)$ with $b + a \in \alpha_{2,4}(b, X)$, where X is the unique element of $\Gamma_3(b, x)$ and $|(b + a) \cap t(b, x)| = 0$ in Ω_b , where $t(b, x) = \Omega_b \cap \Omega_x$. Then there exists $d \in \{b, x\}^\perp$ such that $b + x \in \Gamma_1(X)$ and $(b + d) \cap (b + a) = \emptyset$ in Ω_b . Hence $d \in \Delta_2^3(a)$ and now $x \in \Delta_3^{10}(a)$ by definition. \square

Lemma 7.7. *Suppose that $x_1, x_2 \in \Delta_2(a)$ and $x_1 \in \Delta_1(x_2)$. Let $\Gamma_0(x_1 + x_2) = \{x_1, x_2, x\}$. Then $x_1, x_2 \in \Delta_2^i(a)$ for the same $i \in \{1, 2, 3\}$ and $x \in \Delta_1(a)$.*

Proof. If $x_1 \in \Delta_2^3(a)$, the lemma follows from Lemma 7.3. So we may assume $x_1 \in \Delta_2^1(a) \cup \Delta_2^3(a)$. The point distributions (see, for $\Delta_2^1(a)$, Lemmas 5.5, 7.4 and Section 6 and, for $\Delta_2^3(a)$, Lemmas 5.10, 7.4, 7.5) of lines from $\Delta_2^1(a) \cup \Delta_2^3(a)$ are all known with the exception of $\alpha_{0,2^4}(x_1, \Lambda_1, \Lambda_2)$ when $x_1 \in \Delta_2^1(a)$. (The problem here with $\alpha_{0,2^4}(x_1, \Lambda_1, \Lambda_2)$ is that $\Delta_3^7(a)$ could equal one of the G_a -orbits in $\Delta_2(a)$.) From Lemmas 7.5 and 7.6,

we deduce that $\Delta_3^7(a) \subseteq \Delta_3(a)$. In particular, for $\ell \in \alpha_{0,2^4}(x_1, \Lambda_1, \Lambda_2)$, $\Lambda_0(\ell) \cap \Delta_2^1(a) = \{x_1\} = \Lambda_0(\ell) \cap \Delta_2(a)$, so completing the proof of Lemma 7.7. \square

Lemma 7.8. *Let $x \in \Delta_2^{10}(a)$ and $c \in \Delta_2^3(a) \cap \Delta_1(x)$.*

(i) *We have $\Delta_2(x) \cap \Delta_1(a) = \{a, c\}^\perp$ with $|\Delta_2^1(x) \cap \Delta_1(a)| = 1$, $|\Delta_2^2(x) \cap \Delta_1(a)| = 2$ and $|\Delta_2^3(x) \cap \Delta_1(a)| = 0$.*

(ii) *If $b \in \Delta_2^1(x) \cap \Delta_1(a)$, then $\Delta_2(a) \cap \Delta_1(x) = \{b, x\}^\perp$ with $|\Delta_2^1(a) \cap \Delta_1(x)| = 0$, $|\Delta_2^2(a) \cap \Delta_1(x)| = 4$ and $|\Delta_2^3(a) \cap \Delta_1(x)| = 1$.*

Proof. In Ω_c , for every $b \in \{a, c\}^\perp$, the octad $c + b$ is incident with the trio \mathcal{T}_{ca} and since $c + x \in \alpha_{42^2}(c, \mathcal{T}_{ca})$ we get $|\Delta_2^1(x) \cap \{a, c\}^\perp| = 1$, $|\Delta_2^2(x) \cap \{a, c\}^\perp| = 2$, and $|\Delta_2^3(x) \cap \{a, c\}^\perp| = 0$ from the definitions of $\Delta_2^i(x)$ ($i = 1, 2, 3$). Let $\{b\} = \Delta_2^1(x) \cap \{a, c\}^\perp$. In Ω_b , the two octads $b + a$ and $b + c$ are incident with the trio \mathcal{T}_{ca} and so the octads are disjoint. Let X be the unique element of $\Gamma_3(b, x)$. Then $b + a \in \alpha_{2^4}(b, X)$. Therefore, for every $d \in \{b, x\}^\perp \setminus \{c\}$, the octads $b + a$ and $b + d$ intersect in exactly two elements of Ω_b . So $|\Delta_2^1(a) \cap \{b, x\}^\perp| = 0$, $|\Delta_2^2(x) \cap \{b, x\}^\perp| = 4$, and $|\Delta_2^3(a) \cap \{b, x\}^\perp| = 1$.

To complete the proof by Lemma 7.8, it is enough to show that $\Delta_2(x) \cap \Delta_1(a) = \{a, c\}^\perp$. Assume that $b_1 \in \Delta_2(x) \cap \Delta_1(a)$ with $b_1 \notin \{a, c\}^\perp$. If $a + b_1 \in \Gamma_1(X)$ for some $X \in \Gamma_3(a, c)$, then $a + b_1 \in \Gamma_1(Y) \cup \alpha_{2^4}(a, Y)$ for every $Y \in \Gamma_3(a, c)$ and so $\tau(Y) \in \mathcal{Q}(a)_{b_1}$ by Lemmas 4.2(i) and 4.4. By the definition of $\Delta_3^{10}(a)$, we can find $Y \in \Gamma_3(a, c)$ with $c + x \in \alpha_{1^5_3}(c, Y)$

and then $\tau(Y) \notin G_x$ by Lemma 4.4. So $x, x^{\tau(Y)} \in \Delta_2(b_1) \cap \Gamma_0(c+x)$ and Lemma 7.7 gives $c \in \Delta_1(b_1)$, contrary to the choice of b_1 . Therefore $a+b_1 \notin \Gamma_1(X)$ for all $X \in \Gamma_3(a, c)$ and so in Ω_a , the octad $a+b_1$ intersects the trio \mathcal{T}_{ca} in 42^2 .

We now show that $b_1 \in \Delta_2^2(x)$. Let X be the unique element of $\Gamma_3(b, x)$. Assume $b_1 \notin \Delta_2^2(x)$ for a contradiction. Then there exists $Y \in \Gamma_3(x, b_1)$. If $X \in \beta_i(x, Y)$ for $i = 1, 0$, then there exists $d \in \{x, b_1\}^\perp$ with $x+d \in \alpha_{15_3}(x, X)$. By Lemma 4.4, $d^{\tau(X)} \neq d$. Since $b+a \in \alpha_{2^4}(b, X)$, using Lemma 4.4 again we have $a^{\tau(X)} = a$. Using Lemma 7.7 with d and $d^{\tau(X)}$, we get $x \in \Delta_1(a)$. So we must have $X \in \beta_3(x, Y)$. We can choose $d \in \{b, x\}^\perp$ with $x+d \in \alpha_{4_2}(x, Y)$. So $d \in \Gamma_0(Y)$. If $d \in \Delta_1(b_1)$ then $d \notin \Delta_2^2(a)$ by Lemma 5.8(ii). Then $d = c$ from the first part of the proof. This contradicts the fact that $b_1 \notin \{a, c\}^\perp$. If $d \in \Delta_2(b_1)$, then Lemma 7.7 implies that the point in $\Gamma_0(a+b_1) \setminus \{a, b_1\}$ lies in $\Delta_1(d)$ and using Lemma 7.7 again we get $a \in \Delta_1(x)$. So $d \in \Delta_3^8(b_1)$ because $\Gamma_3(d, b_1) \neq \emptyset$. However Lemma 7.4(iv) now yields $\Gamma_3(a, x) \neq \emptyset$ which contradicts Lemma 7.5. Hence we have shown that $b_1 \in \Delta_2^2(x)$.

Let d be the unique point in $\{x, b_1\}^\perp$. We can choose $Y \in \Gamma_3(a, c)$ such that $c+x \in \alpha_{2^4}(c, Y)$. Then $\tau(Y)$ fixes x by Lemma 4.4. Assume $d \notin \Delta_2^2(a)$ and let $Z \in \Gamma_3(a, d)$. If $Z \notin \Gamma_3(\mathcal{T}_{ca})$ we could choose $Y_1 \in \Gamma_3(\mathcal{T}_{ca})$ such that $Y_1 \in \beta_i(a, Z)$ for $i = 0$ or 1 and $b' \in \{a, d\}^\perp$ with $a+b' \in \alpha_{15_3}(a, Y_1)$. So $\tau(Y_1)$ does not fix b' by Lemma 4.4 and then Lemma 7.7 gives $a \in \Delta_1(x)$, a contradiction. Therefore $Z \in \Gamma_3(\mathcal{T}_{ca})$. Applying a similar argument to the one used to show $b_1 \in \Delta_2^2(x)$, we can prove that $d \in \Delta_2^2(a)$.

Since $b_1 \notin \{a, c\}^\perp$, the octad $a + b_1$ in Ω_a is not incident with the trio \mathcal{T}_{ca} . Therefore, we can choose $Y \in \Gamma_3(a, c)$ with $\tau(Y) \in G_x$ and $b_1 \notin \Gamma_0(Y)$. If $\tau(Y)$ does not fix b_1 Lemma 7.7 would imply that $a \in \Delta_1(x)$ and so $\tau(Y) \in G_{b_1}$. From Lemma 4.8(ii), as $b_1 \notin \Gamma_0(Y)$, $\tau(Y) \notin Q(b_1)$. Hence $1 \neq \tau(Y)^{*b_1} \in O_2(G_{ab_1}^{*b_1}) \cong 2^4$ and because $d \in \Delta_2^2(a)$, the octads (in Ω_{b_1}) $b_1 + a$ and $b_1 + d$ intersect in two elements. As a consequence $\tau(Y)$ does not fix the line $b_1 + d$. In particular, $d^{\tau(Y)} \neq d$ and therefore, as $d, d^{\tau(Y)} \in \{b_1, x\}^\perp$, we have now have $|\{b_1, x\}^\perp| > 1$ which contradicts Lemma 5.8(ii). This completes the proof of the lemma. \square

Lemma 7.9. (i) $|\Delta_3^{10}(a)| = 2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.

(ii) G_a is transitive on $\Delta_3^{10}(a)$.

(iii) For $x \in \Delta_3^{10}(a)$, $G_{ax} \sim [2^9].S_4$ and $G_{ax}^{*x} \sim 2^6 : S_4$ is the stabilizer in G_x^{*x} of the tetrad $t(x, b)$ (where b is the unique element of $\Delta_1(a) \cap \Delta_2^1(x)$) and a partition of $x + c \setminus t(x, b)$ into two pairs of elements.

Proof. Let $x \in \Delta_3^{10}(a)$ and c be the unique point in $\Delta_2^3(a) \cap \Delta_1(x)$ (c exists by Lemma 7.8). Then $|\Delta_3^{10}(a) \cap \Delta_1(c)| = 2|\alpha_{42^2}(c, \mathcal{T}_{ca})| = 2 \cdot 672$ by **O**(2.4). By the uniqueness of c and Lemma 5.11(i), we have

$$|\Delta_3^{10}(a)| = 2 \cdot 672 \cdot |\Delta_2^3(a)| = 2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

For part (ii), working in Ω_c , there are four sextets $X \in \Gamma_3(a, c)$ such that $c + x \in \alpha_{15_3}(c, X)$ and so $\tau(X) \notin G_x$ by Lemma 4.4. Therefore G_{ac} is transitive on $\Gamma_0(c + x) \setminus \{c\}$. Now part (ii) follows because $\alpha_{15_3}(c, X)$ is a G_{ac} -orbit on $\Gamma_1(c)$ and $\Delta_2^3(a)$ is a G_a -orbit of points by Lemma 5.11(ii).

Turning to part (iii) we have $G_{ax} \leq G_{ac}$. Let $b \in \{a, c\}^\perp \cap \Delta_2^1(x)$ (b exists and is unique by Lemma 7.8(i)). By Lemma 7.8(ii) there exists $c_1, c_2 \in \{b, x\}^\perp \cap \Delta_2^2(a)$ with $c_1 \neq c_2$. We show that $|Q(a)_x| \leq 2^3$ by first proving that $Q(a)_x \leq Q(a)_{c_i}$ for $i = 1, 2$. Assume $g \in Q(a)_x \setminus Q(a)_{c_i}$ for a contradiction. If $b^g = b$, then in Ω_b the octads $b + c_1$ and $b + c_1^g$ contain the same two elements of $b + a$. However Lemma 5.5 implies that $t(b, x) \subseteq b + c$ which gives $(b + a) \cap (b + c) \neq \emptyset$, contrary to Lemma 5.9. So $b^g \neq b$ and we can use Lemma 7.7 to show that $a \in \Delta_1(x)$, a contradiction. So $Q(a)_x \leq Q(a)_{c_i}$ for $i = 1, 2$. Since $(b + a) \cap (b + c_i) = \emptyset$ in Ω_b , there are seven hyperplanes $Y_i \in \Gamma_3(a, b)$ ($i = 1, \dots, 7$) with $\tau(Y_i) \in G_{c_1c_2}$ and the subgroup generated by the elements $\tau(Y_i)$ has order at least 2^4 . Further we can show that, up to relabelling $Q(a) \cap Q(c_1) = \langle \tau(Y_1)\tau(Y_2)\tau(Y_3) \rangle \leq Q(a)_{c_1c_2}$ (see Lemma 6.15 in [17] for details). Since $Q(a)_{c_1c_2} \neq Q(a)_{c_1}$ we have $|Q(a)_{c_1c_2}| = 2^4$ by Theorem 4. Therefore $|Q(a)_{c_1c_2}^{*c_1}| = 2^3$. In Ω_{c_1} , the octads $c_1 + b$ and $c_1 + x$ intersect in four elements and the subgroup of $O_2(G_{c_1b}^{*c_1})$ fixing $c_1 + x$ is of order 2^2 . Therefore $|Q(a)_{c_1c_2x}^{*c_1}| \leq 2^2$ and so $|Q(a)_x| \leq 2^3$, as required.

By parts (i) and (ii), $[G_{ac} : G_{ax}] = 2^6.3.7$. Since $|Q(a)_c| = 2^9$ by Theorem 5 we must have $|Q(a)_x| \leq 2^3$ and so $|Q(a)_x| = 2^3$ and $[G_{ac}^{*a} : G_{ax}^{*a}] = 3.7$. Using the ATLAS [2] and Theorem 5 we get $G_{ax}^{*a} \sim 2^6 : S_4$. This completes the proof of the lemma. □

Lemmas 7.6 and 7.8 now imply

Lemma 7.10. *Let $x \in \Delta_3^7(a)$ and $c \in \Delta_2^1(a) \cap \Delta_1(x)$. Then*

(i) $\Delta_2(x) \cap \Delta_1(a) = \{a, c\}^\perp$ with $|\Delta_2^1(x) \cap \Delta_1(a)| = 0$, $|\Delta_2^2(x) \cap \Delta_1(a)| = 4$ and $|\Delta_2^3(x) \cap \Delta_1(a)| = 1$.

(ii) *If $b \in \Delta_2^3(x) \cap \Delta_1(a)$, then $\Delta_2(a) \cap \Delta_1(x) = \{b, x\}^\perp$ with $|\Delta_2^1(a) \cap \Delta_1(x)| = 1$, $|\Delta_2^2(a) \cap \Delta_1(x)| = 2$ and $|\Delta_2^3(a) \cap \Delta_1(x)| = 0$.*

Lemma 7.11. (i) $|\Delta_3^7(a)| = 2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.

(ii) G_a is transitive on $\Delta_3^7(a)$.

(iii) *For $x \in \Delta_3^7(a)$, $G_{ax} \sim [2^9] \cdot S_4$ and $G_{ax}^{*x} \sim 2^5 : S_4$ is the stabilizer in G_x^{*x} of the octad $x + d$ (where d is the unique element of $\Delta_2^2(a) \cap \Delta_1(x)$), the trio T_0 which is the unique element of $\Gamma_3(b, x)$ for $b \in \Delta_2^3(x) \cap \Delta_1(a)$ and a partition of the octad $x + d$ into four 2-element sets.*

Proof. Let $x \in \Delta_3^7(a)$ and $c \in \Delta_2^1(a) \cap \Delta_1(x)$. Then $|\Delta_3^3(a) \cap \Delta_1(c)|$ is twice the number of octads in Ω_c lying in $\alpha_{24}(c, X(c, a))$ that have an empty intersection with $t(c, a)$. This number is 240. Therefore Lemmas 5.6 and 7.10 give

$$|\Delta_3^7| = 2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

Let $x' \in \Gamma_0(c + x) \setminus \{c, x\}$. Then by definition $x' \in \Delta_3^7(a)$. Since $c + x \notin \Gamma_1(X(c, a))$, there exists $Y \in \beta_3(c, X(c, a))$ with $c + x \in \alpha_{15_3}(c, Y)$. For this Y we have $x^{\tau(Y)} = x'$ by Lemma 4.4. By **O**(2.2) and Lemma 5.5(iii), G_{ca}^{*c} is transitive on the lines in $\alpha_{24}(c, X(c, a))$ that have an empty intersection with $t(c, a)$ and so part (ii) follows from the transitivity of G_a on $\Delta_2^2(a)$ (see Lemma 5.6(ii)).

For part (iii) we know that $a \in \Delta_3^{10}(x)$ by Lemma 7.6 and hence $G_{ax} \sim [2^9].S_4$ by Lemma 7.9(iii). Let $b \in \Delta_2^3(x) \cap \Delta_1(a)$ and $e_1, e_2 \in \{b, x\}^\perp \cap \Delta_2^2(a)$ with $e_1 \neq e_2$. (Such points exist by Lemma 7.10.) Assume $Q(a)_x \not\leq Q(a)_{e_1}$ and let $g \in Q(a)_y \setminus Q(a)_{e_1}$. If $b^g = b$, then $g^{*b} \in O_2(G_{ba}^{*b})$ and so in Ω_b , the octads $b + e_1$ and $b + e_1^g$ intersect $b + a$ in the same two elements. However $e_1^g \in \{b, x\}^\perp$ and so $(b + e_1) \cap (b + e_1^g) = \emptyset$ because $b \in \Delta_2^3(x)$. Therefore $b^g \neq b$. Since $b^g \in \Gamma_0(a + b)$, Lemma 7.7 implies that $a \in \Delta_1(x)$, a contradiction. Therefore $Q(a)_x \leq Q(a)_{e_1}$ and similarly $Q(a)_x \leq Q(a)_{e_2}$. Using an argument similar to that in the proof of Lemma 7.9(iii), we get $|Q(a)_x| \leq 2^4$.

Since c is the unique point in $\Delta_2^1(a) \cap \Delta_1(x)$, $G_{ax} \leq G_{ac}$. By Lemma 5.6(iii) we have $|Q(c)_a| = 2^7$. Therefore $Q(c)_a^{*a} \leq O_2(G_{ac}^{*a})$. Since $c + x \in \alpha_{2^4}(c, X(c, a))$, there exists $Y \in \beta_3(c, X(c, a))$ such that $c + x \in \alpha_{15_3}(c, Y)$. Then Lemma 4.4 implies that $\tau(Y) \notin G_x$. However $\tau(Y) \in Q(c) \cap Q(X(c, a))$ and $\tau(Y) \notin Q(a)$. Therefore $|O_2(G_{xa}^{*x})| \leq 2^5$ and so $G_{xa}^{*x} \sim 2^5.S_4$ and $|Q(x)_a| = 2^4$. \square

We end this section by examining the set

$$\Delta_3^9(a) = \{x \in \Gamma_0 \mid \text{there exists } c \in \Delta_2^2(a) \cap \Delta_1(x) \text{ such that}$$

$$c + x \in \alpha_{2,0}(c, c + b, \mathcal{D}_{ca}), \text{ where } \{b\} = \{a, c\}^\perp\}.$$

Lemma 7.12. $\Delta_3^9(a) \cap \Delta_3^i(a) = \emptyset$ for $i = 1, \dots, 8$ and $i = 10$.

Proof. Since $\Omega_a \cap \Omega_x = \emptyset$ by definition, $\Delta_3^9(a) \cap \Delta_3^i(a) = \emptyset$ for $i = 1, \dots, 6$. By Lemma 7.4(iii), $\Delta_3^9(a) \cap \Delta_3^8(a) = \emptyset$. By Lemmas 7.8 and 7.10 and **O**(2.3) if $x \in \Delta_3^7(a) \cup \Delta_3^{10}(a)$, then $|(c+x) \cap (c+b)| = 0$ or 4 in Ω_c for any $c \in \Delta_2^2(a) \cap \Delta_1(x)$. Therefore $\Delta_3^9(a) \cap \Delta_3^i(a) = \emptyset$ for $i = 7, 10$ as required. \square

Lemma 7.13. *Let $x \in \Delta_3^9(a)$. Then there exists a unique path of length three between a and x in \mathcal{G} .*

Proof. Let $c \in \Delta_2^2(a) \cap \Delta_1(x)$ with $c+x \in \alpha_{20}(c, c+b, \mathcal{D}_{ca})$, and $\{b\} = \{a, c\}^\perp$. Then $b \in \Delta_2^2(x)$ by definition. Assume that a, b_1, c_1, x is another path of length three in \mathcal{G} . By Lemmas 7.8, 7.10, and 7.4 and **O**(2.2) and **O**(2.4), we must have $c_1 \in \Delta_2^2(a)$ and $b_1 \in \Delta_2^2(x)$. It then follows from Lemma 5.8(ii) that $b_1 \neq b$ and $c_1 \neq c$. Therefore $b_1 \in \Delta_2(b)$ and we consider the three possible choices separately.

First assume that $b_1 \in \Delta_2^3(b)$. Notice that $c_1 \notin \Delta_1(b) \cup \Delta_2(b)$ by Lemma 7.7 and Lemma 5.8(ii) and so $c_1 \in \Delta_3(b)$. Therefore $c_1 \in \Delta_3^8(b) \cup \Delta_3^{10}(b)$ by **O**(2.4). However Lemma 7.4(iii) implies that $c_1 \in \Delta_3^{10}(b)$. We now have $c \in \Delta_2^1(c_1) \cap \{b, b_1\}^\perp$ by Lemma 7.8 and so $c, c_1 \in \{x, b_1\}^\perp$, contrary to Lemma 5.8(ii).

Next suppose that $b_1 \in \Delta_2^1(b)$. Therefore $c \in \Delta_3^7(b_1) \cup \Delta_3^8(b_1)$. Using Lemma 7.4(iii) we must have $c \in \Delta_3^7(b_1)$. This again leads to the contradiction that $c, c_1 \in \{x, b_1\}^\perp$.

Therefore we must have $b_1 \in \Delta_2^2(b)$. In Ω_c , $c+x \cap \mathcal{D}_{ca} = \emptyset$ and $|c+x \cap c+b| = 2$. Using the MOG and Lemma 4.4, we can find a sextet $Y \in \Gamma_3(c, b)$ with $\tau(Y) \in G_{ax}$. Since $\tau(Y) \notin Q(a)$ and $a+b_1 \in \alpha_2(a,$

$a + b$), $\tau(Y)$ does not fix $a + b_1$. However by the above argument we must have $a + b_1^{\tau(Y)} \in \alpha_2(a, a + b) \cap \alpha_2(a, a + b_1)$ and $(a + b) \cap (a + b_1) = (a + b) \cap (a + b_1^{\tau(Y)})$ in Ω_a . As this cannot occur we again get a contradiction. This completes the proof of the lemma. \square

Lemma 7.14. (i) $|\Delta_3^9(a)| = 2^{13} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.

(ii) G_a is transitive on $\Delta_3^9(a)$.

(iii) For $x \in \Delta_3^9(a)$, $G_{ax} \sim [2^5] \cdot S_4$ and $G_{ax}^* \sim 2^4 \cdot S_4$.

Proof. Let $x \in \Delta_3^9(a)$ and let a, b, c, x be the unique path of length three between a and x in \mathcal{G} .

(i) From **O(2.4)**, $\Delta_1(c) \cap \Delta_3^9(a) = 2 \times 240 = 2^5 \cdot 3 \cdot 5$. Using Lemmas 5.8(ii) and Lemma 7.13 we then have $|\Delta_3^9(a)| = 2^{13} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.

(ii) Since $c + x \in \alpha_{2,0}(c, c + b, \mathcal{D}_{ca})$ and using Lemma 4.4, we can find $Y \in \Gamma_3(b, c)$ with $\tau(Y) \in G_a \setminus G_x$. Since $\tau(Y)$ fixes $c + x$, $\tau(Y)$ interchanges the points in $\Gamma_0(c + x) \setminus \{c\}$. Because G_a is transitive on $\Delta_2^2(a)$ and $\alpha_{2,0}(c, c + b, \mathcal{D}_{ca})$ is a G_{ac} -orbit, G_a is transitive on $\Delta_3^9(a)$.

(iii) We have $G_{ax} \leq G_{ac}$. Since $c + x \in \alpha_2(c, c + b)$, then $Q(a)_{cx}^* = 1$ and so $Q(a)_x \leq Q(a) \cap Q(c)$. Using the MOG we see there exist $Y_1, Y_2, Y_3 \in \Gamma_3(c + b)$ with $Q(a) \cap Q(c) = \langle \tau(Y_1)\tau(Y_2)\tau(Y_3) \rangle$. Further, if δ is the duad in Ω_c fixed by G_{ca}^* and t_i is the tetrad in Y_i containing δ ($i = 1, 2, 3$), then of the six elements in $(c + a) \setminus \delta$ in Ω_c , three lie in exactly two of the tetrads t_i and three lie in none of the tetrads t_i . (For details, see Proposition 8.12 in [17] where $\Delta_2^2(a)$ is denoted by $\Delta_2^3(a)$.) Since $\Omega_a \cap \Omega_x = \emptyset$ we have that $c + x \in \alpha_{15_3}(c, Y_i)$ for precisely

two or none of the sextets Y_i , $i = 1, 2, 3$. Therefore $\tau(Y_1)\tau(Y_2)\tau(Y_3) \in G_x$ by Lemma 4.4. Hence $|Q(a)_x| = 2$ and it follows that $[G_{xb}^{*x} : G_{xa}^{*x}] = 2.3.5$. Since $G_{xb}^{*x} \sim 2^4.S_6$ by Lemma 5.8(iii) we must have $G_{xa}^{*x} \sim 2^4.S_4$, as required. □

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